A REMARK ON "UNIFORM O-ESTIMATES OF CERTAIN ERROR FUNCTIONS CONNECTED WITH k-FREE INTEGERS"

D. SURYANARAYANA

(Received 15 January 1971)

Communicated by E. S. Barnes

In this note I follow the same notation adopted in [2]. Let $Q_k(x,n)$ denote the number of k-free integers $\leqq x$ which are prime to n. In [2], I proved the following: For $o \leqq s < 1/k$,

$$Q_k(x,n) = \frac{nx}{\zeta(k)\psi_k(n)} + O(\sigma^*_{-s}(n)x^{1/k}), \tag{1}$$

uniformly, where $\sigma^*_{-s}(n)$ is the sum of the reciprocals of the sth powers of the square-free divisor of n.

Also, I proved in an earlier paper [1] (with a different notation and a different argument) that

$$Q_k(x,n) = \frac{nx}{\zeta(k)\psi_k(n)} + O\left(\frac{\phi(n)\theta(n)}{n}x^{1/k}\right), \tag{2}$$

uniformly, where $\theta(n) = \sigma^*_0(n)$.

Professor Subbarao raised the question, which of the above two uniform O-estimates is a better one. Actually, for some values of n, (1) is better than (2) and for some other values of n, (2) is better than (1).

The object of this note is to improve the O-estimate in $Q_k(x,n)$ to $O\left(\frac{\sigma^*_{-s}(n)\phi(n)}{n}x^{1/k}\right)$, where s is any number such that $0 \leqq s < 1/k$. This uniform O-estimate is better than both the O-estimates given in (1) and (2) above.

In the proof of Theorem 1, p. 247, line 9 of [2], we actually get the O-term to be

$$O\left(x^s\sigma^*_{-s}(n) \sum_{\substack{d \leqq \sqrt[k]{x} \\ (d,n)=1}} d^{-sk}\right). \tag{3}$$

By partial summation, we have for $0 \leqq s < 1/k$,

https://doi.org/10.1017/S1446788700012933 Published online by Cambridge University Press

$$\sum_{\substack{d \leqq \sqrt[k]{x} \\ (d,n)=1}} d^{-sk} = \sum_{d \leqq \sqrt[k]{x}} \frac{\varepsilon((d,n))}{d^{sk}} = \frac{\phi(\sqrt[k]{x}, n)}{x} + sk \int_1^{\sqrt[k]{x}} \frac{\phi(t,n)}{t^{sk+1}} dt$$

$$= O\left(x^{1/k-s} \frac{\phi(n)}{n}\right) + O\left(\frac{\phi(n)}{n} \int_1^{\sqrt[k]{x}} \frac{dt}{t^{sk}}\right)$$

$$= O\left(x^{1/k-s} \frac{\phi(n)}{n}\right) + O\left(x^{1/k-s} \frac{\phi(n)}{n}\right)$$

$$= O\left(x^{1/k-s} \frac{\phi(n)}{n}\right).$$

Substituting in (3) above, we get the O - term to be

$$O\left(\frac{\sigma^*_{-s}(n)\phi(n)}{n} x^{1/k}\right).$$

Consequently, the O - term in Theorem 2 of [2] can be replaced by

$$O\left(\frac{\sigma^*_{-s}(n)\phi(n)}{n} \cdot \frac{1}{x^{1-1/k}}\right)$$

and the O - term in Theorem 2 of [1] can be replaced by

$$O\left(\frac{\sigma^*_{-s}(n)\phi(n)}{n} x^{1+1/k}\right).$$

I take this opportunity to correct the following misprints in [2]: on page 247, line 5 should read $(\delta, n) = 1$ instead of $(\delta, d) = 1$ and on page 250 lines 5 and 6 should read, Math. Student 37 (1969), 147–157 instead of Math. Student 36 (1968), 171–181.

Finally, I thank Professor M. V. Subbarao for drawing my attention to the problem and for his useful comments.

References

[1] D. Suryanarayana, 'The number and sum of k-free integers $\leqq x$ which are prime to n', *Indian J. Math.* 11 (1969), 131–139.

[2] D. Suryanarayana, 'Uniform O-estimates of certain error functions connected with k-free integers', *J. Austr. Math. Soc.* 11(2) (1970), 242–250.

Department of Mathematics
Andhra University,
Waltair, India.