ON NORMAL DERIVATIONS OF HILBERT–SCHMIDT TYPE

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Let H denote a separable, infinite dimensional Hilbert space. Let $B(H)$, C_2 and C_1 denote the algebra of all bounded linear operators acting on H, the Hilbert–Schmidt class and the trace class in $B(H)$ respectively. It is well known that C_2 and C_1 each form a two-sided *-ideal in $B(H)$ and C_2 is itself a Hilbert space with the inner product

$$(X, Y) = \sum (Xe_i, Ye_i) = \operatorname{tr}(Y^*X) = \operatorname{tr}(XY^*),$$

where $\{e_i\}$ is any orthonormal basis of H and tr(.) is the natural trace on C_1. The Hilbert–Schmidt norm of $X \in C_2$ is given by $\|X\|_2 = (X, X)^{1/2}$.

For a normal operator $N \in B(H)$, we define the normal derivation δ_N on $B(H)$ as follows:

$$\delta_N(X) = NX - XN \quad \text{for} \quad X \in B(H).$$

In [**1**], Anderson proved that if $N \in B(H)$ is normal, S is an operator such that $NS = SN$, then

$$\|\delta_N(X) + S\| \geqslant \|S\| \quad \text{for} \quad X \in B(H),$$

where $\|.\|$ is the usual operator norm. Hence the range of δ_N is orthogonal to the null space of δ_N which coincides with the commutant of N. The orthogonality here is understood to be in the sense of Definition 1.2 in [**1**].

The purpose of this paper is to prove a similar orthogonality result for δ_N in the usual Hilbert space sense. The basic tools in the main theorems are to treat C_2 as a Hilbert space in its own right and to utilize a result of Weiss [**6**] which asserts that if N is a normal operator, $X \in C_2$ such that $NX - XN \in C_1$ then $\operatorname{tr}(NX - XN) = 0$. We also give an extension of the orthogonality result to certain subnormal operators.

THEOREM 1. *If N is a normal operator and $S \in C_2$ is an operator such that $NS = SN$, then $\|NX - XN + S\|_2^2 = \|NX - XN\|_2^2 + \|S\|_2^2$ for all $X \in B(H)$.*

Proof. With no loss of generality we may assume that $NX - XN + S \in C_2$; otherwise, both sides of the last equation in the theorem are infinite. Hence $NX - XN \in C_2$. Therefore

$$\|NX - XN + S\|_2^2 = \|NX - XN\|_2^2 + 2\operatorname{Re}(NX - XN, S) + \|S\|_2^2.$$

We claim that $(NX - XN, S) = 0$. Now $NS = SN$ implies that $N^*S = SN^*$ and so

$NS^* = S^*N$ by Fuglede's theorem [**3**]. Thus

$$\begin{aligned}(NX - XN, S) &= \operatorname{tr}((NX - XN)S^*)\\ &= \operatorname{tr}(NXS^* - XNS^*)\\ &= \operatorname{tr}(NXS^* - XS^*N)\\ &= 0 \text{ by Weiss's theorem} \quad (XS^* \in C_2).\end{aligned}$$

The claim is verified and the proof is complete.

We now give an example to show that $NS = SN$ is necessary for Theorem 1 to hold.

EXAMPLE. Let U be the unilateral shift, let $N = U + U^*$ and $S = 1 - UU^*$. Then N is self-adjoint, $S \in C_2$ and $NS \neq SN$. If $X = U$, then

$$\|NX - XN + S\|_2^2 = \|2(1 - UU^*)\|_2^2 = 4,$$

while

$$\|NX - XN\|_2^2 + \|S\|_2^2 = \|1 - UU^*\|_2^2 + \|1 - UU^*\|_2^2 = 2.$$

Using Berberian's trick we have the following result.

THEOREM 2. *If N and M are normal operators and $S \in C_2$ is an operator such that $NS = SM$, then for all $X \in B(H)$ we have*

$$\|NX - XM + S\|_2^2 = \|NX - XM\|_2^2 + \|S\|_2^2.$$

Proof. On $H \oplus H$, let $L = \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix}$. Let $T = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}$, and let $Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$. Then L is normal and $T \in C_2$. Since $NS = SM$, it follows that $LT = TL$. Now $LY - YL = \begin{bmatrix} 0 & NX - XM \\ 0 & 0 \end{bmatrix}$. By Theorem 1, we have

$$\|LY - YL + T\|_2^2 = \|LY - YL\|_2^2 + \|T\|_2^2.$$

Therefore

$$\|NX - XM + S\|_2^2 = \|NX - XM\|_2^2 + \|S\|_2^2$$

as required.

An operator A is called subnormal if A has a normal extension. Following the idea of the proof of Theorem 1 in [**4**] enables us to generalize Theorem 2 as follows.

THEOREM 3. *If A and B^* are subnormal operators and $S \in C_2$ is an operator such that $AS = SB$, then for all $X \in B(H)$ we have*

$$\|AX - XB + S\|_2^2 = \|AX - XB\|_2^2 + \|S\|_2^2.$$

Proof. By assumption there exists a Hilbert space H_1 and there exist normal operators N and M on $H \oplus H_1$ such that $N = \begin{bmatrix} A & C \\ 0 & A_1 \end{bmatrix}$ and $M = \begin{bmatrix} B & 0 \\ D & B_1 \end{bmatrix}$. Let $T = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$, and let $Y = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$. Then $T \in C_2$ and $NY - YM = \begin{bmatrix} AX - XB & 0 \\ 0 & 0 \end{bmatrix}$. Since

$AS = SB$, it follows that $NT = TM$. By Theorem 2 we have

$$\|NY - YM + T\|_2^2 = \|NY - YM\|_2^2 + \|T\|_2^2.$$

Therefore

$$\|AX - XB + S\|_2^2 = \|AX - XB\|_2^2 + \|S\|_2^2.$$

In the proof of Theorem 1, a crucial role is played by Weiss's result [**6**]. In order to generalize Theorem 1 to certain subnormal operators an extension of Weiss's result to subnormal operators is needed. Fortunately, the following lemma [**5**] is good enough for our purpose. For convenience, we provide a proof of this result.

LEMMA. *If A is a subnormal operator with $A^*A - AA^* \in C_1$, then for $X \in C_2$, $AX - XA \in C_1$ implies* $\operatorname{tr}(AX - XA) = 0$.

Proof. Let $N = \begin{bmatrix} A & R \\ 0 & A_1 \end{bmatrix}$ be a normal extension of A. Let $Y = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$. Then $Y \in C_2$ and $NY - YN = \begin{bmatrix} AX - XA & -XR \\ 0 & 0 \end{bmatrix}$. Since N is normal, it follows that $A^*A - AA^* = RR^* \in C_1$. Therefore $R \in C_2$ and so $XR \in C_1$. Hence $NY - YN \in C_1$. Now $\operatorname{tr}(AX - XA) = \operatorname{tr}(NY - YN) = 0$ by Weiss's result.

THEOREM 4. *If A is a cyclic subnormal operator and $S \in C_2$ is an operator such that $AS = SA$, then for all $X \in B(H)$ we have*

$$\|AX - XA + S\|_2^2 = \|AX - XA\|_2^2 + \|S\|_2^2.$$

Proof. Since A is a cyclic subnormal operator, it follows that $A^*A - AA^* \in C_1$ by a result of Berger and Shaw [**2**]. Since S commutes with A, it follows that S is subnormal by Yoshino's theorem [**7**]. But any Hilbert–Schmidt subnormal operator is normal. Hence S is normal. Now $AS = SA$ implies $AS^* = S^*A$ by Fuglede's theorem. To complete the proof it is now sufficient to show that $(AX - XA, S) = 0$. By the lemma we have

$$(AX - XA, S) = \operatorname{tr}(AXS^* - XAS^*) = \operatorname{tr}(AXS^* - XS^*A) = 0.$$

REMARK. One should notice that Theorem 3 does not involve symmetric hypotheses on A and B, but rather on A and B^*.

We close by asking the following question. Is it necessary to assume A cyclic in Theorem 4?

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