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Averaging and integral manifolds (II)

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In the first part of this paper (written jointly with W.A. Coppel) the existence and properties of an integral manifold were established for the system

$$x' = f(t, x, y)$$
$$y' = A(t)y + g(t, x, y)$$

where f and g are "integrally small". In this second part of the paper the stability properties of the integral manifold are investigated. Solutions are found which are bounded on the positive half of the real line and it is shown that these solutions approach the manifold exponentially and, moreover, that they are asymptotic to particular solutions on the manifold.

1.

This paper is a continuation of Coppel and Palmer [1]. We consider once more the system of differential equations

$$\text{(1)} \qquad \begin{aligned} x' &= f(t, x, y) \\ y' &= A(t)y + g(t, x, y) , \end{aligned}$$

where $x \in R^m$, $y \in R^n$, $' = d/dt$, and where the linear equation

$$\text{(2)} \qquad y' = A(t)y$$

has a fundamental matrix $Y(t)$ such that

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$$(3)\qquad \begin{aligned} |Y(t)PY^{-1}(s)| &\le Ke^{-2\alpha(t-s)} \quad \text{for} \quad t \ge s \ , \\ |Y(t)(I-P)Y^{-1}(s)| &\le Ke^{-2\alpha(s-t)} \quad \text{for} \quad s \ge t \ , \end{aligned}$$

where the matrix P is a projection and K, α are positive constants. However, instead of supposing this system to be defined on the whole real line we now suppose it to be defined on a half-line $[\tau, \infty)$. This enables us to impose a 'partial' initial condition on y . Given a vector $\xi \in R^m$ and a vector $\eta \in R^n$ we look for a solution $x(t) = x(t, \xi, \eta, \tau)$, $y(t) = y(t, \xi, \eta, \tau)$ of (1) such that $y(t)$ is bounded and

$$x(\tau) = \xi \ , \quad P(\tau)y(\tau) = P(\tau)\eta \ ,$$

where $P(\tau)$ denotes the projection $Y(\tau)PY^{-1}(\tau)$. For each fixed τ, ξ the set of all points $y(\tau, \xi, \eta, \tau)$ defines a submanifold of R^n , of dimension equal to the rank of P , such that only solutions starting from this submanifold remain bounded. The argument of [1] carries over to the present problem with some complication but without essential change, so much of the detail will be omitted. However the treatment of [1] is improved in that no use is made of the roughness property of exponential dichotomies (Lemma 2 of [1]). This means that everything carries over to the case where R^m and R^n are replaced by arbitrary Banach spaces. In §3 the present results are used to discuss the stability of the integral manifold considered in the previous paper. The concluding section contains a discussion of related work by other authors and some remarks on the smoothness of the integral manifold.

We again set

$$\|f\| = \sup_{t \ge \tau} \left\{ e^{-\beta(t-\tau)} |f(t)| \right\} .$$

The following two lemmas correspond to Lemmas 4 and 5 of [1] and are proved in a similar way.

LEMMA 1. *Let* $A(t)$ *be a continuous matrix function such that the linear equation* (2) *has a fundamental matrix* $Y(t)$ *satisfying* (3). *If* $f(t)$ *is a continuous vector function such that* $\|f\| < \infty$, *where* $|\beta| < 2\alpha$, *then the inhomogeneous equation*

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(4) $$y' = A(t)y + f(t)$$

has a unique solution $y(t)$ *such that* $P(\tau)y(\tau) = 0$ *and* $\|y\| < \infty$. *Moreover*

$$\|y\| \le \left\{(2\alpha+\beta)^{-1} + (2\alpha-\beta)^{-1}\right\}K\|f\| .$$

Thus if $|\beta| \le \alpha$ then

$$\|y\| \le \frac{4}{3}\alpha^{-1}K\|f\| .$$

The solution $y(t)$ is given explicitly by

$$y(t) = \int_\tau^t Y(t)PY^{-1}(s)f(s)ds - \int_t^\infty Y(t)(I-P)Y^{-1}(s)f(s)ds .$$

If we add $Y(t)PY^{-1}(\tau)\eta$ to the right side we obtain the unique solution $y(t)$ of (4) with $P(\tau)y(\tau) = P(\tau)\eta$ and $\|y\| < \infty$.

LEMMA 2. *Let* $A(t)$ *be a continuous matrix function such that* $|A(t)| \le N$ *for* $t \ge \tau$, *where* $N \ge 1$, *and suppose the linear equation* (2) *has a fundamental matrix* $Y(t)$ *satisfying* (3). *If* $f(t)$ *is a continuous vector function such that*

$$\left|\int_t^{t+h} f(s)ds\right| \le re^{\beta(t-\tau)}$$

for $0 \le h \le 1$ *and* $t \ge \tau$, *where* $|\beta| < 2\alpha$, *then the inhomogeneous equation* (4) *has a unique solution* $y(t)$ *such that* $P(\tau)y(\tau) = 0$ *and* $\|y\| < \infty$. *Moreover*

$$\|y\| \le \left\{\frac{1}{1-e^{-(2\alpha+\beta)}} + \frac{1}{1-e^{-(2\alpha-\beta)}} + \frac{N}{2\alpha+\beta} + \frac{N}{2\alpha-\beta}\right\}Kr .$$

Thus if $|\beta| \le \alpha$ then

$$\|y\| \le \left(2\gamma + \frac{4}{3}\alpha^{-1}N\right)Kr$$
$$\le 4NK\gamma r ,$$

where $\gamma = \left(1-e^{-\alpha}\right)^{-1}$.

We impose the same conditions on the system (1) as in the previous

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paper, except that the functions involved need only be defined for $t \geq \tau$. That is, we assume f and g are continuous vector functions with continuous partial derivatives in x and y such that

$$|f(t, x, y)| \leq N, \quad |g(t, x, y)| \leq N,$$
$$|f_x(t, x, y)| \leq N,$$
$$|f_x(t, x_1, y_1) - f_x(t, x_2, y_2)| \leq L\left[|x_1-x_2| + |y_1-y_2|\right],$$

where $N \geq 1$, and the same inequalities with f_x replaced by f_y, g_x, g_y. Furthermore we assume

$$\left|\int_{t_1}^{t_2} g(t, x, 0)dt\right| \leq q \quad \text{for all } x \text{ if } |t_2-t_1| \leq 1,$$

and the same inequality with g replaced by f_x, g_x, g_y. Finally we suppose that $A(t)$ is a continuous matrix function, with $|A(t)| \leq N$, such that the linear equation (2) has a fundamental matrix $Y(t)$ satisfying (3). Under these assumptions we will prove

THEOREM 1. *For any* $\beta\left(0 < \beta \leq \frac{1}{2}\alpha\right)$ *there exists a positive constant* $\mu_0 = \mu_0(N, K, L, \alpha, \beta)$ *such that if* $\mu \leq \mu_0$ *and if* $q \leq q_0(N, K, L, \alpha, \beta, \mu)$ *then for any vector* η *with* $|\eta| < \mu/4K$ *the system of differential equations*

$$x' = f(t, x, y)$$
$$(1) \qquad y' = A(t)y + g(t, x, y)$$

has a unique solution $x(t) = x(t, \xi, \eta, \tau)$, $y(t) = y(t, \xi, \eta, \tau)$ *for which*

$$(5) \qquad x(\tau) = \xi, \quad P(\tau)y(\tau) = P(\tau)\eta, \quad |y(t)| \leq \mu \quad \text{for } t \geq \tau.$$

Moreover the partial derivatives x_ξ, x_η, y_ξ, y_η *exist and satisfy*

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$$|x_\xi(t,\ \xi,\ \eta,\ \tau)| \le 2e^{\beta(t-\tau)}\ , \qquad |x_\eta(t,\ \xi,\ \eta,\ \tau)| \le 2CKe^{\beta(t-\tau)}\ ,$$

$$|y_\xi(t,\ \xi,\ \eta,\ \tau)| \le 2C^{-1}e^{\beta(t-\tau)}\ , \quad |y_\eta(t,\ \xi,\ \eta,\ \tau)| \le 2Ke^{\beta(t-\tau)}\ ,$$

$$|x_\xi(t,\ \xi_1,\ \eta_1,\ \tau)-x_\xi(t,\ \xi_2,\ \eta_2,\ \tau)| \le CD\big[|\xi_1-\xi_2|+CK|\eta_1-\eta_2|\big]e^{2\beta(t-\tau)}\ ,$$

$$|y_\xi(t,\ \xi_1,\ \eta_1,\ \tau)-y_\xi(t,\ \xi_2,\ \eta_2,\ \tau)| \le D\big[|\xi_1-\xi_2|+CK|\eta_1-\eta_2|\big]e^{2\beta(t-\tau)}\ ,$$

$$|x_\eta(t,\ \xi_1,\ \eta_1,\ \tau)-x_\eta(t,\ \xi_2,\ \eta_2,\ \tau)| \le C^2DK\big[|\xi_1-\xi_2|+CK|\eta_1-\eta_2|\big]e^{2\beta(t-\tau)}\ ,$$

$$|y_\eta(t,\ \xi_1,\ \eta_1,\ \tau)-y_\eta(t,\ \xi_2,\ \eta_2,\ \tau)| \le CDK\big[|\xi_1-\xi_2|+CK|\eta_1-\eta_2|\big]e^{2\beta(t-\tau)}\ ,$$

where $C = 4N\left(1-e^{-\beta}\right)^{-1}$ *and* $D = 8L\left(N^{-1}+2\alpha^{-1}K\right)$.

Set

$$\nu = (8N)^{-1}\ , \quad R = 4NK\alpha^{-1}e^{\alpha}\ , \quad \gamma = \left(1-e^{-\alpha}\right)^{-1}\ ,$$

and choose $\mu_0 > 0$ so that

$$64\mu_0 LNC(R+\gamma) \le 1\ .$$

Next for any μ $\left(0 < \mu \le \mu_0\right)$ we choose $p_0 > 0$ so that

$$64p_0 NC(R+\nu) \le 1\ , \quad 24p_0(1+\mu)\gamma N^2K \le \mu\ .$$

Then we take

$$q_0 = \frac{1}{2}\,p_0^2\min\left[\left(p_0+LN\right)^{-1},\ \left(p_0+N^2\right)^{-1}\right]\ .$$

Let $x(t)$ and $y(t)$ be continuously differentiable functions such that

(6) $\quad |x'(t)| \le N\ , \quad |y(t)| \le \mu\ , \quad |y'(t)| \le (1+\mu)N$ for $t \ge \tau$.

Put

$$\hat{x}(t) = \xi + \int_\tau^t f\big[s,\ x(s),\ y(s)\big]ds$$

and let $\hat{y}(t)$ denote the bounded solution of the equation

$$\hat{y}' = A(t)\hat{y} + g\big[t,\ x(t),\ y(t)\big]$$

such that $P(\tau)\hat{y}(\tau) = P(\tau)\eta$. Such a solution exists and is unique, by

Lemma 1 with $\beta = 0$. If we write

$$(7)\quad \begin{aligned} f(t, x, y) &= f(t, x, 0) + f_y(t, x, 0)y + F(t, x, y) \\ g(t, x, y) &= g(t, x, 0) + g_y(t, x, 0)y + G(t, x, y) \end{aligned}$$

then

$$|F(t, x, y)| \le \frac{1}{2} L|y|^2 , \quad |G(t, x, y)| \le \frac{1}{2} L|y|^2 ,$$

and if $|y_1| \le \mu$, $|y_2| \le \mu$

$$\begin{aligned} |F(t, x_1, y_1) - F(t, x_2, y_2)| &\le 2\mu L\left[|x_1-x_2| + |y_1-y_2|\right] , \\ |G(t, x_1, y_1) - G(t, x_2, y_2)| &\le 2\mu L\left[|x_1-x_2| + |y_1-y_2|\right] . \end{aligned}$$

Thus

$$|G[t, x(t), y(t)]| \le \frac{1}{2} \mu^2 L .$$

For $|t_2-t_1| \le 1$ we have, by Lemma 3 of [1],

$$\left|\int_{t_1}^{t_2} g[t, x(t), 0]dt\right| \le p_0$$

and, by Lemma 7 of [1] with $\beta = 0$,

$$\left|\int_{t_1}^{t_2} g_y[t, x(t), 0]y(t)dt\right| \le p_0\{\mu + (1+\mu)N\} .$$

Therefore, since $N \ge 1$,

$$\left|\int_{t_1}^{t_2} \left\{g[t, x(t), 0] + g_y[t, x(t), 0]y(t)\right\}dt\right| \le (1+\mu)2Np_0 .$$

It follows from the present Lemmas 1 and 2, with $\beta = 0$, and from the superposition principle that

$$\begin{aligned} |\hat{y}(t)| &\le |Y(t)PY^{-1}(\tau)\eta| + \frac{1}{2} \mu^2 L.\alpha^{-1}K + (1+\mu)2Np_0.\left(2\gamma+N\alpha^{-1}\right)K \\ &\le K|\eta| + \frac{1}{2} \mu^2\gamma KL + 6(1+\mu)\gamma N^2 Kp_0 \\ &\le \tfrac{1}{4}\mu + \tfrac{1}{4}\mu + \tfrac{1}{4}\mu = \tfrac{3}{4}\mu . \end{aligned}$$

Hence $|\hat{y}'(t)| \le (1+\mu)N$, and it is obvious that $|\hat{x}'(t)| \le N$.

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For a fixed β such that $0 < \beta \leq \alpha$ let $\mathcal{B}$ denote the set of all pairs (x, y), where $x = x(t)$ and $y = y(t)$ are continuously differentiable functions for $t \geq \tau$ with $\|x\| + \|x'\| + \|y\| + \|y'\| < \infty$. The set $\mathcal{B}$ becomes a Banach space if we define

$$(x_1+x_2,\ y_1+y_2) = (x_1,\ y_1) + (x_2,\ y_2)$$
$$\lambda(x,\ y) = (\lambda x,\ \lambda y)$$
$$|(x,\ y)| = \|x\| + \nu\|x'\| + C\|y\| + \nu\|y'\| ,$$

where ν, C are the positive constants defined above. The set S of all pairs (x, y) in $\mathcal{B}$ satisfying (6) is a closed subset of $\mathcal{B}$ and, by what we have just proved, the transformation $T : (x, y) \to (\hat{x}, \hat{y})$ maps S into itself. We show next that T is a contraction on S.

Let (x_1, y_1) and (x_2, y_2) be any two points in S, and let $(\hat{x}_1, \hat{y}_1)$ and $(\hat{x}_2, \hat{y}_2)$ be their images under T. We set

$$(z,\ w) = (x_1,\ y_1) - (x_2,\ y_2) ,\quad (\hat{z},\ \hat{w}) = (\hat{x}_1,\ \hat{y}_1) - (\hat{x}_2,\ \hat{y}_2) .$$

Then

$$\hat{z}(t) = \int_\tau^t \left\{f[s,\ x_1(s),\ y_1(s)] - f[s,\ x_2(s),\ y_2(s)]\right\}ds .$$

By exactly the same argument as in the previous paper it follows that

$$\|\hat{z}\| \leq H\left\{3\mu L[\|z\|+\|w\|] + N\|w\| + p_0[\|z\|+\|z'\|]\right\} ,$$
$$\|\hat{z}'\| \leq N[\|z\| + \|w\|] ,$$

where $H = \left(1-e^{-\beta}\right)^{-1}$. The difference $\hat{w}(t) = \hat{y}_1(t) - \hat{y}_2(t)$ is a bounded solution, with $P(\tau)\hat{w}(\tau) = 0$, of the equation

$$\hat{w}' = A(t)\hat{w} + \varphi(t) + \psi(t) ,$$

where

$$\varphi(t) = g[t,\ x_1(t),\ 0] - g[t,\ x_2(t),\ 0] + g_y[t,\ x_1(t),\ 0]w(t) ,$$
$$\psi(t) = G[t,\ x_1(t),\ y_1(t)] - G[t,\ x_2(t),\ y_2(t)] + \left\{g_y[t,\ x_1(t),\ 0] - g_y[t,\ x_2(t),\ 0]\right\}y_2(t) .$$

It follows at once that

$$|\psi(t)| \leq 3\mu L[|z(t)| + |w(t)|] .$$

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Also, using Lemma 7 of [1] we obtain for $0 \le h \le 1$

$$\left|\int_t^{t+h} \varphi(s)ds\right| \le p_0\beta^{-1}(e^{\beta}-1)\left[\|z\|+\|z'\|+\|w\|+\|w'\|\right]e^{\beta(t-\tau)} .$$

By Lemmas 1 and 2 and the superposition principle it follows that

$$\begin{aligned}\|\hat{w}\| &\le \frac{4}{3}\alpha^{-1}K.3\mu L\left[\|z\|+\|w\|\right] + 4NK\gamma p_0\beta^{-1}(e^{\beta}-1)\left[\|z\|+\|z'\|+\|w\|+\|w'\|\right] \\ &\le 4\alpha^{-1}\mu LK\left[\|z\|+\|w\|\right] + p_0R\left[\|z\|+\|z'\|+\|w\|+\|w'\|\right] .\end{aligned}$$

Also

$$\|\hat{w}'\| \le N\|\hat{w}\| + N\left[\|z\|+\|w\|\right]$$

Combining these estimates and using the inequalities imposed on μ and p_0 we get

$$|(\hat{z}, \hat{w})| \le \frac{1}{2}|(z, w)| .$$

Thus the mapping T is a contraction. Its fixed point is the required solution $x(t, \xi, \eta, \tau)$, $y(t, \xi, \eta, \tau)$.

If (x_1, y_1) and (x_2, y_2) are the fixed points corresponding to the initial values (ξ_1, η_1) and (ξ_2, η_2) respectively we obtain in the same way

$$|(z, w)| \le |\xi_1-\xi_2| + CK|\eta_1-\eta_2| + \frac{1}{2}|(z, w)| ,$$

and hence

$$|(z, w)| \le 2|\xi_1-\xi_2| + 2CK|\eta_1-\eta_2| . \tag{8}$$

Thus

$$\begin{aligned}|x(t, \xi_1, \eta_1, \tau) - x(t, \xi_2, \eta_2, \tau)| &\le 2\left[|\xi_1-\xi_2| + CK|\eta_1-\eta_2|\right]e^{\beta(t-\tau)} , \\ |y(t, \xi_1, \eta_1, \tau) - y(t, \xi_2, \eta_2, \tau)| &\le 2\left[C^{-1}|\xi_1-\xi_2| + K|\eta_1-\eta_2|\right]e^{\beta(t-\tau)} .\end{aligned}$$

Similarly let (x_1, y_1) and (x_2, y_2) be the fixed points corresponding to the initial values ξ, η at times τ and $\tau+h$ respectively, where $h > 0$. If we again set $(z, w) = (x_1, y_1) - (x_2, y_2)$ then

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$$z(t) = \int_\tau^{\tau+h} f[s,\, x_2(s),\, y_2(s)]ds$$
$$+ \int_\tau^t \{f[s,\, x_1(s),\, y_1(s)] - f[s,\, x_2(s),\, y_2(s)]\}ds$$

and

$$w(t) = Y(t)P\left[Y^{-1}(\tau)-Y^{-1}(\tau+h)\right]\eta + \int_\tau^{\tau+h} Y(t)PY^{-1}(s)g[s,\, x_2(s),\, y_2(s)]ds$$
$$+ \int_\tau^t Y(t)PY^{-1}(s)\zeta(s)ds - \int_t^\infty Y(t)(I-P)Y^{-1}(s)\zeta(s)ds \ ,$$

where

$$\zeta(t) = g[t,\, x_1(t),\, y_1(t)] - g[t,\, x_2(t),\, y_2(t)] \ .$$

We have

$$\left|\int_\tau^{\tau+h} f[s,\, x_2(s),\, y_2(s)]ds\right| \le Nh \ ,$$
$$\left|Y(t)P\left[Y^{-1}(\tau)-Y^{-1}(\tau+h)\right]\right| = \left|\int_\tau^{\tau+h} Y(t)PY^{-1}(s)A(s)ds\right|$$
$$\le \int_\tau^{\tau+h} Ke^{N(s-\tau)}Nds$$
$$\le KNe^{Nh}h \ ,$$

and similarly

$$\left|\int_\tau^{\tau+h} Y(t)PY^{-1}(s)g[s,\, x_2(s),\, y_2(s)]ds\right| \le KNe^{Nh}h \ .$$

It follows that

$$|(z,\, w)| \le 2Nh + 4CKNe^{Nh}h \ . \tag{9}$$

Thus

$$|x(t,\, \xi,\, \eta,\, \tau+h) - x(t,\, \xi,\, \eta,\, \tau)| \le 2N\left(1+2CKe^{Nh}\right)he^{\beta(t-\tau)}$$
$$|y(t,\, \xi,\, \eta,\, \tau+h) - y(t,\, \xi,\, \eta,\, \tau)| \le \tfrac{1}{2}\left(1-e^{-\beta}\right)\left(1+2CKe^{Nh}\right)he^{\beta(t-\tau)} \ .$$

This shows, *a posteriori*, that if $0 \le h \le h_0$, the solutions $x(t, \xi, \eta, \tau+h)$, $y(t, \xi, \eta, \tau+h)$ can be continued over the interval $[\tau, \tau+h]$ without leaving the region $|y| < \mu$.

2.

To prove the existence and Lipschitzian nature of the partial derivatives we consider first a linear system of differential equations

$$(10) \qquad \begin{aligned} x' &= F_1(t)x + F_2(t)y + \chi(t) \\ y' &= G_1(t)x + [A(t) + G_2(t)]y + \zeta(t) , \end{aligned}$$

on $[\tau, \infty)$, where the matrix functions F_k, G_k $(k = 1, 2)$ are continuous and bounded by N , and the vector functions χ, ζ are continuous with $\|\chi\| < \infty$, $\|\zeta\| < \infty$. We assume also that for $|t_2-t_1| \le 1$

$$(11) \qquad \left|\int_{t_1}^{t_2} F_1(t)dt\right|, \ \left|\int_{t_1}^{t_2} G_1(t)dt\right|, \ \left|\int_{t_1}^{t_2} G_2(t)dt\right| \le r .$$

We wish to show that if r is so small that

$$(12) \qquad 32NC(R+\nu)r \le 1 ,$$

then the system (10) has a unique solution $x(t)$, $y(t)$ in $\mathcal{B}$ such that $x(\tau) = \xi$ and $P(\tau)y(\tau) = P(\tau)\eta$.

For any (x, y) in $\mathcal{B}$ set

$$\hat{x}(t) = \xi + \int_\tau^t \{F_1(s)x(s) + F_2(s)y(s) + \chi(s)\}ds$$

and let $\hat{y}(t)$ denote the unique solution with $P(\tau)\hat{y}(\tau) = P(\tau)\eta$ and $\|\hat{y}\| < \infty$ of the equation

$$\hat{y}' = A(t)\hat{y} + G_1(t)x(t) + G_2(t)y(t) + \zeta(t) .$$

Using Lemma 2 and proceeding as in [1] we obtain, with the same notation as above,

$$\|\hat{z}\| \le rH\big[\|z\|+\|z'\|\big] + N\beta^{-1}\|w\| ,$$
$$\|\hat{z}'\| \le N\big[\|z\|+\|w\|\big] ,$$
$$\|\hat{w}\| \le 4NK\gamma r\beta^{-1}\big(e^{\beta}-1\big)\big[\|z\|+\|z'\|+\|w\|+\|w'\|\big] ,$$
$$\le rR\big[\|z\|+\|z'\|+\|w\|+\|w'\|\big] ,$$
$$\|\hat{w}'\| \le N\big[\|\hat{w}\|+\|z\|+\|w\|\big] ,$$

and hence

$$|(\hat{z}, \hat{w})| \le \tfrac{1}{2}|(z, w)| .$$

Therefore the mapping $(x, y) \to (\hat{x}, \hat{y})$ is a contraction, and the result follows. Moreover the solution $x(t)$, $y(t)$ satisfies the inequality

$$(13) \qquad |(x, y)| \le 2|\xi| + 2CK|\eta| + N^{-1}C\|\chi\| + 2\big(2\alpha^{-1}CK+\nu\big)\|\zeta\| .$$

If

$$F_1(t) = f_x\big[t, x(t, \xi, \eta, \tau), y(t, \xi, \eta, \tau)\big] ,$$
$$F_2(t) = f_y\big[t, x(t, \xi, \eta, \tau), y(t, \xi, \eta, \tau)\big] ,$$
$$G_1(t) = g_x\big[t, x(t, \xi, \eta, \tau), y(t, \xi, \eta, \tau)\big] ,$$
$$G_2(t) = g_y\big[t, x(t, \xi, \eta, \tau), y(t, \xi, \eta, \tau)\big] ,$$

then we can take $r = p_0 + \mu_0 L$ and the inequality (12) is satisfied. Let $X_1(t)$, $Y_1(t)$ denote the corresponding solution in $\mathcal{B}$ of the matrix system

$$(14) \qquad \begin{aligned} X' &= F_1(t)X + F_2(t)Y \\ Y' &= G_1(t)X + \big[A(t) + G_2(t)\big]Y \end{aligned}$$

with $X_1(\tau) = I$, $P(\tau)Y_1(\tau) = 0$. Then by (13),

$$(15) \qquad \|X_1\| + C\|Y_1\| \le 2 .$$

As in [1] we can show that the partial derivatives $x_\xi(t, \xi, \eta, \tau)$, $y_\xi(t, \xi, \eta, \tau)$ exist and equal $X_1(t)$, $Y_1(t)$ respectively. Similarly, if $X_2(t)$, $Y_2(t)$ is the solution in $\mathcal{B}$ of (14) with $X_2(\tau) = 0$, $P(\tau)Y_2(\tau) = P(\tau)$ then

$$(16) \qquad \|X_2\| + C\|Y_2\| \le 2CK ,$$

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and we can show that the partial derivatives $x_\eta(t, \xi, \eta, \tau)$, $y_\eta(t, \xi, \eta, \tau)$ exist and equal $X_2(t)$, $Y_2(t)$ respectively.

It remains to show that the partial derivatives satisfy Lipschitz conditions. If we set

$$\tilde{X}_1(t) = x_\xi(t, \tilde{\xi}, \tilde{\eta}, \tau) , \quad \tilde{Y}_1(t) = y_\xi(t, \tilde{\xi}, \tilde{\eta}, \tau)$$

then $(Z, W) = (X_1, Y_1) - (\tilde{X}_1, \tilde{Y}_1)$ is the solution in B of a system (10) with

$$\chi(t) = [F_1(t)-\tilde{F}_1(t)]\tilde{X}_1(t) + [F_2(t)-\tilde{F}_2(t)]\tilde{Y}_1(t)$$
$$\zeta(t) = [G_1(t)-\tilde{G}_1(t)]\tilde{X}_1(t) + [G_2(t)-\tilde{G}_2(t)]\tilde{Y}_1(t) .$$

By (8) and (15)

$$\begin{aligned}|\chi(t)|, \ |\zeta(t)| &\le L[|z(t)|+|w(t)|][|\tilde{X}_1(t)|+|\tilde{Y}_1(t)|] \\ &\le 4L[|\xi-\tilde{\xi}| + CK|\eta-\tilde{\eta}|]e^{2\beta(t-\tau)} .\end{aligned}$$

Put

$$\|f\|_2 = \sup_{t\ge\tau} \left\{e^{-2\beta(t-\tau)}|f(t)|\right\} .$$

Since $Z(\tau) = 0$ and $P(\tau)W(\tau) = 0$ it follows from (13) that

$$\|Z\|_2 + C_2\|W\|_2 \le 2C_2(N^{-1}+2\alpha^{-1}K).4L[|\xi-\tilde{\xi}|+CK|\eta-\tilde{\eta}|] .$$

Therefore, since $C_2 = 4N(1-e^{-2\beta})^{-1} \le C$,

$$|x_\xi(t, \xi, \eta, \tau) - x_\xi(t, \tilde{\xi}, \tilde{\eta}, \tau)| \le CD[|\xi-\tilde{\xi}|+CK|\eta-\tilde{\eta}|]e^{2\beta(t-\tau)}$$
$$|y_\xi(t, \xi, \eta, \tau) - y_\xi(t, \tilde{\xi}, \tilde{\eta}, \tau)| \le D[|\xi-\tilde{\xi}|+CK|\eta-\tilde{\eta}|]e^{2\beta(t-\tau)} .$$

The Lipschitz conditions for x_η , y_η are proved similarly, and thus the proof of Theorem 1 is complete.

In the same way, for any h $(0 \le h \le h_0)$ the functions

$$Z(t) = x_\xi(t, \xi, \eta, \tau) - x_\xi(t, \xi, \eta, \tau+h)$$
$$W(t) = y_\xi(t, \xi, \eta, \tau) - y_\xi(t, \xi, \eta, \tau+h)$$

are the solutions of a system (10), where by (9) and (15)

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$$\|\chi\|_2,\ \|\zeta\|_2 \le 4LN\left(1+2CKe^{Nh}\right)h\ .$$

Since the coefficients of (14) are bounded by N the scalar function

$$\lambda(t) = |\tilde{X}_1(t)| + 2|\tilde{Y}_1(t)|$$

satisfies the integral inequality

$$\lambda(t) \le \lambda(\tau+h) + 3N\int_t^{\tau+h} \lambda(s)ds\ .$$

Moreover $\lambda(\tau+h) \le 2$ by (15), and so by Gronwall's inequality

$$\lambda(t) \le 2e^{3N(\tau+h-t)} \quad \text{for} \quad \tau \le t \le \tau+h\ .$$

Since

$$Z(\tau) = I - \tilde{X}_1(\tau) = \int_\tau^{\tau+h} (\tilde{F}_1\tilde{X}_1+\tilde{F}_2\tilde{Y}_1)dt$$

it follows that

$$|Z(\tau)| \le N\int_\tau^{\tau+h} 2e^{3N(\tau+h-t)}dt \le 2Ne^{3Nh}h\ . \tag{17}$$

Similarly we obtain

$$|\tilde{Y}_1(\tau+h)-\tilde{Y}_1(\tau)| \le 2Ne^{3Nh}h\ .$$

We have

$$\begin{aligned} P(\tau)W(\tau) &= -P(\tau)\tilde{Y}_1(\tau) \\ &= P(\tau)\left[\tilde{Y}_1(\tau+h)-\tilde{Y}_1(\tau)\right] - \left[P(\tau)-P(\tau+h)\right]\tilde{Y}_1(\tau+h) \end{aligned}$$

and

$$P(\tau) - P(\tau+h) = \left[I-Y(\tau+h)Y^{-1}(\tau)\right]P(\tau) - P(\tau+h)\left[I-Y(\tau+h)Y^{-1}(\tau)\right]\ .$$

Since $|P(\tau)| \le K$ and

$$\begin{aligned} \left|Y(\tau+h)Y^{-1}(\tau)-I\right| &= \left|\int_\tau^{\tau+h} A(t)Y(t)Y^{-1}(\tau)dt\right| \\ &\le \int_\tau^{\tau+h} Ne^{N(t-\tau)}dt \\ &\le Ne^{Nh}h\ , \end{aligned}$$

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it follows that

$$|P(\tau)W(\tau)| \le 2NKe^{3Nh}h + 2NKe^{Nh}h|\tilde{Y}_1(\tau+h)| .$$

But by (15), $|\tilde{Y}_1(\tau+h)| \le 2C^{-1} \le \frac{1}{2}$. Hence

$$|P(\tau)W(\tau)| \le 3NKe^{3Nh}h . \tag{18}$$

It now follows from (13) that for $0 \le h \le h_0$ (≤ 1) .

$$\|Z\|_2 + C_2\|W\|_2 = O(h)$$

where the constant involved in the O-notation depends only on N, K, L, α, β . Thus, for $0 \le h \le h_0$,

$$\begin{aligned} &|x_\xi(t, \xi, \eta, \tau+h) - x_\xi(t, \xi, \eta, \tau)| = O(h)e^{2\beta(t-\tau)} , \\ &|y_\xi(t, \xi, \eta, \tau+h) - y_\xi(t, \xi, \eta, \tau)| = O(h)e^{2\beta(t-\tau)} . \end{aligned} \tag{19}$$

Similarly we obtain

$$\begin{aligned} &|x_\eta(t, \xi, \eta, \tau+h) - x_\eta(t, \xi, \eta, \tau)| = O(h)e^{2\beta(t-\tau)} , \\ &|y_\eta(t, \xi, \eta, \tau+h) - y_\eta(t, \xi, \eta, \tau)| = O(h)e^{2\beta(t-\tau)} . \end{aligned} \tag{20}$$

Now let us look at the function

$$\psi(\tau, \xi, \eta) = y(\tau, \xi, \eta, \tau) .$$

We have

$$|\psi(\tau, \xi, \eta)| \le \mu ,$$

the partial derivatives ψ_ξ, ψ_η exist and

$$|\psi_\xi(\tau, \xi, \eta)| \le 2C^{-1} , \quad |\psi_\eta(\tau, \xi, \eta)| \le 2K ,$$

$$|\psi_\xi(\tau, \xi_1, \eta_1) - \psi_\xi(\tau, \xi_2, \eta_2)| \le D[|\xi_1-\xi_2|+CK|\eta_1-\eta_2|]$$

$$|\psi_\eta(\tau, \xi_1, \eta_1) - \psi_\eta(\tau, \xi_2, \eta_2)| \le CKD[|\xi_1-\xi_2|+CK|\eta_1-\eta_2|] .$$

We now show that the partial derivatives $x_\tau(t, \xi, \eta, \tau)$ and $y_\tau(t, \xi, \eta, \tau)$ exist. Let $x_3(t)$, $y_3(t)$ denote the solution in B of the vector system (14), with

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$$x_3(\tau) = -f[\tau,\ \xi,\ \psi(\tau,\ \xi,\ \eta)]\ ,$$
$$P(\tau)y_3(\tau) = -P(\tau)\left\{A(\tau)\eta + g[\tau,\ \xi,\ \psi(\tau,\ \xi,\ \eta)]\right\}\ .$$

By the superposition principle and the initial conditions for the partial derivatives x_ξ, y_ξ, x_η, y_η we have

$$(21)\quad \begin{aligned} x_3(t) &= -x_\xi(t,\ \xi,\ \eta,\ \tau)f[\tau,\ \xi,\ \psi(\tau,\ \xi,\ \eta)] \\ &\qquad - x_\eta(t,\ \xi,\ \eta,\ \tau)\left\{A(\tau)\eta + g[\tau,\ \xi,\ \psi(\tau,\ \xi,\ \eta)]\right\} \\ y_3(t) &= -y_\xi(t,\ \xi,\ \eta,\ \tau)f[\tau,\ \xi,\ \psi(\tau,\ \xi,\ \eta)] \\ &\qquad - y_\eta(t,\ \xi,\ \eta,\ \tau)\left\{A(\tau)\eta + g[\tau,\ \xi,\ \psi(\tau,\ \xi,\ \eta)]\right\}\ . \end{aligned}$$

Put

$$z(t) = x(t,\ \xi,\ \eta,\ \tau+h) - x(t,\ \xi,\ \eta,\ \tau)$$
$$w(t) = y(t,\ \xi,\ \eta,\ \tau+h) - y(t,\ \xi,\ \eta,\ \tau)\ ,$$

and set

$$\varphi(t) = z(t) - hx_3(t)\ ,\quad \psi(t) = w(t) - hy_3(t)\ .$$

Then

$$\varphi'(t) = F_1(t)\varphi(t) + F_2(t)\psi(t) + \chi(t)\ ,$$
$$\psi'(t) = G_1(t)\varphi(t) + [A(t)+G_2(t)]\psi(t) + \zeta(t)\ ,$$

where

$$|\chi(t)|,\ |\zeta(t)| \le L[|z(t)|+|w(t)|]^2\ .$$

Therefore, by (9),

$$\|\chi\|_2,\ \|\zeta\|_2 = O(h^2)\quad \text{for}\quad h \to 0\ .$$

We have

$$\varphi(\tau) = x(\tau,\ \xi,\ \eta,\ \tau+h) - \xi + f[\tau,\ \xi,\ \psi(\tau,\ \xi,\ \eta)]h$$
$$P(\tau)\psi(\tau) = P(\tau)y(\tau,\ \xi,\ \eta,\ \tau+h) - P(\tau)\eta + P(\tau)\left\{A(\tau)\eta+g[\tau,\ \xi,\ \psi(\tau,\ \xi,\ \eta)]\right\}h\ .$$

Now, by (9),

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$$\begin{aligned} x(\tau, \xi, \eta, \tau+h) - \xi &= -\int_{\tau}^{\tau+h} f[s, x(s, \xi, \eta, \tau+h), y(s, \xi, \eta, \tau+h)]ds \\ &= -\int_{\tau}^{\tau+h} f[s, x(s, \xi, \eta, \tau), y(s, \xi, \eta, \tau)]ds + O(h^2) \\ &= -\int_{\tau}^{\tau+h} f[s, \xi, \psi(\tau, \xi, \eta)]ds + O(h^2) \\ &= -f[\tau, \xi, \psi(\tau, \xi, \eta)]h + o(h) . \end{aligned}$$

Also

$$\begin{aligned} y(t, \xi, \eta, \tau+h) &= Y(t)PY^{-1}(\tau+h)\eta \\ &\quad + \int_{\tau+h}^{t} Y(t)PY^{-1}(s)g[s, x(s, \xi, \eta, \tau+h), y(s, \xi, \eta, \tau+h)]ds \\ &\quad - \int_{t}^{\infty} Y(t)(I-P)Y^{-1}(s)g[s, x(s, \xi, \eta, \tau+h), y(s, \xi, \eta, \tau+h)]ds \end{aligned}$$

and hence

$$\begin{aligned} P(\tau)y(\tau, \xi, \eta, \tau+h) &= P(\tau)Y(\tau)Y^{-1}(\tau+h)\eta \\ &\quad - \int_{\tau}^{\tau+h} Y(\tau)PY^{-1}(s)g[s, x(s, \xi, \eta, \tau+h), y(s, \xi, \eta, \tau+h)]ds . \end{aligned}$$

Since

$$Y(\tau)Y^{-1}(\tau+h) = I - A(\tau)h + o(h)$$

it follows that

$$P(\tau)y(\tau, \xi, \eta, \tau+h) = P(\tau)[I-A(\tau)h]\eta - P(\tau)g[\tau, \xi, \psi(\tau, \xi, \eta)]h + o(h) .$$

Thus $\varphi(\tau) = o(h)$, $P(\tau)\psi(\tau) = o(h)$. Applying the inequality (13) we obtain

$$\|\varphi\|_2 + C_2\|\psi\| = o(h) .$$

By the definition of φ and ψ this shows that the partial derivatives $x_\tau(t, \xi, \eta, \tau)$, $y_\tau(t, \xi, \eta, \tau)$ exist and equal $x_3(t)$, $y_3(t)$ respectively.

It follows that $\psi_\tau(\tau, \xi, \eta)$ exists and, by (21), is equal to

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$$A(\tau)y(\tau, \xi, \eta, \tau) + g[\tau, x(\tau, \xi, \eta, \tau), y(\tau, \xi, \eta, \tau)] - y_\xi(\tau, \xi, \eta, \tau)f[\tau, \xi, \psi(\tau, \xi, \eta)] - y_\eta(\tau, \xi, \eta, \tau)\left\{A(\tau)\eta+g[\tau, \xi, \psi(\tau, \xi, \eta)]\right\} .$$

Hence $\psi(\tau, \xi, \eta)$ is a solution of the partial differential equation

$$(22) \quad \psi_\tau + \psi_\xi f(\tau, \xi, \psi) + \psi_\eta\{A(\tau)\eta+g(\tau, \xi, \psi)\} = A(\tau)\psi + g(\tau, \xi, \psi) .$$

We can now conclude that

$$|\psi_\tau(\tau, \xi, \eta)| = O(1) ,$$

$$|\psi_\tau(\tau, \xi_1, \eta_1) - \psi_\tau(\tau, \xi_2, \eta_2)| = O[|\xi_1-\xi_2|+|\eta_1-\eta_2|] .$$

Also

$$\psi_\xi(\tau+h, \xi, \eta) - \psi_\xi(\tau, \xi, \eta) = y_\xi(\tau+h, \xi, \eta, \tau+h) - y_\xi(\tau, \xi, \eta, \tau+h) + y_\xi(\tau, \xi, \eta, \tau+h) - y_\xi(\tau, \xi, \eta, \tau) .$$

From the differential equation satisfied by y_ξ and from (19) and (15) we obtain for $0 \le h \le h_0$

$$\begin{aligned} |y_\xi(\tau+h, \xi, \eta, \tau+h) - y_\xi(\tau, \xi, \eta, \tau+h)| &\le N \int_\tau^{\tau+h} [|x_\xi(s, \xi, \eta, \tau+h)|+2|y_\xi(s, \xi, \eta, \tau+h)|]ds \\ &= N \int_\tau^{\tau+h} [|x_\xi(s, \xi, \eta, \tau)|+2|y_\xi(s, \xi, \eta, \tau)|]ds + O(h) \\ &= O(h) . \end{aligned}$$

Therefore, by (19) again,

$$|\psi_\xi(\tau+h, \xi, \eta) - \psi_\xi(\tau, \xi, \eta)| = O(h) .$$

This inequality has been established for $0 \le h \le h_0$ but then extends, with the same constant, to arbitrary $h > 0$. Similarly we have

$$|\psi_\eta(\tau+h, \xi, \eta) - \psi_\eta(\tau, \xi, \eta)| = O(h) .$$

The four Lipschitz conditions show that ψ_ξ, ψ_η are continuous functions of (τ, ξ, η) , and hence ψ_τ is also by the partial differential

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equation (22).

Altogether we have proved

THEOREM 2. *Under the hypotheses of Theorem* 1 *there exists a bounded continuous function* $\psi(t, x, y)$ *defined for* $t \geq \tau$, $|x| < \infty$, $|y| < \mu/4K$ *with bounded continuous partial derivatives such that* ψ_t *satisfies a Lipschitz condition in* (x, y) *and* ψ_x, ψ_y *satisfy Lipschitz conditions in* (t, x, y) . *This function has the property that if* $x(t)$, $y(t)$ *is a solution of the system*

$$(1) \qquad \begin{aligned} x' &= f(t, x, y) \\ y' &= A(t)y + g(t, x, y) \end{aligned}$$

for which $x(\tau) = \xi$, $|y(\tau)| < \mu/4K$ *then* $|y(t)| \leq \mu$ *for* $t \geq \tau$ *if and only if* $y(\tau) = \psi(\tau, \xi, \eta)$ *for some* η *such that* $|\eta| < \mu/4K$.

For each fixed pair (τ, ξ) the set $M(\tau, \xi)$ of all points $\psi(\tau, \xi, \eta)$ with $|\eta| < \mu/4K$ is a C^1-submanifold of R^n with dimension equal to the rank of P . In fact, since $\psi(\tau, \xi, \eta_1) = \psi(\tau, \xi, \eta_2)$ if $P(\tau)\eta_1 = P(\tau)\eta_2$ we can restrict attention to η such that $P(\tau)\eta = \eta$. Then the mapping $\eta \to \psi(\tau, \xi, \eta)$ is continuously differentiable and, since $P(\tau)\psi(\tau, \xi, \eta) = \eta$, it has a continuously differentiable inverse. Thus $M(\tau, \xi)$ is the diffeomorphic image of the intersection of the ball $|\eta| < \mu/4K$ with the subspace of η satisfying $P\eta = \eta$.

3.

We suppose now that the system (1) is defined and satisfies our assumptions over the whole real line. Then, as shown in [1], the system (1) has an integral manifold $y = v(t, x)$, where v is a bounded continuous function with bounded continuous partial derivatives. We will first derive more precise estimates for the function v and its partial derivative v_x than were given in [1].

The function $y(t) = y(t, \xi, \tau)$ is the unique bounded solution of the equation

$$y' = A(t)y + \varphi(t) + \psi(t) ,$$

where

$$\varphi(t) = g[t, x(t, \xi, \tau), 0] + g_y[t, x(t, \xi, \tau), 0]y(t, \xi, \tau) ,$$

$$\psi(t) = G[t, x(t, \xi, \tau), y(t, \xi, \tau)] .$$

For any function $f(t)$ write

$$|f| = \sup_{-\infty<t<\infty} |f(t)| .$$

Then

$$|\psi| \leq \frac{1}{2} \mu L |y| .$$

Also, by Lemma 7 of [1] with $\beta = 0$,

$$\left| \int_t^{t+h} \varphi(s)ds \right| \leq p_0 + p_0(|y| + |y'|)$$

for $|h| \leq 1$ and either $t \geq \tau, h > 0$ or $\tau \geq t, h < 0$. So, by Lemmas 4 and 5 of [1] with $\beta = 0$, and by the superposition principle,

$$|y| \leq \alpha^{-1}K.\frac{1}{2} \mu L|y| + 4NK\gamma p_0(1+|y|+|y'|) .$$

Since $|y'| \leq N(|y|+1)$ it follows that

$$|y| \leq \frac{1}{2} \alpha^{-1}\mu LK|y| + 8N^2K\gamma p_0(1+|y|)$$
$$\leq \frac{1}{2}|y| + 8N^2K\gamma p_0 .$$

Therefore

(23) $$|y(t, \xi, \tau)| \leq 16\gamma N^2Kp_0 \quad \text{for all} \quad t, \xi, \tau .$$

Similarly, if we set

$$z(t) = x(t, \xi_1, \tau) - x(t, \xi_2, \tau) , \quad w(t) = y(t, \xi_1, \tau) - y(t, \xi_2, \tau)$$

then $w(t)$ is the unique bounded solution of an equation

$$w' = A(t)w + \varphi(t) + \psi(t) ,$$

where, by (23),

$$|\psi(t)| \leq 48\gamma N^2KLp_0[|z(t)|+|w(t)|] ,$$

and

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$$\left|\int_t^{t+h} \varphi(s)ds\right| \le p_0\beta^{-1}(e^\beta-1)\,[\|z\|+\|z'\|+\|w\|+\|w'\|]e^{\beta(t-\tau)}$$

for $|h| \le 1$ and either $t \ge \tau$, $h > 0$ or $\tau \ge t$, $h < 0$. Hence, by Lemmas 4 and 5 of [1],

$$\|w\| \le 6p_0R^2L[\|z\|+\|w\|] + p_0R[\|z\|+\|z'\|+\|w\|+\|w'\|] .$$

But by (15) of [1]

$$\|z\| + \|w\| \le 2|\xi_1-\xi_2| .$$

Since

$$\|z'\| \le N[\|z\|+\|w\|] , \quad \|w'\| \le 2N[\|z\|+\|w\|]$$

it follows that

$$\|w\| \le 4p_0R(3RL+2N)|\xi_1-\xi_2| .$$

Hence

(24) $$|y_\xi(t, \xi, \tau)| \le 4p_0R(3RL+2N)e^{\beta|t-\tau|} .$$

From (23) and (24) we obtain in particular

(25) $$|v(\tau, \xi)| \le 16\gamma N^2Kp_0 ,$$

(26) $$|v_\xi(\tau, \xi)| \le 4R(3RL+2N)p_0 .$$

It will now be proved that the solutions considered in Theorem 2 of the previous section all converge exponentially to the integral manifold $y = v(t, x)$ as $t \to \infty$. Let $x(t)$, $y(t)$ be a solution of (1) such that $|y(t)| \le \mu$ for $t \ge \tau$. By the partial differential equation which v satisfies, (29) of [1],

$$\begin{aligned} dv[t, x(t)]/dt &= v_t[t, x(t)] + v_x[t, x(t)]x'(t) \\ &= A(t)v[t, x(t)] + g\{t, x(t), v[t, x(t)]\} \\ &\qquad - v_x[t, x(t)]f\{t, x(t), v[t, x(t)]\} \\ &\qquad + v_x[t, x(t)]f\{t, x(t), y(t)\} . \end{aligned}$$

Therefore, if we put

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$$z(t) = v[t, x(t)] - y(t) ,$$

we will have

$$z'(t) = A(t)z(t) + g[t, x(t), y(t)+z(t)] - g[t, x(t), y(t)]$$
$$- v_x[t, x(t)]\{f[t, x(t), y(t)+z(t)] - f[t, x(t), y(t)]\} .$$

Thus $z(t)$ is a solution of the equation

$$z' = A(t)z + h(t, z) ,$$

where

$$h(t, z) = g[t, x(t), y(t)+z] - g[t, x(t), y(t)]$$
$$- v_x[t, x(t)]\{f[t, x(t), y(t)+z] - f[t, x(t), y(t)]\} .$$

The function h is continuous, vanishes when z vanishes, and has a continuous partial derivative h_z . Moreover, the properties of v established in [1] imply that

$$|h_z(t, z)| \le 2N ,$$
$$|h_z(t, z_1) - h_z(t, z_2)| \le 2L|z_1-z_2| ,$$

while by (26) above

$$\left|\int_{t_1}^{t_2} h_z(t, 0)dt\right| = O(1)p_0 + O(1)\mu_0 \quad \text{for} \quad |t_2-t_1| \le 1 .$$

If we set

$$h(t, z) = h_z(t, 0)z + H(t, z)$$

then $|H(t, z)| \le L|z|^2$. It follows from the Lemma proved below that with a suitable choice of μ_0 and q_0

$$|z(t)| \le 4Ke^{-\alpha(t-s)}|z(s)| \quad \text{for} \quad t \ge s \ge \tau ,$$

that is,

$$(27) \quad |y(t)-v[t, x(t)]| \le 4Ke^{-\alpha(t-s)}|y(s)-v[s, x(s)]| \quad \text{for} \quad t \ge s \ge \tau .$$

LEMMA 3. *Let* $A(t)$ *and* $B(t)$ *be continuous matrix functions with* $|A(t)| \le N$, $|B(t)| \le N$ *for* $t \ge \tau$ *and suppose the linear equation* (2)

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has a fundamental matrix $Y(t)$ *satisfying* (3). *Suppose also that*

$$\left|\int_{t_1}^{t_2} B(t)dt\right| \le r \quad for \quad |t_2-t_1| \le 1 ,$$

and let $f(t, y)$ *be a continuous vector function such that*

$$|f(t, y)| \le \delta|y| .$$

If the positive constants r, δ *are so small that* $r \le 1$ *and*

$$\theta = \alpha^{-1}K\left[\delta + r\left(3N+\delta+e^{2\alpha}\right)\right] \le \frac{1}{2} ,$$

then any bounded solution $y(t)$ *of the nonlinear equation*

$$y' = [A(t)+B(t)]y + f(t, y) \tag{28}$$

satisfies

$$|y(t)| \le 4Ke^{-\alpha(t-s)}|y(s)| \quad for \quad t \ge s .$$

In fact any bounded solution $y(t)$ of (28) satisfies

$$y(t) = Y(t)PY^{-1}(s)y(s) + \int_s^t Y(t)PY^{-1}(u)\left\{B(u)y(u)+f[u, y(u)]\right\}du - \int_t^\infty Y(t)(I-P)Y^{-1}(u)\left\{B(u)y(u)+f[u, y(u)]\right\}du . \tag{29}$$

Put

$$\mu(t) = \sup_{u \ge t} |y(u)| .$$

By integrating by parts in the accustomed way we obtain

$$\left|\int_t^\infty Y(t)(I-P)Y^{-1}(u)B(u)y(u)du\right| \le \frac{1}{2} r\alpha^{-1}K(3N+\delta+1)\mu(t) .$$

Similarly, if $s+m \le t < s+m+1$ then

$$\left|\int_s^t Y(t)PY^{-1}(u)B(u)y(u)du\right| \le rK \sum_{j=0}^{m} e^{-2\alpha(t-s-j)}|y(s+j)| + rK(3N+\delta) \int_s^t e^{-2\alpha(t-u)}|y(u)|du .$$

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But if $j \geq 1$ then

$$e^{2\alpha(s+j)}|y(s+j)| \leq e^{2\alpha}\int_{s+j-1}^{s+j} e^{2\alpha u}\mu(u)du \ .$$

Hence

$$\left|\int_s^t Y(t)PY^{-1}(u)B(u)y(u)du\right| \leq rKe^{-2\alpha(t-s)}|y(s)|$$

$$+ rK\left(3N+\delta+e^{2\alpha}\right)\int_s^t e^{-2\alpha(t-u)}\mu(u)du \ .$$

Estimating the other terms in (29) in the crudest way we obtain

$$(30) \quad |y(t)| \leq (1+r)Ke^{-2\alpha(t-s)}|y(s)| + \theta\alpha\int_s^t e^{-2\alpha(t-u)}\mu(u)du + \frac{1}{2}\theta\mu(t) \ .$$

Choose $h > 0$ so large that $8Ke^{-2\alpha h} \leq 1$. Then, since μ is a non-increasing function, for $t \geq s+h$

$$|y(t)| \leq \frac{1}{4}\mu(s) + \frac{1}{2}\theta\mu(s) + \frac{1}{2}\theta\mu(s)$$
$$\leq \frac{3}{4}\mu(s) \ .$$

Hence $\mu(s+h) \leq \frac{3}{4}\mu(s)$, which shows that $|y(t)| \to 0$ as $t \to \infty$. Therefore for any t there exists $t' \geq t$ such that

$$\mu(t) = \mu(s) = |y(t')| \quad \text{for} \quad t \leq s \leq t' \ .$$

By (30), with t' in place of t ,

$$\mu(t) \leq 2Ke^{-2\alpha(t'-s)}|y(s)| + \theta\alpha\int_s^t e^{-2\alpha(t'-u)}\mu(u)du + \theta\mu(t) \ .$$

Thus $\varphi(t) = e^{2\alpha t}\mu(t)$ satisfies

$$\varphi(t) \leq 4Ke^{2\alpha s}|y(s)| + \alpha\int_s^t \varphi(u)du \ .$$

Therefore by Gronwall's inequality,

$$\varphi(t) \leq 4Ke^{2\alpha s}|y(s)|e^{\alpha(t-s)} \ .$$

Hence

$$|y(t)| \le 4Ke^{-\alpha(t-s)}|y(s)| .$$

Finally it will be shown that the solutions of Theorem 2 are asymptotic not only to the integral manifold but also to particular solutions on the manifold. As in [1] let

$$k(t, x) = f[t, x, v(t, x)] .$$

The function k is continuous and has a continuous partial derivative k_x such that $|k_x| \le 2N$. Let $x(t)$, $y(t)$ be a solution of (1) such that $|y(t)| \le \mu$ for $t \ge \tau$, and put

$$\lambda(t) = |y(t) - v[t, x(t)]| .$$

By (27) we have

$$\lambda(t) \le 4Ke^{-\alpha(t-\tau)}\lambda(\tau) \quad \text{for} \quad t \ge \tau .$$

Let $x_n(t)$ be the solution of the equation

$$x' = k(t, x)$$

such that $x_n(\tau+n) = x(\tau+n)$. Since

$$x'(t) = k[t, x(t)] + \ell(t) ,$$

where

$$\ell(t) = f[t, x(t), y(t)] - f\Big[t, x(t), v[t, x(t)]\Big] ,$$

the difference $z_n(t) = x(t) - x_{n-1}(t)$ has the representation

$$z_n(t) = \int_{\tau+n-1}^{t} \Big\{k[s, x(s)] - k[s, x_{n-1}(s)] + \ell(s)\Big\}ds .$$

Therefore, since $|\ell(t)| \le N\lambda(t)$,

$$|z_n(t)| \le 2N \int_{\tau+n-1}^{t} |z_n(s)|ds + N \int_{\tau+n-1}^{t} \lambda(s)ds .$$

Thus for $\tau+n-1 \le t \le \tau+n$

$$|z_n(t)| \le 2N \int_{\tau+n-1}^{t} |z_n(s)|ds + \gamma^{-1}R\lambda(\tau)e^{-\alpha n}$$

and so, by Gronwall's inequality,

$$|x(t)-x_{n-1}(t)| = |z_n(t)| \le \gamma^{-1}Re^{2N}\lambda(\tau)e^{-\alpha n} . \tag{31}$$

In particular, for $t = \tau+n$,

$$|x_n(\tau+n) - x_{n-1}(\tau+n)| \le \gamma^{-1}Re^{2N}\lambda(\tau)e^{-\alpha n} .$$

Therefore, by (15) of [1], for $\tau \le t \le \tau+n$

$$|x_n(t) - x_{n-1}(t)| \le 2\gamma^{-1}Re^{2N}\lambda(\tau)e^{-(\alpha-\beta)n} . \tag{32}$$

This shows that the sequence $\{x_n(t)\}$ converges uniformly on any bounded subinterval of $[\tau, \infty)$. Let $x_\infty(t)$ denote its limit. Summing (32) for $n \ge m$ we get for $\tau \le t \le \tau+m$

$$\begin{aligned} |x_\infty(t)-x_{m-1}(t)| &\le 2\gamma^{-1}Re^{2N}\lambda(\tau)\left(1-e^{-\alpha/2}\right)^{-1}e^{-(\alpha-\beta)m} \\ &\le 4Re^{2N}\lambda(\tau)e^{-(\alpha-\beta)m} . \end{aligned}$$

Taking $n = m$ in (31) we deduce that for $\tau+m-1 \le t \le \tau+m$

$$|x(t) - x_\infty(t)| \le 5Re^{2N}\lambda(\tau)e^{-(\alpha-\beta)m} .$$

Thus for all $t \ge \tau$

$$|x(t) - x_\infty(t)| \le 5Re^{2N}\lambda(\tau)e^{-(\alpha-\beta)(t-\tau)} . \tag{33}$$

If we set $y_\infty(t) = v[t, x_\infty(t)]$ then

$$\begin{aligned} |y(t)-y_\infty(t)| &\le |y(t)-v[t, x(t)]| + |v[t, x(t)]-v[t, x_\infty(t)]| \\ &= O(1)\lambda(\tau)e^{-(\alpha-\beta)(t-\tau)} , \end{aligned}$$

by (27) and (33). Altogether we have proved

THEOREM 3. *Let the hypotheses of Theorems* 1 *and* 2 *hold for* $-\infty < t < \infty$ *and let* $y = v(t, x)$ *be the integral manifold whose existence was established in* [1]. *If* $x(t)$, $y(t)$ *is a solution of the system* (1) *such that* $|y(t)| \le \mu$ *for* $t \ge \tau$ *then*

$$|y(t)-v[t,\ x(t)]| \le 4Ke^{-\alpha(t-s)}|y(s)-v[s,\ x(s)]| \quad \textit{for} \quad t \ge s \ge \tau .$$

Moreover there exists a solution $x_\infty(t)$, $y_\infty(t) = v[t,\ x_\infty(t)]$ *of* (1) *such that for* $t \ge \tau$

$$|x(t)-x_\infty(t)|+|y(t)-y_\infty(t)| = O(1)|y(\tau)-v[\tau,\ x(\tau)]|e^{-(\alpha-\beta)(t-\tau)} .$$

4.

My first work on this subject was contained in a joint paper with W.A. Coppel, communicated to the *Journal of Mathematical Analysis and Applications* in September 1968. In April 1969 the Editor of the Journal wrote that they had no record of having received the article. Since the earlier treatment has now been superseded by [*1*], it has not been resubmitted for publication. However, its introduction contained some motivation for the study of the problem and comparison with related work which it seems worthwhile to include here, in a slightly extended form.

The method of averaging, for non-conservative systems, was first applied to some special second order equations by Krylov and Bogolyubov in 1934. Their results were considerably generalized by Bogolyubov in 1945 and given a finished form in Bogolyubov and Mitropolskii [*2*]. An account of their methods, with numerous applications, was given by Hale [*3*] and [*4*]. There are three main results. The first says that solutions of the original equation and solutions of the averaged equation with the same initial point remain close over a large, but finite, interval of time. The second says that if the averaged equation has a constant solution then the original equation has in its neighbourhood a unique bounded solution, with the same stability properties. The third, and most remarkable, says that if the averaged equation has a periodic solution then the original equation has in its neighbourhood a unique integral manifold, with the same stability properties. The introduction of moving orthonormal coordinates near the given periodic orbit reduces the problem to the study of a system (1). The proofs of these results were rather indirect and depended on changes of variables combined with smoothing operations.

It was shown by Gihman [*5*] in 1952, and by others after him, that the first result is a simple consequence of a theorem about the continuous

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dependence of solutions on a parameter when the variation is not small in the usual sense but is 'integrally small'. A version of his result will be presented here, using Lemma 3 of [1].

THEOREM 4. *Let* $f(t, x)$ *and* $g(t, x)$ *be continuous functions such that*

$$|f(t, x_1)-f(t, x_2)| + |g(t, x_1)-g(t, x_2)| \le L|x_1-x_2| ,$$
$$|g(t, x)| \le N ,$$
$$\left|\int_{t_1}^{t_2} \{f(t, x)-g(t, x)\}dt\right| \le q \quad \text{for} \quad |t_2-t_1| \le 1 .$$

If $x(t)$ *and* $y(t)$ *are solutions of the differential equations*

$$x' = f(t, x) \ , \quad y' = g(t, y)$$

for $0 \le t \le T$ *, with* $x(0) = y(0)$ *, then*

$$|x(t)-y(t)| \le pe^{Lt} + p\left(e^{Lt}-1\right)/L \quad \text{for} \quad 0 \le t \le T ,$$

provided $q \le \frac{1}{2} p^2(p+LN)^{-1}$.

If we set $h(t, x) = f(t, x) - g(t, x)$ then, by Lemma 3 of [1],

$$\left|\int_{t_1}^{t_2} h[s, y(s)]ds\right| \le p \quad \text{for} \quad |t_2-t_1| \le 1 .$$

Therefore

$$\left|\int_0^t h[s, y(s)]ds\right| \le p(t+1) .$$

The difference $z(t) = x(t) - y(t)$ has the representation

$$z(t) = \int_0^t \left\{f[s, x(s)]-f[s, y(s)]\right\}ds + \int_0^t h[s, y(s)]ds .$$

Hence

$$|z(t)| \le L \int_0^t |z(s)|ds + p(t+1) ,$$

and the result follows by the extended Gronwall lemma.

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It was shown by Coppel [6] that the second result of the method of averaging could also be treated, more generally and more directly, as an integrally small perturbation problem. His use of the roughness of exponential dichotomies can be avoided, as in the present paper, and the result thus extended to arbitrary Banach spaces.

The third result, on integral manifolds, has been treated by different methods by a number of authors, e.g. Levinson [7], Diliberto and Hufford [8], Diliberto [9] and [10], Sacker [11] and [12], and Kurzweil [13], [14], [15]. In many cases, however, the problems treated are less general or stronger assumptions are imposed than in the work of Bogolyubov and Mitropolskii. Only Kurzweil has treated the problem as one of integrally small perturbations. The generality of his approach has perhaps obscured some of his contributions. In particular he showed that a solution asymptotic to the integral manifold was also asymptotic to a particular solution on the manifold, which Bogolyubov and Mitropolskii proved only in a special case. Our proof of this property was modelled on his. Otherwise this work has been essentially independent of that of Kurzweil. A detailed comparison of the differences in hypotheses, conclusions and methods will not be attempted here.

Sacker's main contribution is connected with higher order smoothness of the manifold. This can also be treated by the present methods. It can be shown that *if* f *and* g *have* r *continuous partial derivatives with respect to* x *and* y *, the* r-th *derivatives being Lipschitzian in* x *and* y *, then for* q *sufficiently small,* $\partial^s v/\partial x^s$ *and* $\partial^s v/\partial t \partial x^{s-1}$ *exist for* $1 \le s \le r$ *,* $\partial^r v/\partial x^r$ *is Lipschitzian in* t *and* x *and* $\partial^r v/\partial t \partial x^{r-1}$ *is Lipschitzian in* x *and continuous in* t *. Also, for* $1 \le s \le r$ *,* $|\partial^s v/\partial x^s| = O(1)q^{2-s}$.

In order to obtain the latter estimates we need the integral smallness of the higher derivatives. However, it is not necessary to impose this as an additional hypothesis since it follows from the assumptions already made. For let us suppose that $h(t, x)$ has a continuous partial derivative $h_x(t, x)$ such that

$|h_x(t, x_1)-h_x(t, x_2)| \le L|x_1-x_2|$ and

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$$\left|\int_{t_1}^{t_2} h(t, x)dt\right| \le q \quad \text{if} \quad |t_1-t_2| \le 1 \ .$$

Then if

$$R = h(t, x+k) - h(t, x) - h_x(t, x)k \ ,$$

$|R| \le L|k|^2$. If we assume $q \le q_1^2/8L$ and take k to be *any* vector with $|k| = q_1/2L$ then for $|t_2-t_1| \le 1$

$$\begin{aligned}\left|\int_{t_1}^{t_2} h_x(t, x)dtk\right| &\le 2q + L|k|^2 \\ &\le q_1^2/4L + q_1^2/4L = q_1^2/2L \ .\end{aligned}$$

Hence

$$\left|\int_{t_1}^{t_2} h_x(t, x)dt\right| \le q_1 \quad \text{for} \quad |t_2-t_1| \le 1 \ .$$

This was suggested by Kurzweil [*13*], Lemma 1.1. It also shows that in [*1*] the assumption that g_x is integrally small is redundant.

Finally as a by-product, the present methods yield a generalization of Bogolyubov's and Mitropolskii's theorem. They assumed that the averaged equation

$$x' = \varepsilon X_0(x)$$

has a solution $\xi(\varepsilon t)$ with period $2\pi/\varepsilon$ such that the variational equation

$$x' = X_0'[\xi(t)]x$$

has $(n-1)$ characteristic exponents with nonzero real parts. This can be replaced by the assumption that the averaged equation has a *bounded* solution $\xi(\varepsilon t)$ such that the variational equation has a fundamental matrix $Z(t)$ satisfying

$$|Z(t)P_1Z^{-1}(s)| \le Me^{-\sigma(t-s)} \quad \text{if} \quad s \le t \ ,$$
$$|Z(t)P_2Z^{-1}(s)| \le Me^{-\sigma(s-t)} \quad \text{if} \quad s \ge t \ ,$$

where M, σ are positive constants and P_1, P_2 are mutually orthogonal

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projections such that $P_1 + P_2$ has rank $n - 1$.

Under these assumptions the existence of the integral manifold can be established and almost periodic properties of $\xi(t)$ induce similar properties in the manifold.

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