BULL. AUSTRAL. MATH. SOC. 60G40
VOL. 38 (1988) [83–85]

A CONVERGENCE PROPERTY OF DUBINS' REPRESENTATION OF DISTRIBUTIONS

SHEY SHIUNG SHEU

Let $(X_n)_{n\geqslant 1}$ be a sequence of random variables with zero means and uniformly bounded variences. Let τ_n be the stopping time defined on a given Brownian motion $(B_t)_{t\geqslant 0}$, $B_0 = 0$, by Dubins' method such that $B(\tau_n)$ has the same distribution as X_n. We prove that X_n converging to 0 in distribution implies that τ_n converges to 0 in probability. Examples are presented to illustrate the result is the best possible.

1. INTRODUCTION

This is an extension of my paper (Sheu [**7**]) which dealt with problems related to the Skorohod representation (Skorohod [**8**]). More specifically, one is seeking a propability space supporting a Brownian motion $(B_t)_{t\geqslant 0}$ starting at the origin and a stopping time τ so that B_τ will be distributed according to a given distribution (or random variable). Special interest is focussed on whether such τ can be required to depend only on Brownian paths without furhter randomisation (see Root [**6**], Dubins [**4**], Chacon and Walsh [**3**], Azéma and Yor [**1**], Bass [**2**], Vallois [**9**], etcetera).

For convenience, we use the notation $B_\tau \sim X$ to mean B_τ and X have the same distribution and say B_τ *represents* X. In an investigation of Root's method of representation, Loynes [**5**] posed a problem of convergence. He asked whether the stopping time τ depends on the distribution of X continuously in a certain sense. In fact, he showed that if (X_n) is a sequence of random variables with zero means and uniformly bounded variences and if X_n converges in distribution, then the stopping time τ_n constructed by Roots' method, $B_{\tau_n} \sim X_n$, converges in probability. Motivated by his results, we shall answer the same question for the Dubins method of representation in this paper.

2. MAIN RESULTS

The Dubins method can be described as follows (Dubins [**4**]). Let μ be a distribution on the real line with finite expectation $E\mu$, let μ^+ and μ^- denote the conditional distribution of μ given $[E\mu, \infty)$ and $(-\infty, E\mu)$, respectively. If μ is degenerate, set $\mu^+ = \mu^- = \mu$. For any set K of n-tuples, let $(m; K)$ be the set of $(n+1)$-tuples of the form (m, x), $x \in K$. Now introduce for each μ and each $n \geqslant 1$, a finite set of

Received 23 November, 1987

$n-$-tuples of real numbers $H_n(\mu)$ as follows: $H_1(\mu) = \{E\mu\}$, $K_n(\mu) = H_n(\mu^+) \cup H_n(\mu^-)$, $H_{n+1}(\mu) = (E\mu; K_n(\mu))$. Let $(B_t)_{t \geqslant 0}$ be a given Brownian motion, $B_0 = 0$, and μ a distribution with finite expectation $E\mu$. The Dubins stopping time τ is defined to be the least t such that for all $n \geqslant 1$, there is an n-tuple $t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n \leqslant t$ for which $(B_{t_1}, B_{t_2}, \ldots, B_{t_n}) \in H_n(\mu)$. Dubins showed that B_τ is distributed according to μ and if $E\mu = 0$, then $E(\tau) = \int_{-\infty}^{\infty} x^2 \, d\mu$. Turning to the question of convergence, we have

THEOREM 1. *Let $(X_n)_{n \geqslant 1}$ be a sequence of random variables with zero means and uniformly bounded variances. Let τ_n be the Dubins stopping time for representing X_n, $B_{\tau_n} \sim X_n$, $n \geqslant 1$. Then X_n converging to 0 in distribution implies τ_n converges to 0 in probability.*

PROOF: Let μ_n be the distribution of X_n, $n \geqslant 1$, and let

$$T_a = \inf\{t \geqslant 0 \colon B_t = a\}.$$

Since X_n converges to 0 in distribution and $E(X_n^2)$ is uniformly bounded, we conclude $E|X_n| \to 0$; that is

$$E(X_n^+) \to 0 \text{ and } E(X_n^-) \to 0.$$

Given $\varepsilon > 0$, choose $\delta > 0$ so that $P(T_\delta \geqslant \frac{\varepsilon}{2}) \leqslant \frac{\varepsilon}{2}$. Then there exists an $N = N(\varepsilon)$ such that , if $n \geqslant N$, either

$$E\mu_n^+ = E(X_n^+)/P(0 \leqslant X_n < \infty) \leqslant \delta$$

or

$$E\mu_n^- = E(X_n^-)/P(-\infty < X_n < 0) \leqslant \delta.$$

By Dubins' construction, we see

$$\tau_n \leqslant \inf\{t \geqslant 0 \colon B_t = 0,\ B_s = \delta, \text{ some } s \leqslant t\}$$

or

$$\tau_n \leqslant \inf\{t \geqslant 0 \colon B_t = 0,\ B_s = -\delta, \text{ some } s \leqslant t\}.$$

By the strong Markov property of Brownian motion, we have $P(\tau_n \geqslant \varepsilon) \leqslant 2P(T_\varepsilon \geqslant \frac{\varepsilon}{2}) \leqslant \varepsilon$. That is, τ_n converges to 0 in probability. ∎

The following examples indicate the result obtained is the best possible.

Example 1. Let $P(X_n = 0) = 1 - \frac{2}{n}$, $P(X_n = n) = P(X_n = -n) = \frac{1}{n}$. Then $E(X_n) = 0$, $E(X_n^2) = 2n$. By computation, $E\mu_n^+ = \frac{n}{n-1}$, $E\mu_n^- = -n$. Clearly, X_n converges to 0 in distribution but τ_n does not. This example shows that the assumption of uniformly bounded variances can not be omitted.

Example 2. Let $P\left(X_n = \frac{3n+1}{n}\right) = \frac{2}{10}$, $P\left(X_n = -\frac{1}{n}\right) = \frac{1}{2}$, $P(X_n) = \frac{-2n+1}{n} = \frac{3}{10}$ and let $P(X = 3) = \frac{2}{10}$, $P(X = 0) = \frac{1}{2}$, $P(X = -2) = \frac{3}{10}$. Then $E(X) = E(X_n) = 0$, $E\left(X_n^2\right) \leqslant \frac{32}{5}$. By computation, μ_n^+, μ_n^{-+}, μ_n^{--} are degenerate and $E\mu_n^+ = \frac{3n+1}{n} \to 3$, $E\mu_n^- = \frac{-3n-1}{4n} \to -\frac{3}{4}$, $E\mu_n^{-+} = -\frac{1}{n} \to 0$, $E\mu_n^{--} = \frac{-2n+1}{n} \to -2$. Also, μ^-, μ^{++}, μ^{+-} are degenerate and $E\mu^+ = \frac{6}{7}$, $E\mu^- = -2$, $E\mu^{++} = 3$, $E\mu^{--} = 0$. Therefore, X_n converges to X in distribution but the Dubins stopping time, τ_n, $B_{\tau_n} \sim X_n$, does not converge to τ in probability, where $B_\tau \sim X$. This example shows that the assumption that the limiting distribution of X_n is degenerate can not be omitted.

References

[1] J. Azéma and M. Yor, 'Une solution simple au problème de Skorohod', *Sém. Prob XIII, Lecture Notes in Math* **721** (1979), 90–115.

[2] R. F. Bass, 'Skorohod imbedding via stochastic integrals', *Sém. Prob. XVII, Lecture Notes in Math.* **986** (1983), 221–224.

[3] R.V. Chacon and J.B. Walsh, 'One-dimensional potential embedding', *Sém. Prob. X, Lecture Notes in Math.* **511** (1976), 19–23.

[4] L.E. Dubins, 'On a theorem of Skorohod', *Ann. Math. Statist.* **39** (1968), 2094-2097.

[5] R.M. Loynes, 'Stopping times on Brownian motion: some properties of Root's construction', *Z. Wahrsch. verw. Gebiete* **16** (1970), 211–218.

[6] D.H. Root, 'The existence of certain stopping times on Brownian motion', *Ann. Math. Statist.* **40** (1967), 715–718.

[7] S.S. Sheu, 'Representing a distribution by stopping a Brownian motion: Root's construction', *Bull. Austral.Math. Soc.* **34** (1986), 427–431.

[8] A.V. Skorohod, *Studies in the Theory of Random Processes* (Addison-Wesley, Reading, Mass., 1965).

[9] P. Vallois, 'Le problème de Skorohod sur **R**: une approche avec le temps local', *Sém. Prob. XVII, Lecture Notes in Math.* **986** (1983), 227–239.

Institute of Applied Mathematics,
National Tsing Hua University,
Hsinchu, Taiwan, 300043
Republic of China