

ON COHERENCE OF ENDOMORPHISM RINGS

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Abstract

Let R be a ring and U a left R -module with $S = \text{End}({}_R U)$. The aim of this paper is to characterize when S is coherent. We first show that a left R -module F is T_U -flat if and only if $\text{Hom}_R(U, F)$ is a flat left S -module. This removes the unnecessary hypothesis that U is Σ -quasiprojective from Proposition 2.7 of Gomez Pardo and Hernandez [‘Coherence of endomorphism rings’, *Arch. Math. (Basel)* **48**(1) (1987), 40–52]. Then it is shown that S is a right coherent ring if and only if all direct products of T_U -flat left R -modules are T_U -flat if and only if all direct products of copies of ${}_R U$ are T_U -flat. Finally, we prove that every left R -module is T_U -flat if and only if S is right coherent with $\text{wD}(S) \leq 2$ and U_S is FP -injective.

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1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary modules. For a ring R , ${}_R M$ (M_R) denotes a left (right) R -module. In what follows, U is a left R -module and $S = \text{End}({}_R U)$. We denote by $\text{add } {}_R U$ the category consisting of all left R -modules isomorphic to direct summands of finite direct sums of copies of ${}_R U$ and by $\text{pres}(U)$ the category of all finitely U -presented left R -modules, that is, of all left R -modules M admitting an exact sequence $U^n \rightarrow U^m \rightarrow M \rightarrow 0$ with m, n positive integers. Here H denotes $\text{Hom}_R(U, -)$ and T means $U \otimes_S -$. Given a left R -module M and a left S -module A , define $v_M : TH(M) \rightarrow M$ and $\eta_A : A \rightarrow HT(A)$ via $v_M(u \otimes f) = f(u)$ and $\eta_A(a)(u) = u \otimes a$ for any $u \in U$, $f \in H(M)$ and $a \in A$. For a module M , M^I ($M^{(I)}$) is the direct product (sum) of copies of M indexed by a set I , $\text{pd}(M)$ denotes the projective dimension of M , and the character module $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+ . As usual, we use $\text{wD}(S)$ to denote the weak global dimension of a ring S . General background material can be found in [1, 7, 13, 16].

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Gomez Pardo and Hernandez [11] have given conditions under which S is a coherent ring assuming that ${}_R U$ is (Σ) -quasiprojective. Our aim is to characterize when S is coherent for a general left R -module ${}_R U$. We start by proving that a left R -module F is T_U -flat if and only if $H(F)$ is a flat left S -module. This removes the unnecessary hypothesis that U is Σ -quasiprojective from [11, Proposition 2.7]. Then it is shown that S is a right coherent ring if and only if all direct products of T_U -flat left R -modules are T_U -flat if and only if all direct products of copies of ${}_R U$ are T_U -flat. Moreover, if both ${}_R U$ and U_S are finitely presented, then we obtain that S is a right coherent ring if and only if F^{++} is T_U -flat for every T_U -flat left R -module F . Finally, we prove that every left R -module is T_U -flat if and only if S is right coherent with $\text{wD}(S) \leq 2$ and U_S is FP -injective.

Next we recall some known notions and facts required in the paper.

A left R -module M is *quasiprojective* [1] if, for every quotient module L of M , the canonical homomorphism $\text{Hom}_R(M, M) \rightarrow \text{Hom}_R(M, L)$ is epic. On the other hand, M is called Σ -*quasiprojective* when every direct sum $M^{(I)}$ is quasiprojective. A left R -module F is called T_U -*flat* (see [11]) if for every homomorphism $f : K \rightarrow F$ with $K \in \text{pres}(U)$, there exist homomorphisms $g : K \rightarrow U^n$ and $h : U^n \rightarrow F$ for some integer n such that $f = hg$. Note that if U is a finitely generated projective generator of the category of all left R -modules, the M is T_U -flat if and only if M is flat.

Let \mathcal{C} be a class of left R -modules and M a left R -module. A homomorphism $\phi : M \rightarrow F$ with $F \in \mathcal{C}$ is called a \mathcal{C} -*preenvelope* of M [8] if for any homomorphism $f : M \rightarrow F'$ with $F' \in \mathcal{C}$, there is a homomorphism $g : F \rightarrow F'$ such that $g\phi = f$.

A left R -module M is *small* [7, p. 6] if the covariant functor $\text{Hom}(M, -)$ commutes with arbitrary direct sums. It is well known that finitely generated modules are always small.

A right S -module N is called FP -*injective* [14] if $\text{Ext}_S^1(F, N) = 0$ for every finitely presented right S -module F . When S_S is FP -injective, S is said to be right FP -injective.

A ring R is *right coherent* [4] when every finitely generated left ideal of R is finitely presented and *left IF* [6] when every injective left R -module is flat.

2. Coherence of endomorphism rings

Let U be a Σ -quasiprojective left R -module and F a left R -module, then $H(F)$ is a flat left S -module if and only if F is a T_U -flat module (see [11, Proposition 2.7]). In fact, this result is true for any left R -module U as shown by the following proposition.

PROPOSITION 2.1. *Let ${}_R U$ be a module with $S = \text{End}({}_R U)$ and F be a left R -module. Then $H(F)$ is a flat left S -module if and only if F is a T_U -flat module.*

PROOF. Assume that $H(F)$ is a flat left S -module. Let $M \in \text{pres}(U)$ and $\alpha : M \rightarrow F$ be an R -homomorphism. Then there is an exact sequence $0 \rightarrow K \rightarrow U^k \rightarrow U^l \rightarrow M \rightarrow 0$ with k, l some positive integers. Let $Y = \text{Coker}(H(U^k) \rightarrow H(U^l))$, then

Y is a finitely presented left S -module. The exactness of $0 \rightarrow H(K) \rightarrow H(U^k) \rightarrow H(U^l) \rightarrow Y \rightarrow 0$ induces the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
 TH(U^k) & \longrightarrow & TH(U^l) & \longrightarrow & T(Y) & \longrightarrow & 0 \\
 \downarrow v_{U^k} & & \downarrow v_{U^l} & & \downarrow \sigma & & \\
 U^k & \longrightarrow & U^l & \longrightarrow & M & \longrightarrow & 0
 \end{array}$$

Note that v_{U^k} and v_{U^l} are isomorphisms, and so the induced homomorphism σ is an isomorphism. Since $H(F)$ is a flat left S -module, there exist homomorphisms $f : Y \rightarrow S^n$ and $g : S^n \rightarrow H(F)$ for some integer n such that $H(\alpha)H(\sigma)\eta_Y = gf$. Note that $v_{T(Y)}T(\eta_Y) = 1_{T(Y)}$ by [7, Equality 2.1, p. 13] and the diagram

$$\begin{array}{ccccc}
 THT(Y) & \xrightarrow{TH(\sigma)} & TH(M) & \xrightarrow{TH(\alpha)} & TH(F) \\
 \downarrow v_{T(Y)} & & \downarrow v_M & & \downarrow v_F \\
 T(Y) & \xrightarrow{\sigma} & M & \xrightarrow{\alpha} & F
 \end{array}$$

is commutative. Thus,

$$\begin{aligned}
 v_F T(g)T(f)\sigma^{-1} &= v_F T(gf)\sigma^{-1} \\
 &= v_F T(H(\alpha)H(\sigma)\eta_Y)\sigma^{-1} \\
 &= v_F TH(\alpha)TH(\sigma)T(\eta_Y)\sigma^{-1} \\
 &= \alpha v_M TH(\sigma)T(\eta_Y)\sigma^{-1} \\
 &= \alpha \sigma v_{T(Y)}T(\eta_Y)\sigma^{-1} \\
 &= \alpha \sigma \sigma^{-1} = \alpha.
 \end{aligned}$$

Clearly, $T(f)\sigma^{-1} : M \rightarrow T(S^n)$ and $v_F T(g) : T(S^n) \rightarrow F$ are homomorphisms, and $T(S^n) \cong U^n$. So F is T_U -flat.

Conversely, suppose that F is T_U -flat and $f : X \rightarrow H(F)$ is an S -homomorphism with X a finitely presented left S -module. Note that $T(X) \in \text{pres}(U)$, then there are R -homomorphisms $g : T(X) \rightarrow U^n$ and $h : U^n \rightarrow F$ satisfying $v_F T(f) = hg$. Since $H(v_F)\eta_{H(F)} = 1_{H(F)}$ by [7, Equality 2.1, p. 13], it follows that

$$H(h)(H(g)\eta_X) = H(hg)\eta_X = H(v_F)HT(f)\eta_X = H(v_F)\eta_{H(F)}f = f,$$

and hence f factors through $H(U^n) \cong S^n$. So $H(F)$ is a flat left S -module. □

The following corollary is an immediate consequence of Proposition 2.1.

COROLLARY 2.2. *Let U be a left R -module.*

- (1) $\bigoplus_{i=1}^n F_i$ is T_U -flat if and only if each F_i is T_U -flat for any positive integer n .
- (2) If ${}_R U$ is small, then $\bigoplus_{i \in I} F_i$ is T_U -flat if and only if each F_i is T_U -flat for any index set I .

PROPOSITION 2.3. *Let ${}_R U$ be a module with $S = \text{End}({}_R U)$. The following are equivalent.*

- (1) *Every injective left R -module is T_U -flat.*
- (2) *For any $M \in \text{pres}(U)$, the injective envelope of M is T_U -flat.*
- (3) *Any $M \in \text{pres}(U)$ is finitely cogenerated by U .*

Moreover, if S is right coherent, then the above conditions are equivalent to:

- (4) *U_S is FP-injective.*

PROOF. That condition (1) implies (2) is clear.

(2) \Rightarrow (3). Let $M \in \text{pres}(U)$ and $i : M \hookrightarrow E(M)$ be an injective envelope of M . By condition (2), there exist homomorphisms $\alpha : M \rightarrow U^n$ and $\beta : U^n \rightarrow E(M)$ for some positive integer n such that $\beta\alpha = i$. Note that α is monic, and so condition (3) holds.

(3) \Rightarrow (1). For any homomorphism $\varphi : M \rightarrow E$ with $M \in \text{pres}(U)$ and E injective, by condition (3) there is a monomorphism $M \rightarrow U^n$ for some integer n , and hence φ factors through U^n . So condition (1) follows.

Moreover, if S is right coherent, then by [13, Theorem 9.51] and the remark following it, we have U_S is FP-injective if and only if $H(E)$ is flat for any injective left R -module E . So the equivalence of (1) and (4) follows from Proposition 2.1. \square

Specializing Proposition 2.3 to the case ${}_R U = {}_R R$ gives the following corollaries.

COROLLARY 2.4 (Part of [6, Theorem 1]). *The following are equivalent for a ring R .*

- (1) *R is left IF.*
- (2) *The injective envelope of every finitely presented left R -module is flat.*
- (3) *Every finitely presented left R -module is a submodule of a free module.*

COROLLARY 2.5 [12, Theorem 3.10]. *If R is a right coherent ring, then R is left IF if and only if R is right FP-injective.*

Let M and N be left R -modules. There is a natural homomorphism

$$\sigma = \sigma_{M,N} : \text{Hom}_R(M, U) \bigotimes_S \text{Hom}_R(U, N) \rightarrow \text{Hom}_R(M, N)$$

defined via $\sigma(f \otimes g)(m) = g(f(m))$ for all $f \in \text{Hom}_R(M, U)$ and $g \in \text{Hom}_R(U, N)$, $m \in M$.

It is easy to check that $\sigma_{M,N}$ is an isomorphism if $M \in \text{add}_R U$ or $N \in \text{add}_R U$.

LEMMA 2.6. *The following are equivalent.*

- (1) *A left R -module F is T_U -flat.*
- (2) *For any left R -module $M \in \text{pres}(U)$, $\sigma_{M,F}$ is an epimorphism (isomorphism).*

PROOF. (1) \Rightarrow (2). Let $M \in \text{pres}(U)$ and F be T_U -flat. Then there is an exact sequence $U^n \rightarrow U^m \rightarrow M \rightarrow 0$ with m, n some positive integers, and

so $0 \rightarrow \text{Hom}_R(M, U) \rightarrow S^m \rightarrow S^n$ and $0 \rightarrow \text{Hom}_R(M, F) \rightarrow \text{Hom}_R(U^m, F) \rightarrow \text{Hom}_R(U^n, F)$ are exact. Note that $\text{Hom}_R(U, F) = H(F)$ is a flat left S -module by Proposition 2.1, and hence we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Hom}_R(M, U) \otimes_S \text{Hom}_R(U, F) & \longrightarrow & \text{Hom}_R(U, F)^m & \longrightarrow & \text{Hom}_R(U, F)^n \\
 & & \downarrow \sigma_{M,F} & & \downarrow \cong & & \downarrow \cong \\
 0 & \longrightarrow & \text{Hom}_R(M, F) & \longrightarrow & \text{Hom}_R(U^m, F) & \longrightarrow & \text{Hom}_R(U^n, F)
 \end{array}$$

Thus condition (2) follows.

(2) \Rightarrow (1). Let $M \in \text{pres}(U)$ and $\alpha \in \text{Hom}_R(M, F)$. By condition (2), there are $f_i \in \text{Hom}_R(M, U)$ and $g_i \in \text{Hom}_R(U, F)$ for all $i = 1, 2, \dots, n$, such that $\alpha = \sigma_{M,F}(\sum_{i=1}^n f_i \otimes g_i)$. Define $f : M \rightarrow U^n$ via $f(m) = (f_i(m))$ for any $m \in M$ and $g : U^n \rightarrow F$ via $g((a_i)) = \sum_{i=1}^n g_i(a_i)$ for all $a_i \in U$. It is easy to check that $\alpha = gf$, as required. \square

LEMMA 2.7. *Let U be a finitely presented left R -module. Then the class of T_U -flat left R -modules is closed under pure submodules and direct limits.*

PROOF. Let F be a T_U -flat left R -module and K a pure module of F , then there is an exact sequence $0 \rightarrow K \xrightarrow{i} F \xrightarrow{\pi} F/K \rightarrow 0$, where i is the canonical injection and π is the canonical projection. For any left R -module $M \in \text{pres}(U)$ and any homomorphism $f : M \rightarrow K$, there are homomorphisms $g : M \rightarrow U^n$ and $h : U^n \rightarrow F$ for some integer n such that $if = hg$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 M & \xrightarrow{g} & U^n & \xrightarrow{p} & \text{Coker}(g) & \longrightarrow & 0 \\
 \downarrow f & \swarrow \gamma & \downarrow h & \swarrow \beta & \downarrow \alpha & & \\
 0 & \longrightarrow & K & \xrightarrow{i} & F & \xrightarrow{\pi} & F/K \longrightarrow 0
 \end{array}$$

where α is the induced homomorphism. Note that $\text{Coker}(g)$ is a finitely presented left R -module, then there exists a homomorphism $\beta : \text{Coker}(g) \rightarrow F$ satisfying $\pi\beta = \alpha$. It follows that there is a homomorphism $\gamma : U^n \rightarrow K$ such that $\gamma g = f$, and so K is T_U -flat.

Suppose that $\{F_i\}_{i \in I}$ is a direct system of T_U -flat left R -modules over a directed index set I . Let $M \in \text{pres}(U)$ and $f : M \rightarrow \lim_{\rightarrow} F_i$ be a homomorphism. Since U is finitely presented, so is M . By [10, Corollary 1.2.7], the epimorphism $\pi : \bigoplus_{i \in I} F_i \rightarrow \lim_{\rightarrow} F_i$ is pure. Thus, there is $g : M \rightarrow \bigoplus_{i \in I} F_i$ with $f = \pi g$. It follows that $\lim_{\rightarrow} F_i$ is T_U -flat since $\bigoplus_{i \in I} F_i$ is T_U -flat by Corollary 2.2(2). \square

THEOREM 2.8. *Let ${}_R U$ be a module with $S = \text{End}({}_R U)$. The following are equivalent.*

- (1) S is a right coherent ring.

- (2) All direct products of T_U -flat left R -modules are T_U -flat.
- (3) All direct products of copies of ${}_R U$ are T_U -flat.

Moreover, if ${}_R U$ and U_S are finitely presented, then the above conditions are also equivalent to the following.

- (4) Every left R -module has a T_U -flat preenvelope.
- (5) F^{++} is T_U -flat for every T_U -flat left R -module F .

PROOF. (1) \Rightarrow (2). Let $\{F_i\}_{i \in I}$ be a family of T_U -flat left R -modules. Then

$$H\left(\prod_{i \in I} F_i\right) = \text{Hom}_R\left(U, \prod_{i \in I} F_i\right) \cong \prod_{i \in I} \text{Hom}_R(U, F_i) = \prod_{i \in I} H(F_i)$$

is a flat left S -module by Proposition 2.1 and condition (1). Thus, $\prod_{i \in I} F_i$ is T_U -flat by Proposition 2.1 again.

The implication (2) implies (3) is clear.

(3) \Rightarrow (1). Note that, for any index set I , $S^I \cong \text{Hom}_R(U, U^I)$ is a flat left S -module by Proposition 2.1 and condition (3). So condition (1) follows.

(2) \Rightarrow (4). Let N be any left R -module. By [9, Lemma 5.3.12], for any homomorphism $f : N \rightarrow M$ where M is T_U -flat, there is a cardinal number \aleph_α and a pure submodule L of M such that $\text{Card}(L) \leq \aleph_\alpha$ and $f(N) \subseteq L$. Note that L is T_U -flat by Lemma 2.7, and so N has a T_U -flat preenvelope by condition (2) and [9, Proposition 6.2.1].

(4) \Rightarrow (1). Let $M \in \text{pres}({}_R U)$. Then M has a T_U -flat preenvelope $f : M \rightarrow F$ by condition (4). It follows that there are homomorphisms $\alpha : M \rightarrow \bar{U}$ and $\beta : \bar{U} \rightarrow F$ such that $f = \beta\alpha$ with $\bar{U} \in \text{add}_R U$. It is easy to check that $\alpha : M \rightarrow \bar{U}$ is just an $\text{add}_R U$ -preenvelope of M . Thus condition (1) holds by [2, Proposition 5].

(1) \Rightarrow (5). Let F be a T_U -flat left R -module. Then $\text{Hom}_R(U, F) = H(F)$ is a flat left S -module by Proposition 2.1. Since S is right coherent by condition (1), $\text{Hom}_R(U, F)^{++}$ is also a flat left S -module by [5, Theorem 1]. Note that $\text{Hom}_R(U, F^{++}) \cong (F^+ \otimes_R U)^+ \cong \text{Hom}_R(U, F)^{++}$, and hence F^{++} is T_U -flat by Proposition 2.1 again.

(5) \Rightarrow (3). Note that $U^{(I)}$ is T_U -flat by Corollary 2.2, then $(U^{(I)})^{++}$ is T_U -flat by condition (5). Since $(U^+)^{(I)}$ is a pure submodule of $(U^+)^I$, $((U^+)^{(I)})^+$ is a direct summand of $((U^+)^I)^+ \cong (U^{(I)})^{++}$. It follows that $(U^{++})^I \cong ((U^+)^{(I)})^+$ is T_U -flat by Corollary 2.2 again. Note that U^I is a pure submodule of $(U^{++})^I$ by [5, Lemma 1(2)], so U^I is T_U -flat by Lemma 2.7. \square

REMARK 2.9. Recall that a module ${}_R U$ is called a *generalized tilting module* [15] (now it is also called a *Wakamatsu tilting module* [3]) if it has the following properties:

(T1) there exists an exact sequence

$$\dots \rightarrow P_i \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow U \rightarrow 0$$

with each P_i finitely generated and projective for $i \geq 0$;

- (T2) ${}_R U$ is self-orthogonal, that is, $\text{Ext}_R^i(U, U) = 0$ for $i \geq 1$;
- (T3) there exists a $\text{Hom}_R(-, U)$ exact sequence

$$0 \rightarrow {}_R R \rightarrow U_0 \rightarrow U_1 \rightarrow \cdots \rightarrow U_i \rightarrow \cdots$$

where each $U_i \in \text{add } {}_R U$ for $i \geq 0$.

Wakamatsu [15] proved that ${}_R U$ is a Wakamatsu tilting module with $S = \text{End}({}_R U)$ if and only if U_S is a Wakamatsu tilting module with $R = \text{End}(U_S)$. So, for a Wakamatsu tilting module ${}_R U$, both ${}_R U$ and U_S are finitely presented.

REMARK 2.10. Let ${}_R U = {}_R R$ in Theorem 2.8, one obtains some known equivalent conditions for a ring to be right coherent.

We conclude this paper with the following theorem.

THEOREM 2.11. *Let ${}_R U$ be a module with $S = \text{End}({}_R U)$. The following are equivalent.*

- (1) Every left R -module is T_U -flat.
- (2) Every finitely U -presented left R -module belongs to $\text{add } {}_R U$.
- (3) If ${}_S A$ is finitely presented, then $HT(A)$ is a finitely generated projective left S -module.
- (4) S is right coherent with $\text{wD}(S) \leq 2$ and U_S is FP -injective.

PROOF. The equivalence of (1) and (2) holds by definition.

(2) \Rightarrow (3). Let ${}_S A$ be finitely presented. Then $T(A)$ is finitely U -presented, and so $T(A) \in \text{add}_R U$ by condition (2). Thus, $HT(A)$ is a finitely generated projective left S -module.

(3) \Rightarrow (2). Let M be a finitely U -presented left R -module, then there is an exact sequence $0 \rightarrow K \rightarrow U^n \rightarrow U^m \rightarrow M \rightarrow 0$ with n, m positive integers. Note that $H(U^n) \cong S^n$ and $H(U^m) \cong S^m$, then we obtain an exact sequence $0 \rightarrow H(K) \rightarrow S^n \rightarrow S^m$ of left S -modules. Thus, $D = \text{Coker}(S^n \rightarrow S^m)$ is a finitely presented left S -module, and so $HT(D)$ is a finitely generated projective left S -module by condition (3). It follows that $THT(D) \in \text{add}_R U$. Since there is the commutative diagram with exact rows:

$$\begin{array}{ccccccc} U^n & \longrightarrow & U^m & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cdots & & \\ U^n & \longrightarrow & U^m & \longrightarrow & T(D) & \longrightarrow & 0 \end{array}$$

we have $M \cong T(D)$. Note that $T(D)$ is a direct summand of $THT(D)$ by [7, Equality 2.1, p. 13], so $M \in \text{add}_R U$.

(2) \Rightarrow (4). Since condition (2) is equivalent to condition (1) by the foregoing proof, every left R -module is T_U -flat. So S is right coherent by Theorem 2.8. Thus, U_S is FP -injective by Proposition 2.3. Let ${}_S A$ be finitely presented, then there is an exact sequence $S^k \rightarrow S^l \rightarrow A \rightarrow 0$ of right S -modules with k, l positive integers. Now we

obtain an exact sequence $0 \rightarrow \text{Hom}_S(A, U) \rightarrow U^l \rightarrow U^k$ of left R -modules which induces a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K & \longrightarrow & S^k & \longrightarrow & S^l \\
 & & \downarrow h & & \downarrow \cong & & \downarrow \cong \\
 0 & \longrightarrow & \text{Hom}_R(D, U) & \longrightarrow & \text{Hom}_R(U^k, U) & \longrightarrow & \text{Hom}_R(U^l, U)
 \end{array}$$

where $K = \text{Ker}(S^k \rightarrow S^l)$, $D = \text{Coker}(U^l \rightarrow U^k)$ and h is the induced homomorphism. Thus, $K \cong \text{Hom}_R(D, U)$. Note that D is a finitely U -presented left R -module, then $D \in \text{add}_R U$ by condition (2). It follows that K is a finitely generated projective right S -module, and hence $\text{pd}(A_S) \leq 2$. Therefore, $\text{wD}(S) = \sup\{\text{pd}(A_S) \mid A_S \text{ is finitely presented}\} \leq 2$ by [14, Theorem 3.3].

(4) \Rightarrow (1). Let M be any left R -module and E the injective envelope of M , then there is an exact sequence $0 \rightarrow M \rightarrow E \rightarrow C \rightarrow 0$ which induces the following exact commutative diagram.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H(M) & \longrightarrow & H(E) & \longrightarrow & H(C) \longrightarrow D \longrightarrow 0 \\
 & & & & \searrow \pi & & \nearrow i \\
 & & & & & K &
 \end{array}$$

Since $\text{wD}(S) \leq 2$, there are exact sequences

$$0 \rightarrow \text{Tor}_2^S(A, H(M)) \rightarrow \text{Tor}_2^S(A, H(E)) \rightarrow \text{Tor}_2^S(A, K) \rightarrow \text{Tor}_1^S(A, H(M)) \rightarrow \text{Tor}_1^S(A, H(E)) \quad (*)$$

$$0 \rightarrow \text{Tor}_2^S(A, K) \rightarrow \text{Tor}_2^S(A, H(C)) \quad (**)$$

for any right S -module A . Since S is right coherent and U_S is FP -injective, E is T_U -flat by Proposition 2.3. Hence, $H(E)$ is flat by Proposition 2.1. Thus, $\text{Tor}_2^S(A, H(M)) = 0$ and $\text{Tor}_2^S(A, K) \cong \text{Tor}_1^S(A, H(M))$ by the exactness of the sequence (*). Similarly, we have $\text{Tor}_2^S(A, H(C)) = 0$. Thus, $\text{Tor}_2^S(A, K) = 0$ by the exactness of the sequence (**), and hence $\text{Tor}_1^S(A, H(M)) = 0$. It follows that $H(M)$ is flat, and so M is T_U -flat by Proposition 2.1. \square

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