NORMAL LATTICES

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1. Introduction

If L is a distributive lattice with 0 then it is shown that each prime ideal contains a unique minimal prime ideal if and only if, for any x and y in L, $x \wedge y = 0$ implies $(x]^* \vee (y]^* = L$. A distributive lattice with 0 is called *normal* if it satisfies the conditions of this result. This terminology is appropriate for the following reasons. Firstly the lattice of closed subsets of a T_1-space is normal if and only if the space is normal. Secondly lattices satisfying the above annihilator condition are sometimes called normal by those mathematicians interested in (Wallman-) compactications, for example see [2].

The above result is applied to give (1) a very simple and natural proof of known characterizations of Stone lattices and generalized Stone lattices as discussed in [4], [11], [13] and [5], (2) clarification and extensions of Mandelker's ([8]) results on distributive lattices in which any pair of incomparable prime ideals is comaximal and (3) a discussion of necessary and sufficient conditions for the space of ultrafilters of a distributive lattice to be either Hausdorff or totally disconnected.

2. The basic theorem

Throught this paper all lattices are assumed to be distributive.

For an ideal I in a lattice L with 0,

$$J^* = \{y \in L : y \wedge x = 0 \textit{ for all } x \in J\}.$$

$(x] = \{y \in L : y \leqq x\}$ denotes the *principal ideal generated by* x; it is clear that

$$(x]^* = \{y \in L :\ y \wedge x = 0\}.$$

Ideals I and J of lattice L are said to be *comaximal* if $I \vee J = L$.

If P is a prime ideal in lattice L with 0 then $0(P)$ is used to donte the ideal

$$\{y \in L;\ y \wedge x = 0 \textit{ for some } x \in L \backslash P\}.$$

Clearly $0(P) \subseteq P$.

A prime ideal P is said to be a *minimal prime ideal belonging to ideal* I, if (1) $I \subseteq P$ and (2) there exists no prime ideal Q such that $Q \neq P$ and $I \subseteq Q \subseteq P$. A minimal prime ideal belonging to the zero ideal of a lattice with 0 is called a *minimal prime ideal.*

LEMMA 2.1 *Let P be a prime ideal in lattice L with* 0. *Then each minimal prime ideal belonging to* $0(P)$ *is contained in* P.

PROOF. Let Q be a minimal prime ideal belonging to $0(P)$. If $Q \nsubseteq P$ then choose $y \in Q \backslash P$. From [6, Lemma 3.1] and the distributivity of L it follows that $y \wedge z \in 0(P)$ for some $z \notin Q$. Hence $y \wedge z \wedge x = 0$ for a suitable $x \notin P$. As P is prime $y \wedge x \notin P$ so $z \in 0(P) \subseteq Q$. This is a contradiction. Hence $Q \subseteq P$.

PROPOSITION 2.2. *If P is a prime ideal in a lattice with* 0 *then the ideal* $0(P)$ *is the intersection of all the minimal prime ideals contained in* P.

PROOF. It is clear that $0(P)$ is contained in any prime ideal which is contained in P. Hence $0(P)$ is contained in the intersection of all minimal prime ideals contained in P. As L is distributive $0(P)$ is the intersection of all minimal primes belonging to it. As each prime contains a minimal prime ideal these remarks and Lemma 2.1 establish the proposition.

DEFINITION 2.3. A lattice with 0 is called *normal* if each prime ideal contains a unique minimal prime ideal.

THEOREM 2.4. *For a lattice L with* 0 *the following conditions are equivalent:*

(a) *any two distinct minimal prime ideals are comaximal*

(b) *L is normal,*

(c) *for each prime ideal P, $0(P)$ is a prime ideal,*

(d) *for any $x\,y \in L$, $x \wedge y = 0$ implies $(x]^*$ and $(y]^*$ are comaximal*

(e) *for any $x, y \in L$, $(x \wedge y]^* = (x]^* \vee (y]^*$.*

Moreover, when L has a largest element 1, *each of the above conditions is equivalent to*:

(f) *each maximal ideal contains a unique minimal prime ideal,*

(g) *for each maximal ideal M, $0(M)$ is a prime ideal,*

(h) *for any $x, y \in L$, $x \wedge y = 0$ implies there exist x_1, $y_1 \in L$ such that $x \wedge x_1 = 0 = y \wedge y_1$ and $x_1 \vee y_1 = 1$.*

PROOF. (a) $\Rightarrow$ (b) is trivial, and $(b) \Rightarrow (c)$ is a direct consequence of Proposition 2.2.

(c) $\Rightarrow$ (d). Suppose (c) holds and yet (d) does not. Then there exist $x, y \in L$ with $x \wedge y = 0$ and such that $(x^*] \vee (y]^* \neq L$. As L is distributive there is a prime ideal P such that $(x]^* \vee (y]^* \subseteq P$. Then $(x]^* \subseteq P$ and $(y]^* \subseteq P$ imply $x \notin 0(P)$ and $y \notin 0(P)$. But $0(P)$ is prime and so $x \wedge y = 0 \in 0(P)$ is contradictory. Thus (c) implies (d).

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(d) $\Rightarrow$ (e). The inequality $(x]^* \vee (y]^* \subseteq (x \vee y]^*$ always holds. Thus suppose $w \in (x \wedge y]^*$. Then $w \wedge x \wedge y = 0$, so by (d),

$$(w \wedge x) \wedge r = 0 = y \wedge s$$

and $r \vee s = w$ for suitable r and s. Also $y \wedge s = 0$ implies that there exist p and q such that

$$p \wedge y = 0 = q \wedge s \text{ and } p \vee q = w.$$

Then $w = (w \wedge r) \vee (w \wedge s)$ so $w \wedge q = w \wedge r \wedge q$. Thus

$$w = (w \wedge p) \vee (w \wedge q) = (w \wedge p) \vee (w \wedge r \wedge q)$$

and $w \wedge r \wedge q \wedge x = 0$ as $(w \wedge r) \wedge x = 0$ and $w \wedge p \wedge y = 0$. Hence $w \in (x^*] \vee (y]^*$ and (e) follows.

The rest is either obvious or follows in a manner similar to the above.

Varlet [13, section 3, p.82] has shown that conditions (a) and (b) of the above theorem are equivalent by a different method; the rest of the theorem seems to be novel.

3. Relatively normal lattices

The terminology used in the following definitions follows current usage in lattice theory. For $x < y$ in a lattice L $[x, y]$ denotes the interval $\{z \in L;\ x \leqq z \leqq y\}$; it is considered to be a sublattice of L.

Definition 3.1. A lattice L with 0 is called *sectionally normal* if each interval $[0,x]$ with $0 < x$ is a normal lattice.

Definition 3.2. A lattice L is called *relatively normal* if each interval $[x,y]$ with $x < y$ is a normal lattice.

Katriňák [5, lemma 9,p.135] has shown that a normal lattice is sectionally normal. The next theorem both improves and yields an alternative proof of his result.

Theorem 3.3. *Let L be a lattice with 0. Then the following are equivalent*

(a) *L is normal,*

(b) *each ideal $J \neq L$ is a normal sublattice,*

(c) *L is sectionally normal.*

Proof. (a) $\Rightarrow$ (b), If J is an ideal and $x, y \in J$ with $x \wedge y = 0$ then $(x]^* \vee (y]^* = L$ because of Theorem 2.3 providing (a) holds. Hence

$$J = J \cap L = (J \cap (x]^*) \vee (J \cap (y]^*)$$

as the lattice of ideals of a distributive lattice is itself distributive. But $J \cap (x]^*$ and

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$J \cap (y]^*$ are respectively $\{z \in J;\ z \wedge x = 0\}$ and $\{z \in J:\ z \wedge y = 0\}$ and it follows from theorem 2.3 that J is normal. (b) $\Rightarrow$ (c) trivial and (c) $\Rightarrow$ (a) follows from Theorem 2.4 (d).

The following theorem and its accompanying lemma were inspired by a result of Grätzer and Schmidt [4, Theorem 3, p.459] and our proofs are related to theirs. The lemma which is of independent interest does not seem to appear in the literature.

LEMMA 3.4. *If L_1 is a sublattice of lattice L and P_1 is a prime ideal in L_1 then there exists a prime ideal P in L such that $P_1 = L_1 \cap P$.*

PROOF. Let I be the ideal generated by P_1 in L. Clearly

$$I = \{z \in L:\ z \leqq x \text{ for some } x \in P_1\}.$$

Clearly $I \cap (L_1 \backslash P_1) = \varnothing$. Then as $L_1 \backslash P_1$ is closed under $\wedge$, a modification of Stone‘s theorem (for Stone's theorem see [8, p. 379]) or Krull's lemma [6, Lemma 1.2] stated for distributive lattices implies that there is a prime ideal P in L such that $I \subseteq P$ and

$$(L_1 \backslash P_1) \cap P = \varnothing.$$

Then $P_1 \subseteq I \cap L_1 \subseteq P \cap L_1$ and $P \cap L_1 \subseteq P_1$ so $P_1 = P \cap L$.

THEOREM 3.5. *A lattice is relatively normal if and only if any two incomparable prime ideals are comaximal.*

PROOF. Suppose L has a pair of prime ideals P and Q such that $P \nsubseteq Q$ and $Q \nsubseteq P$ and yet $P. \vee Q \neq L$. We can then construct an interval which is not normal as follows. Choose $a \in L \backslash (P \vee Q)$, $b \in P \backslash Q$ and $c \in Q \backslash P$ and consider the interval

$$I = [b \wedge c,\ a \vee b \vee c].$$

$b \wedge c$ is the zero of I, $a \vee b \vee c$ is the unit and $b, c \in I$ is the unit and $b, c \in I$ with $b \wedge c =$ the zero of I. Now if I were normal there would exist (by Theorem 2.4) $x, y \in I$ such that

$$x \wedge b = b \wedge c = y \wedge c \text{ and } x \vee y = a \vee b \vee c.$$

Then $x \wedge b = b \wedge c \in Q$, as $c \in Q$, as $c \in Q$, so $x \in Q$ is prime and $b \notin Q$. Similarly, $y \in P$. Hence,

$$a \leqq a \vee b \vee c = x \vee y \in P \vee Q$$

This is contradictory, so the interval I is not normal.

Suppose each pair of incomparable prime ideals is comaximal Let M_1, M_2 be minimal prime ideals in the interval $[x,y]$. By Lemma 3.4, there are prime ideals P_1 and P_2 in L such that $M_i = [x, y] \cap P_i$, for $i = 1,2$. Clearly P_1 and P_2 are incomparable so $P_1 \vee P_2 = L$. Now let $a \in [x,y]$. Then $a = c \vee d$ where

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$c \in P_1, d \in P_2$. So $c, d \leqq a \leqq y$ and $c \vee x$, $d \vee x \in [x,y]$. But $x \in M_1$ and M_2 so $x \in P_1$ and P_2, so $c \vee x \in M_1$ and $d \vee x \in M_2$. Thus

$$a = (c \vee x) \vee (d \vee x) \in M_1 \vee M_2.$$

Thus M_1 and M_2 are comaximal in $[x,y]$.

Theorem 3.5 should be compared with condition (a) of Theorem 2.4. Also Theorem 3.5 shows that relatively normal lattice are well-known objects. In fact combining the theorem with results due to Varlet [13, section 5, p. 83] and Mandelker [8, Theorem 4, p. 380] we see that the following are equivalent for a lattice L: (1) L is relatvely normal, (2) the prime idelas contained in a given prime ideal form a chain, (3) $\langle a, b\rangle \vee \langle b, a\rangle = L$ for any a and b in L (there $\langle a, b\rangle$ is the ideal $\{x \in L\colon\ x \wedge a \leqq b\}$). The second condition should be compared with condition (b) of Theorem 2.4. In an effort to obtain an extension of condition (e) of Theorem 2.4 to relatively normal lattices we extend Mandelker's concept of relative annihilator.

For non-empty subsets A and B of lattice L $\langle A, B\rangle$ denotes

$$\{x \in L\colon\ x \wedge a \in B \quad \text{for all } a \in A\}.$$

$\langle a,b\rangle$ denotes $\langle\{a\},\{b\}\rangle$. As observed by Mandelker $\langle a,b\rangle$ is an ideal due to the distributivity of L. When A and B are ideals it is also clear that $\langle A,B\rangle$ is an ideal. Clearly $\langle (a], (b]\rangle = \langle a, b\rangle$. The following lemma summarizes some useful information,

LEMMA 3.6. *Let L be a lattice. Then the following hold*

(a) $\langle x \vee y,\ x\rangle = \langle y,x\rangle$ *for any* $x,y \in L$,

(b) $\langle (x],J\rangle = \bigvee_{y \in J}\langle x,y\rangle$, *the supremum of ideals* $\langle x,y\rangle$ *in the lattice of ideals of L, for any* $x \in L$ *and any ideal J in L,*

(c) $\{\langle x,a\rangle \vee \langle y,a\rangle\} \cap [a,b] = \{\langle x,a\rangle \cap [a,b] \vee \{\langle y,a\rangle \cap [a,b]\}$, *for any* $x, y \in [a,b], a < b$.

PROOF. (a) and (b) are easily proved.

Let z be a member of the left hand side of (c). Then

$$a \leqq z = c \vee d \leqq b$$

with $c \wedge x \leqq a$ and $d \wedge y \leqq a$. Then

$$(c \vee a) \wedge x = (c \wedge x) \vee (a \wedge x) = (c \wedge x) \vee a \leqq a \vee a \leqq a$$

and similarly $(d \vee a) \wedge y \leqq a$. Thus

$$c \vee a \in \langle x,a\rangle \cap [a,\ b] \text{ and } (d \vee a) \in \langle y,a\rangle \cap [a,b]$$

so $z = (c \vee a) \vee (d \vee a)$ is a member of the right hand side of (c). The reverse inequality is clear and (c) follows.

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THEOREM 3.7. *Let a, b and c be arbitary elements of a lattice L. Let A, B and C be arbitrary ideals in L. Then the following are equivalent*:

(a) *L is relatively normal*

(b) $\langle a, b\rangle \vee \langle b,a\rangle = L.$,

(c) $\langle c, a \vee b\rangle = \langle c, a\rangle \vee \langle c, b\rangle$,

(d) $\langle (c], A \vee B\rangle = \langle (c], A\rangle \vee \langle (c],B\rangle$,

(e) $\langle a \wedge b, c\rangle = \langle a, c\rangle \vee \langle b, c\rangle$,

(f) $\langle (a] \cap (b], C\rangle = \langle (a], C\rangle \vee \langle (b],C\rangle$.

PROOF. $(a) \Rightarrow$ (b). Let $z \in L$ be arbitrary. Consider the interval

$$I = [a \wedge b \wedge z, a \vee b \vee z].$$

Then $a \wedge (b \wedge z)$ is the smallest element of I. As I is normal by (a), Theorem 2.4(h) implies that there are $r,s \in I$ such that

$$a \wedge s = a \wedge b \wedge z = b \wedge z \wedge r$$

and $z = s \vee r$. $a \wedge s \leqq b$ so

$$s \in \langle a,b\rangle. \; b \wedge r = b \wedge z \wedge r = a \wedge b \wedge z \leqq a$$

so $r \in \langle b,a\rangle$. Hence $z = s \vee r \in \langle a,b\rangle \vee \langle b,a\rangle$ and (b) follows.

(b) $\Rightarrow$ (c). Let $z \in \langle c, a \vee b\rangle$. Then $z \wedge c \leqq a \vee b$. Also, as (b) holds, $z = x \vee y$ where $x \wedge a \leqq b$ and $y \wedge b \leqq a$. Then

$$x \wedge c = x \wedge z \wedge c \leqq x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b) \leqq b \vee (x \wedge b) = b.$$

Similarly $y \wedge c \leqq a$. Hence $z = x \vee y \in \langle c, b\rangle \vee \langle c, a\rangle$ and $\langle c, a \vee b\rangle \subseteq \langle c, b\rangle \vee \langle c, a\rangle$. Since the reverse inequality always hclds (c) is proved.,

(c) $\Rightarrow$ (d). This implication follows immediately from part (b) of Lemma 3.6.

$(d) \Rightarrow (c)$ trivially and (c) $\Rightarrow$ (b) follows immediately from part of (a) of Lemma 3.6 if we put $c = a \vee b$.

(b) $\Rightarrow$ (e). Let $z \in \langle a \wedge , c\rangle$. Then $z = x \vee y$ where $x \wedge a \leqq b$ and $y \wedge b \leqq a$. Also

$$x \wedge a = x \wedge a \wedge b \leqq z \wedge a \wedge b \leqq c.$$

Similarly $y \in \langle b, c\rangle$. It follows that $\langle a \wedge b, c\rangle \subseteq \langle a, c\rangle \vee \langle b, c\rangle$. The reverse inequality always holds so (e) is established.

(e) $\Rightarrow$ (f) is a trivial consequence of lemma 3.6.

(f) $\Rightarrow$ (e) is clear so we show that (e) $\Rightarrow$ (a). For $x \in [a, b]$, $a < b$, let

$$x^+ = \{y \in [a, b] : y \wedge x = a, \text{ the zero of } [a, b]\}.$$

Clearly $x^+ = \langle x, a\rangle \cap [a,b]$. (a) now follows (b), due to part (c) of Lemma 3.6.

The theorem is proved.

4. Homomorphic images

The first theorem in this section derives from a scrutiny of the proof of a result by Grätzer and Schmidt [4, Theorem 1].

For a filter F in a distributive lattice a congruence relation $\Psi(F)$ is defined by $x \equiv y\{\Psi(F)\}$ if there exist $t \in F$ such that $x \wedge t = y \wedge t$. The associated quotient lattice is denoted by $L/\Psi(F)$ and ψ denotes the canonical epimorphism of L onto the quotient lattice. For $x \in L$, $\psi(x) = \bar{x} =$ the congruence class of x modulo $\Psi(F)$. As is well-known the elements of F are all congruent under $\Psi(F)$ and the equivalence class of F is the largest element in $L/\Psi(F)$. This congruence has been studied in detail by Speed [12]. Though some of the following results can be proved with the aid of Speed's results we proceed by alternative methods.

Recall that a filter F is *prime* if $x \vee y \in F$ implies $x \in F$ or $y \in F$. A lattice L with 0 is *dense* if $(x]^* = (0]$ for each $x \neq 0$ in L.

We shall also make use of the fact that a prime ideal P is a minimal prime if and only if for each $x \in P$ there is $y \notin P$ such that $x \wedge y = 0$. This follows from [6, Lemma 3.1].

Theorem 4.1 (a). *If F is a filter in a normal lattice L then $L/\Psi(F)$ is a normal lattice.*

(b) *A lattice L with 0 is normal if and only if, for each prime filter F, $L/\Psi(F)$ is a dense lattice.*

Proof. (a) Let $\bar{Q}$ be a prime ideal in $L/\Psi(F)$ and let Q be its inverse image under ψ. Patently Q is a prime ideal. If $\bar{Q}$ is a minimal prime ideal then Q is a minimal prime. For suppose $q \in Q$ then $\bar{q} \in \bar{Q}$ so there exists $\bar{r} \notin \bar{Q}\,(r \in L)$ such that

$$\overline{q \wedge r} = \bar{q} \wedge \bar{r} = \bar{0}.$$

So for some $x \in F$,

$$q \wedge r \wedge x = 0 \wedge x = 0.$$

Now $x \notin Q$, otherwise $\bar{x} \in \bar{Q}$ and then $\bar{Q}$ would be an improper ideal. As Q is prime we have $r \wedge x \notin Q$ and $q \wedge (r \wedge x) = 0$ so Q is a minimal prime.

Now suppose L is normal, and let $\bar{Q}, \bar{R}$ be distinct minimal primes in $L/\Psi(F)$. Then their complete inverse images Q and R, respectively, are distinct minimal primes in L, so $Q \vee R = L$. Whence $\bar{Q} \vee \bar{R} = L/\Psi(F)$ so the quotient lattice is normal.

(b) Suppose F is a prime filter in a normal lattice L. To prove that $L/\Psi(F)$ is dense it suffices to show that the quotient lattice contains a unique minimal prime, for then $\{\bar{0}\}$ is a (minimal) prime as the intersection of the minimal primes in $\{\bar{0}\}$. Thus, assume $\bar{Q}$ and $\bar{R}$ are distinct minimal primes in $L/\Psi(F)$. By part (a) of this theorem, $\bar{Q} \vee \bar{R} = L/\Psi(F)$. Hence if $f \in F$ then $f = \bar{q} \vee \bar{r}$ where $\bar{q} \in \bar{Q}$, $\bar{r} \in \bar{R}$ and $q, r \in L$. Thus, there exists $g \in F$ such that

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$$f \wedge g = (q \vee r) \wedge g.$$

As F is a filter it follows that $q \vee r \in F$ and the primeness of F implies $q \in F$ or $r \in F$. But then either $\bar{Q}$ or $\bar{R}$ is improper. The contradiction shows the quotient lattice possesses a unique minimal prime ideal.

Now assume L is a lattice with 0 for which $L/\Psi(F)$ is dense for each prime filter F. Let P a prime ideal in L. Suppose $x, y \in L$ and that $x \wedge y \in 0(P)$. By definition of this ideal $x \wedge y \wedge z = 0$ for some $z \in L \backslash P$. Now $L \backslash P$ is a prime filter hence in $L/\Psi(L \backslash P)$, $\bar{x} \wedge \bar{y} \wedge \bar{z} = \bar{0}$. i.e. $\bar{x} \wedge \bar{y} = \bar{0}$ as $\bar{z}$ is the identity of the quotient lattice lattice as it is in $L \backslash P$. As the quotient is dense, $\bar{x} = \bar{0}$ or $\bar{y} = \bar{0}$. Thus either

$$x \wedge w = 0 \wedge w = 0 \text{ or } y \wedge w = 0$$

for some $w \in L \backslash P$ i.e. either x or y is in $0(P)$. Thus, $0(P)$ is prime so, by Theorem 2.4, L is normal. $\square$

From Theorem 3.7 and [8, Theorem 4] we have the following result.

Lemma 4.2. *A lattice L is relatively normal if and only if, for any prime filter F and any elements a and b in L, there exists an element x in F such that $a \wedge x$ and $b \wedge x$ are comparable.*

Theorem 4.3. (a) *If F is a filter in a relatively normal lattice then $L/\Psi(F)$ is relatively normal.*

(b) *A lattice L is relatively normal if and only if $L/\Psi(F)$ is a chain for each prime filter in L.*

Proof. (a) follows from Theorem 3.5.

(b) is an easy consequence of Lemma 4.2.

From Theorem 2.4 part (h), we immediately obtain

Proposition 4.4. *Let X be a T_1-space. $\mathscr{F}(X)$ is its lattice of closed sets. Then the following are equivalent*

(a) *X is normal*

(b) *$\mathscr{F}(X)$ is normal.*

Theorem 4.5. *A homomorphic image of normal lattice with 0 is not necessarily normal.*

Proof. Let X be a normal space with a subspace A which is not normal (e.g. X can be the Stone-Čech compatification of a non-normal completely regular space A). $\mathscr{F}(X)$ is normal and $\mathscr{F}(A)$ is not normal by Proposition 4.4. For $F \in \mathscr{F}(X)$, the map $F \leftrightarrow F \cap A$ is the required lattice epimorphism.

5. "Grätzer-Schmidt theorems"

Definitions 5.1. A lattice L with 0 is called quasi-complemented if for each x in L there exists x' in L such that $x \wedge x' = 0$ and $(x \vee x']^* = (0]$.

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5.2. A lattice L with 0 is sectionally quasi-complemented if each interval $[0,x]$, $0 < x$, is quasi-complemented.

5.3. A lattice L with 0 is called a generalized Stone lattice if $(x]^* \vee (x]^{**} = L$ for each x in L.

The terminology of 5.1 is due to Varlet [14] while that of 5.3 is due to Katriňák [5]. Quasi-complemented lattices have been studied by Varlet [14] and Speed [12]. They generalize pseudo complemented lattices (i.e. a lattice with 0 such that $(x]^*$ is a principal for each x).

Katriňák [5, Lemma 8, p. 134] proved the following result.

LEMMA 5.4. *A lattice L with 0 is a generalized Stone lattice if and only if each interval $[0,x], 0 < x \in L$, is a Stone Lattice.*

We remark that a Stone lattice can be considered as either a generalized Stone lattice with 1 or a pseudo-complemented lattice in which $x^* \vee x^{**} = 1$ for each x where $(x]^* = (x]$.

The following three results sum up the connections between lattices satisfying Definitions 5.1, 5.2, and 5.3 and normal lattices.

PROPOSITION 5.5 *Let L be a lattice with 0. Then*

(a) *L is quasi-complemented if and only if it is sectionally quasi-complemented and possesses an element d such that $(d]^* = (0]$,*

(b) *if L is a generalized Stone lattice then it is normal.*

PROOF. (b) was proved by Katriňák [5, Theorem 4].

(a) Suppose L is quasi-complemented. Then there is an element d such that

$$0 \wedge d = 0 \text{ and } (d]^* = (0 \vee d]^* = (0].$$

We now show that an arbitrary interval $[0,x]$ is quasi-complemented. Let $y \in [0,x]$ and choose $y' \in L$ such that $y \wedge y' = 0$ and $(y \vee y']^* = (0]$ in L. Put $z = x \wedge y'$. Then $z \wedge y = 0$ and $z \in [0,x]$. If $w \in [0,x]$ and $w \wedge (y \vee z) = 0$ then

$$w \wedge y = 0 = w \wedge z = w \wedge x \wedge y' = w \wedge y'$$

so $w \in (y \wedge y']^* = (0]$ i.e. $w = 0$. It follows that L is sectionally quasi-complemented.

Suppose L is sectionally quasi-complemented and that there is an element d in L with $(d]^* = (0]$. Let $x \in L$ and consider the interval $[0, x \vee d]$. In this interval there is an element x' with $x \wedge x' = 0$ and such that

$$\{y \in [0,x \vee d]\colon\ y \wedge (x \vee x') = 0\} = \{0\}.$$

For $y \in L$ such that $y \vee (x \wedge x') = 0$, $(y \wedge d) \wedge (x \vee x') = 0$ so $y \wedge d = 0$ as $y \wedge d \in [0,x \vee d]$. But then $y = 0$ so L is quasi-complemented.

THEOREM 5.6. *Let L be a lattice* 0. *Then sufficient conditions for L to be a generalized Stone lattice are*

(*a*) *L is quasi complemented, and*

(*b*) *L is normal.*

When L possesses an element d such that $(d]^* = (0]$, the conditions (a) *and* (b) *are also necessary.*

PROOF. Sufficiency of (a) and (b). For $x \in L$ choose x' such that

$$x \wedge x' = 0 \text{ and } (x \vee x']^* = (x]^* \cap (x'] = (0].$$

Then $(x]^{**} = (x']^*$. (This is easy to show and well-known from [14] and [10]. As L is normal Theorem 2.4 shows that $(x] \vee (x]^* = L$. Combining we get $(x]^* \vee (x]^{**} = L$.

The necessity of the conditions when L has an element d such that $(d]^* = (0]$ follows from Lemma 5.4 and Proposition 5.5.

THEOREM 5.7. *A lattice with O is a generalized Stone lattice if and only if it is both normal and sectionally quasi-complemented.*

PROOF. Apply Lemma 5.4, Proposition 5.5 and Theorems 5.6 and 3.3

Theorem 5.7 can be regarded as a minor improvement of Katriňák's result [5, Theorem 4] which in turn generalizes Varlet's ([14]) and Speed's ([11]) extensions of Grätzer and Schmidt's ([4]) characterization of Stone lattices.

Theorem 3.3 shows that nothing is gained in Theorem 5.7 by supposing L is sectionally normal rather than normal.

Because of the simple characterization of normalcy given in Theorem 2.4, Theorem 5.6 presents a very natural proof of these "Grätzer-Schmidt theorems".

Before leaving this section we would like to emphasize one more point.

THEOREM 5.8. *Let L be a relatively pseudo-complemented lattice. Then the following are equivalent*

(a) *L is a relative Stone lattice,*

(b) *$(a \wedge b) * c = (a * c) \vee (b * c)$ holds for all a, b* and *c* in *L*

(c) *$c*(a \vee b) = (c * a) \vee (c * b)$ holds for all a,b and c in L.*

PROOF. A relatively pseudo-complemented lattice is a lattice with 0 and 1 such that $\langle a,b \rangle$ is a principal ideal for each a and b. $\langle a,b \rangle = (a * b]$. The theorem follows from Theorem 3.7 if we apply Theorem 5.7 and the fact that each interval in L is pseudo-complemented (L is relative Stone lattice if each interval is a Stone lattice.)

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6. Complementedly normal lattices

DEFINITION 6.1. *A complementedly normal* lattice is a lattice L with 0 and 1 such that if $x \wedge y = 0$ then $x \leqq u$ and $y \leqq u'$ for some complemented element u in L.

In the above definition u' is the complement of u. $Z(L)$ will be used to denote the sublattice of complemented elements.

The above condition appears in [1, Proposition 2.4] but is not really studied there.

From Theorem 2.4 (part (h)) we see that a complementedly normal lattice is normal.

The following result has some interest though we omit the easy proof.

PROPOSITION 6.2. *A complementedly normal lattice is a Stone lattice if and only if for each x there is a smallest complemented element w such that $x = x \wedge w$.*

We now propose to generalize and extend Speed's results [1, Proposition 3.2 and Theorem 3.3.].

PROPOSITION 6.3. *L is a complemented normal lattice, $Z(L)$ is its Boolean algebra of complemented elements, $\Pi(L)$ and $\Pi(Z(L))$ are the sets of minimal prime ideals of L and $Z(L)$, respectively. Then the map $P \to L(P) = \{x \in L: x = x \wedge u$ for some $u \in P\}$ is a bijection of $\Pi(Z(L))$ onto $\Pi(L)$.*

PROOF. It is clear that $L(P)$ is an ideal in L and if it is prime it must be a minimal prime due to [6, Lemma 3.1]. Thus we show that $L(P)$ is prime. Suppose $x, y \in L$ are such that $x \wedge y \in L(P)$ and $y \notin L(P)$. Then $x \wedge y \wedge u = x \wedge y$ for some $u \in Z(L)$ so $x \wedge y \wedge u' = 0$. As L is complementedly normal $x = x \wedge v$ and

$$y \wedge u' \wedge v' = y \wedge u'$$

for some $v \in Z(L)$. Also we have $y \wedge u' \wedge v = 0$ so $y \wedge w = y$ and

$$u' \wedge v \wedge w' = u' \wedge v$$

for some $w \in Z(L)$. Since $y \notin L(P)$, $w \notin P$ so $w' \in P$ and then $u' \wedge v \in P$. But $u \in P$ so $u' \notin P$ hence $v \in P$ whence $x = x \wedge v \in L(P)$ and so $L(P)$ is prime.

The map $P \to L(P)$ is an injection since if $L(P) = L(L)$, $P, L \in \Pi(Z(L))$, then $P \subseteq L(P)$ implies that for arbitrary $p \in P$ there is an element $q \in L$, such that $p \leqq q$. That is $P \subseteq L$ whence $P = L$.

The map is surjection since if $H \in \Pi(L)$ then $H \cap Z(L) \in \Pi(Z(L))$ and it is clear that $L(H \cap Z(L)) \subseteq H$. Hence $L(H \cap Z(L)) = H$.

COROLLARY 6.4. *If P is a prime ideal in a complementedly normal lattice L then $P \cap Z(L)$ is the unique prime ideal in the sublattice $Z(L)$ which is contained in P.*

PROOF. This follows readily from the proposition if we recall that all prime ideals in a Boolean algebra are minimal primes.

The next result presumes a knowledge of the hull-kernel topology on the set of minimal ideals of a lattice. For details we refer to [6] and [10].

THEOREM 6.5. *For a complementedly normal lattice L the map $P \to L(P)$ of Proposition 6.3 is a continuous function from $\Pi(Z(L))$ onto $\Pi(L)$, with respect to the hull-kernel topologies, if and only if L is a Stone lattice.*

PROOF. If $P \to L(P)$ is continuous then $\Pi(L)$ is a continuous image of the compact space $\Pi(Z(L))$ (using a well-known result of Stone or [9, Proposition 3.2]) and so is compact. Then L is quasi-complemented by [9] and as L is normal it is a Stone lattice by Theorem 5.6.

Suppose L is a Stone lattice. Let $a \in L$. $h(a)$ is the hull of a in $\Pi(L)$. Then the inverse image of $h(a)$ under the map $P \to L(P)$ is

$$\{P \in \Pi(Z(L)) \colon \ a \in L(P)\} = \{P \in \Pi(Z(L)) \colon a^{**} \in P\}$$

as is readily verified, and this is closed in $\Pi(Z)L)\}$. As the map is a bijection it follows that it is continuous.

7. Spaces of ultra filters

The proof of the following lemma is identical with that presented for commutative rings in [7, Proposition 5, p. 157].

LEMMA 7.1. *Let L be a lattice with 0. Σ is a non-empty set of prime ideals L endowed with the hull-kernel topology and such that $\cap \{P \colon P \in \Sigma\} = (0]$. Then Σ is a Hasdorff space if and only if, for each $P \in \Sigma$, P is the unique ideal in Σ containing $0(P)$.*

For a prime filter F (in particular an ultrafilter) $\omega(F)$ denotes the filter $\{x \in L \colon x \vee y = 1 \text{ for some } y \notin F\}$ when L has a largest element 1. This is the dual of the ideal $0(P)$ in section 2.

A lattice with 0 and 1 is called *semi-complemented* if for each $0 < x < 1$ there is an element $0 < y$ such that $x \wedge y = 0$. It is well-known and easily proved that L is semi-complemented if and only if the intersection of all the ultrafilters is $\{1\}$. From this the dula of Lemma 7.1 gives

LEMMA 7.2 *Let L be a semi-complemented lattice with 0 and 1. Then the set of all ultrafilters in L, endowed with the hull-kernel toplogy, is a Hausdorff space if and only if for any ultrafilter F, the unique ultrafilter containing $\omega(F)$ is F.*

The space of ultrafilters is considered in [1], for example.

THEOREM 7.3. *Let L be a semi-complemented lattice with 0 and 1. Ω is the space of all ultrafilters in the hulllkenel topology. Then the following are equivalent*:

(a) *Ω is Hausdorff*

(b) *L is normal.*

REMARK. Ω is always compact so that Ω is Hausdorff iff it is a normal space, so Ω is normal iff L is normal!

PROOF. (a) $\Rightarrow$ (b). Suppose (a) holds and yet (b) does not. Then some maximal ideal P contains two distinct minimal prime ideals M_1 and M_2. Clearly, $L\backslash P$ is a minimal prime filter contained in the distinct ultrafilters $L\backslash M_1$ and $L\backslash M_2$. But $L\backslash P \subseteq L\backslash M_i$ implies

$$\omega(L\backslash M_i) \subseteq \omega(L\backslash P)$$

for $i = 1,2$. As $L\backslash P$ is a minimal prime filter $L\backslash P = \omega(L\backslash P)$, by [6, Lemma 3.1] (working in the dual of L). Then

$$\omega(L\backslash M_1) \subseteq M_1, M_2$$

which is a contradiction due to Lemma 7.2. Thus (b) holds.

(b) $\Rightarrow$ (a). Suppose (b) holds. If (a) does not hold then by Lemma 7.2 there are two distinct ultrafilters F_1,F_2 such that F_1,$F_2 \supseteq \omega(F_1)$. Choose $x \in F_1\backslash F_2$, then

$$\{z \in L : x \wedge y \leqq z \text{ for some } y \in F_2\}$$

is a filter containing both x and F_2 and hence must be L. Thus there is $y \in F_2\backslash F_1$ such that $x \wedge y = 0$. As L is normal, Theorem 2.4 implies $x \wedge x_1 = 0 = y \wedge y_1$ and $x_1 \vee y_1 = 1$ for suitable $x_1, y_1 \in L$. Then $x_1 \notin F_1$ whence $y_1 \in \omega(F_1) \subseteq F_2$, but this is a contradiction for it implies $0 = y \wedge y_1 \in F_2$. Thus (a) must hold.

We now consider a similar theorem for complementedly normal lattices.

LEMMA 7.4. *Let X be a compact Hausdorff totally disconnected space. Then the lattice of closed subsets of X is complementedly normal i.e. if A and B are distjoint closed sets then there is an open and closed set U such that $A \subseteq U$ and $B \subseteq X\backslash U$.*

PROOF. The proof proceeds in exactly the same way as a compact Hausdorff space is proved to be normal where any open sets are replaced by open and closed sets. We therefore omit details.

Recall that a lattice L with 0 is *disjunctive* if for any $a, b \in L$, $a < b$ implies there is an element $c \in L$ such that $a \wedge c = 0$ and $0 < c < b$. The next lemma is standard; a proof can be found in [1, Lemma 2].

LEMMA 7.5. *Let Ω be set of all ultrafilters in a disjunctive lattice L.*

Then, for any $a \in L$, $\{x : a \leqq x\} = \cap \{F \in h(a)\}$ *where* $h(a)$ *is the hull of* a *in* Ω *i.e.* $h(a) = \{F \in \Omega:\ a \in F\}$.

THEOREM 7.6. *Let* L *be a lattice with* 0 *and* Ω *be its space of ultrafilters with the hull-kernel topology. Then*

(a) *if* L *is complemently normal* Ω *is a compact Hasudorff totally disconnected space, and*

(b) *if* Ω *is compact Hausorff totally disconnected space and* L *is disjunctive then* L *is complementedly normal.*

PROOF. (a) let $F_1 \neq F_2$ be ultrafilters. As in the proof of (b) implies (a) in Theorem 7.3, we can find $x \in F_1 \backslash F_2$ and $y \in F_2 \backslash F_1$ such that $x \wedge y = 0$. As L is complementedly normal there is a complemented element $u \in L$ such that $x \leqq u$ and $y \leqq u'$. Then, $u \in F_1$ and $u' \in F_2$ i.e. $F_1 \in h(u)$ and $F_2 \in h(u')$. Now

$$h(u) \cup h(u') = h(u \vee u') = h(1) = \Omega$$

and $h(u) \cap h(u') = h(u \wedge u') = h(0) = \varnothing$. Thus, $h(u)$ and $h(u')$ form the required disconnection of Ω.

(b) Let $x, y \in L$ with $x \wedge u = 0$. Then

$$h(x) \cap h(y) = h(x \vee y) = h(0) = \varnothing,$$

and $h(x)$ and $h(y)$ are closed in Ω. As Ω is, by hypothesis, a compact Hausdorff totally disconnected space, Lemma 7.4 implies $h(x) \subseteq U$ and $h(y) \subseteq \Omega \backslash U$ for some open and closed subset U of Ω. As U is closed there must be a filter J in L such that $h(J) = U$. Similarly, $\Omega \backslash U = h(K)$ for some filter K. Then $U \cup (\Omega \backslash U) = \varnothing$ and $U \cap (\Omega \backslash U) = \varnothing$ yield $h(J \cap K) = \Omega$ and $h(J \vee K) = \varnothing$, respectively, where

$J \cap K =$ the set theoretic intersection of J and K

$= \{x \in L: x = a \vee b$ for some $a \in J$ and $b \in K\}$ and

$J \vee K =$ the supremum J and K in the lattice of filters

$$= \{x \in L: x = a \wedge b,\ a \in J, b \in K\}.$$

Thus, $J \vee K = \Omega$ and $J \cap K \subseteq \cap \{F : F \in \Omega\} = \{1\}$ since L is disjunctive and hence semi-complemented. Thus, J and K are complementary and it easily follows that

$$J = \{x \in L:\ u \leqq x\} \text{ and } K = \{x \in L:\ u' \leqq x\}$$

for some complemented element $u \in L$. Thus $h(x) \subseteq U = h(u)$ and $h(y) \subseteq \Omega \backslash U = h(u')$ so

$$h(x \wedge u) = h(x) \cap h(u) = h(x) \text{ and } h(y \wedge u') = h(y).$$

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As L is disjunctive, Lemma 7.5 applies and we conclude $x \wedge u = x$ and $y \wedge u' = y$ so L is complementedly normal.

REMARKS 7.7. Theorem 7.3 is both related to and inspired by a result of Banaschewski [1, Proposition 1]. When considering bounded lattices Theorem 7.3 is a stronger result than that of Banaschewski. It is is interesting to note that the methods of proof are entirely different.

7.8. Unlike Theorem 7.3, we have note seen a result in the literature related to Theorem 7.6. Something like the disjunctive property is needed in part (b) of that theorem. Indeed it is not hard to show to that the space of ultrafilters of a quasi-complemented lattice is a compact Hausdorff totally disconnected space.

8. Examples

(A) *Normal Lattices.*

Some examples of normal lattices are: (1) the lattice of all zero sets in a completely regular space, (2) the lattice of closed subsets of a locally compact Hausdorff space which are either compact or are the complements of a set with with compact closure, (3) *any* lattice of subsets of a compact Hausdorff space which is a base for the closed sets. Examples (1) and (2) are considered in [1, pp. 107, 108]. In [8, Theorem 6] Mandelker shows that example (1) is actually relatively normal. Example (3) is proved in [2, Theorem 2.2].

Using familiar results on F-spaces [3, Theorem 14.25, p. 208] it can be shown that the lattice of co-zero sets of a completely regular space is normal if and only if the space is an F-space. This supplements [8, Theorem 8].

(B) *Complementedly normal lattices.*

From [3, Theorem 16.17, p. 274] the lattice of zero sets of completely regular space X is complementedly normal if and only if βX is totally disconnected.

From [9, Proposition 2.3] a completely regular space is a U-space if and only if the lattice of co-zero sets is complementedly normal. For the case when the lattice of co-zero sets is a Stone lattice the reader should refer to [8, Theorem 9].

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