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HARMONIC MOMENTS OF INHOMOGENEOUS BRANCHING PROCESSES

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Abstract

We study the harmonic moments of Galton–Watson processes that are possibly inhomogeneous and have positive values. Good estimates of these are needed to compute unbiased estimators for noncanonical branching Markov processes, which occur, for instance, in the modelling of the polymerase chain reaction. By convexity, the ratio of the harmonic mean to the mean is at most 1. We prove that, for every square-integrable branching mechanism, this ratio lies between $1 - A/k$ and $1 - A'/k$ for every initial population of size $k > A$. The positive constants A and A' are such that $A \geq A'$, are explicit, and depend only on the generation-by-generation branching mechanisms. In particular, we do not use the distribution of the limit of the classical martingale associated with the Galton–Watson process. Thus, emphasis is put on nonasymptotic bounds and on the dependence of the harmonic mean upon the size of the initial population. In the Bernoulli case, which is relevant for the modelling of the polymerase chain reaction, we prove essentially optimal bounds that are valid for every initial population size $k \geq 1$. Finally, in the general case and for sufficiently large initial populations, similar techniques yield sharp estimates of the harmonic moments of higher degree.

Keywords: Branching process; harmonic moment; inhomogeneous Markov chain; polymerase chain reaction

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1. Introduction and results

We study the behaviour of the harmonic means, $1/\mathrm{E}(1/Z_n)$, of Galton–Watson processes, $(Z_n)_{n\geq 0}$, that are possibly inhomogeneous and have positive values. A motivation for this theoretical problem is the construction of unbiased estimators for samples of branching Markov processes when the state of an individual depends on the number of its siblings. An instance, outside the realm of pure probability, where this construction is needed arises in the modelling of the polymerase chain reaction by branching processes; see Sun (1995). In this specific case each individual has either 1 or 2 offspring, the state of the first descendant is identical to the state of its parent, and the state of the other descendant, if any, is a stochastic function of the state of its parent. We wish to estimate, for instance, the mutation rate of the reaction from a uniform sample of a given generation. Any unbiased estimator of the state of such a sample requires us to compute the harmonic-mean size of the corresponding generation. However, there exists no closed-form expression for these harmonic means, except for small initial populations and small numbers of generations. Since the mean sizes of the generations of a branching process are well known, the above problem is usually circumvented by assuming that the initial

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population is very large. Then an averaging effect occurs which results, roughly speaking, in the harmonic-mean size of a generation being close to its mean size.

In the context of the polymerase chain reaction, the author showed in previous papers (see Piau (2001), (2004)) that this approximation is accurate for surprisingly small initial populations, and provided sharp quantitative estimates of the discrepancy between the harmonic mean and the mean, for any initial population. These results also proved useful in establishing rigorous confidence intervals for the estimator of the mutation rate of the polymerase chain reaction; see Piau (2005). Our purpose in the present paper is to give the exact extent of this approximation phenomenon for general, possibly inhomogeneous, Galton–Watson processes with positive values. When the approximation phenomenon indeed occurs, we quantify it through nonasymptotic and essentially optimal bounds.

In what follows, $(Z_n)_{n\geq 0}$ denotes a positive, possibly inhomogeneous, Galton–Watson process. The distribution of this Markov process with values in $\{1, 2, \ldots\}$ is characterized by a sequence, $\Xi := (\xi_n)_{n\geq 1}$, of distributions on $\{1, 2, \ldots\}$, as follows. For every $n \geq 1$, conditionally on the past of the process, Z_n is the sum of Z_{n-1} random variables of law ξ_n which are independent of the past. Hence, ξ_n denotes the law of the offspring number in the nth generation.

Assume that each ξ_n is integrable and has mean $\mu_n \geq 1$. Then Z_n is integrable and, if E_k denotes the expectation when $Z_0 = k$, for any positive integer k, we have

$$\mathrm{E}_k(Z_n) = k M_n, \quad \text{with } M_n := \prod_{i=1}^{n} \mu_i.$$

Furthermore, by convexity, the sequence with general term $M_n \, \mathrm{E}_k(1/Z_n)$ is nondecreasing for $n \geq 0$. Thus, every term is at least $1/k$. Our aim is to provide explicit bounds for the harmonic moments, implying, in particular, that $M_n \, \mathrm{E}_k(1/Z_n)$ is close to $1/k$ when this is so. In other words, we wish to show that the sequence with general term $M_n \, \mathrm{E}_k(1/Z_n)$ is nearly constant. Indeed, for every fixed $n \geq 0$, as $k \to \infty$ the law of large numbers implies that $\mathrm{E}_k(1/Z_n)$ is equivalent to $1/k \, \mathrm{E}_1(Z_n)$, whose value is $1/\mathrm{E}_k(Z_n) = 1/kM_n$. Much more is true, as we show below. To ease the task of the reader, we first state the consequence of our general results, in the homogeneous case.

Theorem 1. *Assume that Ξ is constant and square integrable. Then $\xi_n = \xi$ and $M_n = \mu^n$, where ξ is square integrable and $\mu_n =: \mu \geq 1$ for every $n \geq 1$. Thus, there exists a positive constant, A, depending only on ξ, such that, for every integer $k > A$ and every $n \geq 0$,*

$$\frac{1}{k} \leq \mu^n \, \mathrm{E}_k\left(\frac{1}{Z_n}\right) \leq \frac{1}{k - A}.$$

Assume further that $\mu \neq 1$. There then exists a positive constant, A', depending only on ξ, such that $A' \leq A$ and, for every integer $k > A'$,

$$\lim_{n\to\infty} \mu^n \, \mathrm{E}_k\left(\frac{1}{Z_n}\right) \geq \frac{1}{k - A'}.$$

1.1. Harmonic moments

Theorem 1 is a consequence of a general quantitative result, stated as Theorem 2 below, which deals with inhomogeneous processes. To state and prove this result, we rely on some specific families of distributions, which we now define.

Definition 1. For every $m \geq 1$, the generating function, g_m, of the positive, integer-valued random variable L_m is such that, for any t in [0, 1],

$$\mathrm{E}(t^{L_m}) =: g_m(t) = \frac{t}{m - (m-1)t}.$$

For any positive real number c, the random variable $L_{c,m}$ is such that, for any t in [0, 1],

$$\mathrm{E}(t^{L_{c,m}}) =: g_{c,m}(t) = g_m(t^c)^{1/c} = \frac{t}{(m - (m-1)t^c)^{1/c}}.$$

Thus, $g_m = g_{1,m}$. For every c, $L_{c,m} \geq 1$ almost surely and $\mathrm{E}(L_{c,m}) = m$. If $m = 1$ then $L_m = L_{c,m} = 1$ almost surely. If $m > 1$ then the distribution of $L_m - 1$ is geometric and the distribution of $(L_{c,m} - 1)/c$ is negative binomial.

Definition 2. For any positive real number c, let $\mathcal{A}_c$ denote the set of distributions of integrable random variables $L \geq 1$ such that, for any t in [0, 1],

$$\mathrm{E}(t^L) \leq g_{c,m}(t), \qquad m := \mathrm{E}(L).$$

In addition, let $\mathcal{A}'_c$ denote the set of distributions of integrable random variables $L \geq 1$ such that, for any t in [0, 1],

$$\mathrm{E}(t^L) \geq g_{c,m}(t), \qquad m := \mathrm{E}(L).$$

Note that we compare the distribution of L to distributions of random variables not a-priori integer valued, but with the same mean. We are now able to state our main result.

Theorem 2. (i) *Let $n \geq 1$. Assume that there exists a c such that ξ_i belongs to $\mathcal{A}_c$ for every $i \leq n$. Then, for every $k > c$,*

$$M_n \,\mathrm{E}_k\left(\frac{1}{Z_n}\right) \leq \frac{1}{k-c}.$$

(ii) *Assume that $M_n \to \infty$ as $n \to \infty$ and that there exists a c such that ξ_i belongs to $\mathcal{A}'_c$ for every $i \geq 1$. Then, for every $k > c$,*

$$\lim_{n\to\infty} M_n \,\mathrm{E}_k\left(\frac{1}{Z_n}\right) \geq \frac{1}{k-c}.$$

Recall that, by convexity, the sequence $M_n \,\mathrm{E}_k(1/Z_n)$ is nondecreasing. The existence of the limit as $n \to \infty$ is thus a general fact. Assertion (ii) of Theorem 2 is false if M_n is allowed to stay bounded, or if the limit $n \to \infty$ is replaced with a finite n, since, for instance, the $n = 0$ value is $1/k$. On the other hand, in practical situations, the hypothesis that $M_n \to \infty$ is easy to check since it only involves the first moments of the generation-by-generation mechanisms.

The restriction to $k > c$ is important as well. As Proposition 1 shows, the behaviours of $\mathrm{E}_k(1/Z_n)$ and $1/M_n$ can be quite different if k is not sufficiently large. Proposition 1 deals with one generation of a branching process using random variables distributed as $L_{c,m}$, as $m \to \infty$, and Corollary 1 applies this result to the nth generation of a branching process using random variables distributed as $L_{c,m}$ for a given m, as $n \to \infty$.

Definition 3. Let Z denote a random variable and $\mathrm{P}_k^{c,m}$ a probability measure with respect to which Z is distributed like the sum of k independent, identically distributed copies of the random variable $L_{c,m}$.

Proposition 1. *For any $k \le c$, $m\,\mathrm{E}_k^{c,m}(1/Z) \to \infty$ as $m \to \infty$.*

Corollary 1. *Assume that c is an integer and that ξ_n is the distribution of $L_{c,m}$, for every n; hence, $M_n = Z_0 m^n$. Then the distribution of Z_n coincides with the distribution of the first generation of the branching process based on L_{c,m^n}. As a consequence, $m^n\,\mathrm{E}_k^{c,m}(1/Z_n) \to \infty$ as $n \to \infty$.*

Thus, when $k \le c$, $\mathrm{E}_k(1/Z_n)$ may decay on a different scale than $1/M_n$; see more on this in Section 8. However, Theorem 2 describes every square-integrable Galton–Watson process if k is sufficiently large, as the following theorem shows.

Theorem 3. *Any square-integrable distribution on $[1, \infty)$ belongs to $\mathcal{A}_c$ for sufficiently large c and to $\mathcal{A}'_c$ for sufficiently small c. Conversely, any distribution on $[1, \infty)$ which belongs to $\mathcal{A}_c$ is square integrable and has variance equal to at most $cm(m-1)$, where m denotes its mean. Likewise, the variance of any distribution on $[1, \infty)$ which belongs to $\mathcal{A}'_c$ is either finite and equal to at least $cm(m-1)$, or infinite.*

We shall make precise the optimal values of c for some common distributions.

Finally, Theorem 2 indeed describes the behaviour of $\mathrm{E}_k(1/Z_n)$ when k is sufficiently large, for any square-integrable branching process.

1.2. The Bernoulli case

We apply Theorem 2 to the Bernoulli case, when the number of offspring is always 1 or 2. This case is relevant in the context of the polymerase chain reaction. Our techniques yield accurate bounds of $\mathrm{E}_k(1/Z_n)$ for every positive k, even $k = 1$, for instance in the homogeneous case; see Theorem 5, below. We first state uniform bounds that are simple consequences of the results of Section 1.1. For every j, let δ_j denote the Dirac mass at j.

Theorem 4. *Let $n \ge 0$. Assume that the laws, ξ_i on $\{1, 2\}$, of the offspring numbers are given by $\xi_i = (1 - x_i)\delta_1 + x_i\delta_2$, with x_i in $[0, 1]$, for every $i \le n$. Then, for any $k \ge 2$,*

$$\frac{1}{k} \le M_n\,\mathrm{E}_k\left(\frac{1}{Z_n}\right) \le \frac{1}{k-1}.$$

In the homogeneous case, we can prove tighter bounds. We write E_k^x for E_k when $\xi_i = (1-x)\delta_1 + x\delta_2$ for every $i \ge 1$. For every x in $(0, 1)$, define

$$\alpha_1(x) := \frac{1-x}{1+x}, \qquad \alpha_2(x) := 1 - x.$$

Then $0 \le \alpha_1 \le \alpha_2 \le 1$ and α_2 and α_1 decrease from $\alpha_2(0) = \alpha_1(0) = 1$ to $\alpha_2(1) = \alpha_1(1) = 0$.

Theorem 5. (i) *For any $k \ge 1$ and $n \ge 0$,*

$$\frac{1}{k} \le (1+x)^n\,\mathrm{E}_k^x\left(\frac{1}{Z_n}\right) \le \frac{1}{k - \alpha_2(x)}.$$

(ii) *For any $k \ge 1$,*

$$\lim_{n\to\infty} (1+x)^n\,\mathrm{E}_k^x\left(\frac{1}{Z_n}\right) \ge \frac{1}{k - \alpha_1(x)}.$$

These estimates are precise enough to imply the following secondary result for $k = 1$.

Proposition 2. *There exists no uniform upper bound of* $(1+x)^n \operatorname{E}_1^x(1/Z_n)$ *for* $n \geq 0$ *and* x *in* $(0, 1)$, *since* $\lim_{x\to 0}\lim_{n\to\infty}(1+x)^n \operatorname{E}_1^x(1/Z_n)$ *is infinite. More precisely, for every* x *in* $(0, 1)$ *and every* $n \geq 1$,

$$\frac{c(x)}{x} < \lim_{n\to\infty}(1+x)^n \operatorname{E}_1^x\left(\frac{1}{Z_n}\right) < \frac{1}{x},$$

where $c(x) := 1 - x(1-x)/(1+3x)$ *is such that* $\frac{8}{9} \leq c(x) < 1$.

In Theorem 5, the value of $\alpha_1(x)$ stems from the general construction of Section 1.1, but the value of $\alpha_2(x)$ does not. In other words, a direct application of Section 1.1 to the Bernoulli case yields $\alpha(x)$ instead of $\alpha_2(x)$, with

$$\alpha(x) := -\frac{\log(1+x)}{\log(1-x)}.$$

For every x in $(0, 1)$, we have $\alpha_1(x) < \alpha_2(x) < \alpha(x)$.

Theorem 5 follows from the more general case below.

Theorem 6. *Recall that* ξ_i *denotes the law of the offspring number in the* i*th generation, and assume that* $\xi_i = (1-x_i)\delta_1 + x_i\delta_2$ *for every* i.

(i) *If* $x_i \geq x$ *for every* $i \leq n$ *then, for any* $k \geq 1$,

$$\frac{1}{k} \leq \operatorname{E}_k\left(\frac{1}{Z_n}\right)\prod_{i=1}^{n}(1+x_i) \leq \frac{1}{k-\alpha_2(x)}.$$

(ii) *If* $x_i \leq x$ *for every* i *and* $\sum_{i\geq 1} x_i$ *diverges then, for any* $k \geq 1$,

$$\lim_{n\to\infty}\operatorname{E}_k\left(\frac{1}{Z_n}\right)\prod_{i=1}^{n}(1+x_i) \geq \frac{1}{k-\alpha_1(x)}.$$

1.3. A discontinuity result

In the Bernoulli case, the functions $\alpha_2(x)$ and $\alpha_1(x)$ have a nonzero limit as $x \to 0^+$. Hence, the second part of Theorem 5 shows that the limit of the normalized harmonic moments does not always depend continuously on the parameters of the model. We show in this section that the phenomenon is general. For the sake of simplicity, we deal with the homogeneous case.

Let $\mathcal{M}$ denote a given subset of $(1, \infty)$ that has 1 as a limit point. Below, the limits as $\mu \to 1$ are implicitly restricted to values of μ in $\mathcal{M}$. For each μ in $\mathcal{M}$, let ξ^μ denote a distribution of mean μ. If $\xi_i = \xi^\mu$ for every $i \geq 1$, define a function, h_k, on $\mathcal{M}$ by

$$h_k(\mu) := \lim_{n\to\infty}\mu^n \operatorname{E}_k(1/Z_n).$$

Proposition 3. *Assume that for each* μ *in* $\mathcal{M}$ *there exist functions,* $a(\mu)$ *and* $a'(\mu)$, *such that* ξ^μ *belongs to both* $\mathcal{A}_{a(\mu)}$ *and* $\mathcal{A}'_{a'(\mu)}$. *Then*

$$\frac{1}{k-a'_*} \leq \liminf_{\mu\to 1} h_k(\mu) \leq \limsup_{\mu\to 1} h_k(\mu) \leq \frac{1}{k-a_*},$$

where $a_* := \limsup_{\mu\to 1} a(\mu)$ *and* $a'_* := \liminf_{\mu\to 1} a'(\mu)$. *Thus, if* a'_* *is positive then the function* h_k *is not continuous at* $\mu = 1^+$.

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Theorem 7. *In the homogeneous case, assume that each ξ^μ is the law of $1+X$, where the law of the random variable X is binomial, Poisson, or geometric. Then, for every $k \geq 1$, h_k is discontinuous at $\mu = 1$, since $h_k(1) = 1/k$ and*

$$\lim_{\mu \to 1} h_k(\mu) = \frac{1}{k-1}.$$

If the law of X is geometric, then $h_k(\mu) = 1/(k-1)$ for every $\mu > 1$ and $h_k(1) = 1/k$.

Now assume that each ξ^μ is the law of $L_{c,\mu}$, for a given positive integer c. Then $h_k(1) = 1/k$ for every k. Furthermore, for every $\mu > 1$, $h_k(\mu) = 1/(k-c)$ if $k > c$ and $h_k(\mu) = \infty$ if $k \leq c$.

1.4. Higher harmonic moments

We now state an extension of Theorem 2 to higher harmonic moments. Theorem 8 is simply a special case of Proposition 11, below.

Theorem 8. (i) *Let $n \geq 1$. Assume that there exists a c such that ξ_i belongs to $\mathcal{A}_c$ for every $i \leq n$. Then, for every positive integer r and every integer $k > rc$,*

$$M_n^r \,\mathrm{E}_k\left(\frac{1}{Z_n^r}\right) \leq \frac{1}{(k-c)(k-2c)\cdots(k-rc)}.$$

(ii) *Assume that $M_n \to \infty$ as $n \to \infty$ and that there exists a c such that ξ_i belongs to $\mathcal{A}'_c$ for every i. Then, for every positive integer r and every integer $k > rc$,*

$$\lim_{n \to \infty} M_n^r \,\mathrm{E}_k\left(\frac{1}{Z_n^r}\right) \geq \frac{1}{(k-c)(k-2c)\cdots(k-rc)}.$$

Corollary 2. *Let $n \geq 1$. Assume that there exists a c such that ξ_i belongs to $\mathcal{A}_c$ for every $i \leq n$, and write $\sigma_k^2(1/Z_n)$ for the variance of $1/Z_n$ when $Z_0 = k$. Then, for every integer $k > 2c$,*

$$M_n^2 \sigma_k^2\left(\frac{1}{Z_n}\right) \leq \frac{3c}{k(k-c)(k-2c)}.$$

If, additionally, there exists a c' such that ξ_i belongs to $\mathcal{A}'_{c'}$ for every $i \leq n$, then the sequence $k^3 M_n^2 \sigma_k^2(1/Z_n)$ is bounded above and below by finite positive constants, independently of n and k, for sufficiently large values of k.

1.5. Related studies

As mentioned above, Piau (2001), (2004) used preliminary versions of our results, especially in the Bernoulli case, which is relevant to the study of the polymerase chain reaction. In this specific case, we are now able to deal directly with every initial population size, even $k = 1$.

Ney and Vidyashankar (2003) gave asymptotics of the harmonic moments of every integrable homogeneous branching process starting from $k = 1$ particle. Furthermore, when $L \log L$ is integrable, their results specialize as follows (see also Bingham (1988) for some classical facts that are recalled below).

Let $p_1 := \mathrm{P}(L = 1)$, let $\mu := \mathrm{E}(L)$, and let γ denote the Karlin–McGregor exponent of the distribution of L, defined by the equality $p_1 \mu^\gamma = 1$ (γ is also called the Schröder constant). Let W denote the almost-sure limit of the nonnegative martingale Z_n/μ^n. The Poincaré function is the Laplace transform, $P(s) := \mathrm{E}_1(\exp(-sW))$, of the distribution of W when $k = 1$, and solves Poincaré's functional equation:

$$P(\mu s) = f(P(s)).$$

Three cases may arise. First, if $r > \gamma$ then $\mathrm{E}_1(1/Z_n^r)/p_1^n$ converges to a finite, positive limit whose expression is an integral which involves the Schröder function S, defined for any t in $[0, 1)$ by

$$S(t) := \lim_{n\to\infty} \mathrm{E}_1(t^{Z_n})/p_1^n.$$

Up to a multiplicative constant, S is the unique finite solution in $[0, 1)$ to Schröder's functional equation:

$$S(f(t)) = p_1 S(t).$$

Second, when $r = \gamma$, $\mathrm{E}_1(1/Z_n^\gamma)/np_1^n$ converges to a finite, positive limit whose expression involves Poincaré and Schröder functions. Third, when $r < \gamma$, $\mu^{nr}\,\mathrm{E}_1(1/Z_n^r)$ converges to a finite, positive limit. Ney and Vidyashankar (2003) provided an expression for the limit in terms of an integral of the Poincaré function. It can readily be checked that this limit is in fact $\mathrm{E}_1(1/W^r)$, and that the limit is also an upper bound.

When $L \log L$ is not integrable the results are similar, but the normalizations $np_1^n = n/\mu^\gamma$ (for $r = \gamma$) and $1/\mu^{nr}$ (for $r < \gamma$) must be replaced by similar expressions which involve the Seneta–Heyde constants.

Restricting ourselves henceforth to the $L \log L$ case, we recall that the distribution of W has a density, w, on the nonnegative real line such that $w(x)/x^{\gamma-1}$ is bounded to lie between two finite, positive constants as $x \to 0$; see Dubuc (1971).

The comparison of our results with those recalled above is based on two elementary lemmas.

Lemma 1. *For any distribution, ξ, in $\mathcal{A}_c$, we have $\gamma(\xi) \ge 1/c$. For any distribution, ξ, in $\mathcal{A}'_c$, we have $\gamma(\xi) \le 1/c$.*

In other words (see Definition 4, below),

$$A'(\xi) \le 1/\gamma(\xi) \le A(\xi).$$

Lemma 2. *For any branching process, any $k \ge 1$, any $n \ge 0$, and any positive real number r, we have*

$$k^r\,\mathrm{E}_k(1/Z_n^r) \le \mathrm{E}_1(1/Z_n^{r/k})^k.$$

Corollary 3 is not stated as such in the papers mentioned above, but it follows from results that we have recalled.

Corollary 3. *For any homogeneous branching process of Schröder exponent γ and any $k > r/\gamma$, the sequence $\mu^{nr}\,\mathrm{E}_k(1/Z_n^r)$ is bounded, as n varies, by the finite constant $\mathrm{E}_k(1/W^r)$.*

An interesting feature of Corollary 3 is that it deals with the entire regime wherein such a constraint on $\mu^{nr}\,\mathrm{E}_k(1/Z_n^r)$ may hold, that is, with every initial population size $k > r/\gamma(\xi)$. In other words, when $k \le r/\gamma(\xi)$, $\mu^{nr}\,\mathrm{E}_k(1/Z_n^r)$ is not bounded. Our upper bounds are restricted to higher values of k, namely to the regime in which $k > rA(\xi)$.

One might think of recovering the dependence with respect to k from the results of Ney and Vidyashankar (2003) even when $k \ge 2$, starting from the inequality

$$\mathrm{E}_k\left(\frac{1}{W^r}\right) \le \frac{\mathrm{E}_1(1/W^{r/k})^k}{k^r}.$$

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However, the bounds obtained cannot be optimal for $k \geq 2$, since

$$\mathrm{E}_k^c\left(\frac{1}{W}\right) = \frac{1}{k-c} < \frac{\mathrm{E}_1^c(1/W^{1/k})^k}{k}.$$

Furthermore, as stressed by Bingham (1988), the law of W and, hence, the value of $\mathrm{E}(1/W^r)$ may be explicitly computed only in very specific cases. In contrast with every other paper the author is aware of, the bounds provided here are explicit. The assumptions involve only elementary, step-by-step characteristics of the branching process, namely the distributions of the number of descendants at each generation. Also, we allow for inhomogeneous processes, as long as the reproduction laws all belong to a given space (either $\mathcal{A}_c$ or $\mathcal{A}'_c$), and we make explicit the dependence of the bounds on the initial population.

The introduction of the family of distributions described by $g_{c,m}$, for integer values of c, is hardly new; see Harris (1948), for instance. A key point is that we use them for noninteger values of c and as a reference scale for any square-integrable distribution. For instance, the $k = 1$ Bernoulli distribution requires us to make use of values of c in $(0, 1)$. Although these distributions do not correspond to a branching process for noninteger values of c, they still satisfy a semigroup property, and this property is sufficient for our purposes. Finally, our methods do not determine the behaviour of the harmonic moments of homogeneous processes whose reproduction laws are not square integrable.

1.6. Plan

The remainder of the paper is organized as follows. In Section 2, we reduce the case of branching processes in $\mathcal{A}_c$ and $\mathcal{A}'_c$ to the case of well-chosen distributions $g_{c,m}$, which we solve in Section 3. In Section 4, we show that the $\mathcal{A}_c$ and $\mathcal{A}'_c$ cases imply the result for every square-integrable branching process. In Section 5, we deal with harmonic moments of higher degree. In Section 6, we thoroughly study the Bernoulli case, that is, the case when the offspring number is 1 or 2, sharpening our previous results on this subject. We provide an algorithm to compute the asymptotic harmonic moments to any accuracy, and we present some simulations and conjectures about this specific case. Section 7 is a remark about size-biased offsprings. Finally, in Section 8, we briefly explain how to deal with cases in which the asymptotic behaviours of the harmonic mean and the mean do not coincide.

2. From $\mathcal{A}_c$ and $\mathcal{A}'_c$ to $g_{c,m}$

We show that every branching process whose branching mechanism uses only laws in $\mathcal{A}_c$ can be reduced to the case of $L_{c,m}$ for a suitable m, and we solve this case. Similar results hold regarding the comparison with $\mathcal{A}'_c$.

2.1. Results

Lemma 3 describes the semigroup structure of each family $(g_{c,m})_m$. This is the starting point of our computations. Corollary 5 is a special case of Corollary 4 and Corollary 4 is a consequence of Lemma 3. Corollary 4 uses Definition 3.

Lemma 3. *For any positive real number c and any $m \geq 1$ and $m' \geq 1$, we have $g_{c,m} \circ g_{c,m'} = g_{c,m''}$, with $m'' := mm'$.*

Corollary 4. *Let φ denote a nonnegative, completely monotone function. For every branching process in $\mathcal{A}_c$, every $k \geq 1$, and every $n \geq 0$, we have*

$$\mathrm{E}_k(\varphi(Z_n)) \leq \mathrm{E}_k^{c,m}(\varphi(Z)), \quad \textit{where } m := M_n.$$

For every branching process in $\mathcal{A}'_c$, every $k \geq 1$, and every $n \geq 0$, we have

$$\mathrm{E}_k(\varphi(Z_n)) \geq \mathrm{E}_k^{c,m}(\varphi(Z)), \quad \textit{where } m := M_n.$$

Recall that φ is completely monotone if and only if its derivatives (the orders of which are indicated by superscripts in parentheses) are such that $(-1)^i \varphi^{(i)}$ is nonnegative for every positive integer i. Nonnegative, completely monotone functions are Laplace transforms of nonnegative measures on $[0, \infty)$; see Chapter IV of Widder (1941).

Corollary 5. *For every branching process in $\mathcal{A}_c$ and every positive real number r, we have*

$$\mathrm{E}_k(1/Z_n^r) \leq \mathrm{E}_k^{c,m}(1/Z^r), \quad \textit{where } m := M_n.$$

For every branching process in $\mathcal{A}'_c$ and every positive real number r, we have

$$\mathrm{E}_k(1/Z_n^r) \geq \mathrm{E}_k^{c,m}(1/Z^r), \quad \textit{where } m := M_n.$$

2.2. Proofs

Proof of Lemma 3. Since each $g_{c,m}$ is the conjugate of g_m by the bijection $t \mapsto t^c$, the case $c = 1$ implies the general case. When $c = 1$, $1/(1 - g_m(t))$ is an affine function of $1/(1 - t)$. By composition, $g_m \circ g_{m'}$ is also an affine function of $1/(1 - t)$ and it only remains to compute its coefficients to prove the semigroup property.

Proof of Corollary 4. The representation of completely monotone functions which we recalled after the statement of the corollary shows that

$$\varphi(z) = \int_0^1 t^z \,\mathrm{d}\pi(t),$$

for a given measure, π, on $[0, 1]$. Thus, $\mathrm{E}_k(\varphi(Z_n))$ is a positive, linear functional of the generating function, $\mathrm{E}_k(t^{Z_n})$, of Z_n. The function $\mathrm{E}_k(t^{Z_n})$ is the kth power of the composition from $i = 1$ to $i = n$ of the generating functions of the ξ_i. When the branching process belongs to $\mathcal{A}_c$, the generating function of ξ_i is bounded above by g_{c,μ_i} and, thus, the composition is bounded above by the composition of the functions g_{c,μ_i}, $1 \leq i \leq n$, which equals $g_{c,m}$. Finally,

$$\mathrm{E}_k(\varphi(Z_n)) \leq \int_0^1 g_{c,m}(t)^k \,\mathrm{d}\pi(t) = \mathrm{E}_k^{c,m}(\varphi(Z)).$$

The proof of the result for branching processes in $\mathcal{A}'_c$ is similar.

Proof of Corollary 5. For every positive real number r, $\varphi(z) := 1/z^r$ is completely monotone. To see this, choose $\mathrm{d}\pi(t) = (\log 1/t)^{r-1}\,\mathrm{d}t/\Gamma(r)t$ in the representation of φ which we used to prove Corollary 4.

3. The case $g_{c,m}$

Our task in this section is to evaluate the moments of $1/Z$ under the measure $\mathrm{P}_k^{c,m}$. The cases $k > c$ and $k \leq c$ yield different asymptotic behaviours for the first moment of $1/Z$. We begin with the direct way to deal with $\mathrm{P}_k^{c,m}$ when k is sufficiently large, namely the computation of factorial moments of Z instead of the usual moments; see Proposition 4. Starting with Lemma 4, which gives a representation formula valid for every k, we study in depth the first harmonic moment, both in the small-k and large-k regimes. Corollary 5 and Lemma 5 below then imply the results of Theorem 2. Lemma 6 deals with the case $k = c$. Lemma 7 provides an alternative formulation of the integral of Lemma 4, a formulation used in Lemma 8 to treat the case $k < c$. Proposition 1, which applies when $k \le c$, is then an easy consequence.

3.1. Results

We begin with exact formulae. Theorem 8 is a consequence of Corollary 6, below.

Proposition 4. (i) *For every nonnegative integer r,*

$$\mathrm{E}_k^{c,m}(Z(Z+c)\cdots(Z+rc)) = m^{r+1}k(k+c)\cdots(k+rc).$$

(ii) *For every real number r such that $k > rc$,*

$$m^r\,\mathrm{E}_k^{c,m}\left(\frac{\Gamma(Z/c-r)}{\Gamma(Z/c)}\right) = \frac{\Gamma(k/c-r)}{\Gamma(k/c)}.$$

(iii) *In particular, for every nonnegative integer r such that $k > rc$,*

$$m^r\,\mathrm{E}_k^{c,m}\left(\frac{1}{(Z-c)\cdots(Z-rc)}\right) = \frac{1}{(k-c)\cdots(k-rc)}.$$

Corollary 6. *For every nonnegative integer r such that $k > rc$,*

$$\frac{1}{k^r} \le m^r\,\mathrm{E}_k^{c,m}\left(\frac{1}{Z^r}\right) \le \frac{1}{(k-c)\cdots(k-rc)}.$$

In particular, for every $k > c$,

$$\frac{1}{k} \le m\,\mathrm{E}_k^{c,m}\left(\frac{1}{Z}\right) < m\,\mathrm{E}_k^{c,m}\left(\frac{1}{Z-c}\right) = \frac{1}{k-c}.$$

The following is a slight generalization of the assertion for $r = 1$ in Corollary 6.

Proposition 5. *For every $c > 0$, every $m \ge 1$, every $u \ge 0$, and every positive integer $k > u$,*

$$\frac{1}{k} \le m\,\mathrm{E}_k^{c,m}\left(\frac{1}{Z-u}\right) \le \frac{1}{k-\sup\{c,u\}}.$$

Proposition 4 and Corollary 6 are the results that we use to treat the case $k > c$ in the rest of the paper. We now turn to the evaluation of the exact harmonic moment of Z with respect to $\mathrm{P}_k^{c,m}$. The results below are mostly used to deal with the case $k \le c$.

Lemma 4. *For every positive integer k, every positive real number c, and every $m > 1$,*

$$\mathrm{E}_k^{c,m}\left(\frac{1}{Z}\right) = \frac{G(k/c,m)}{c}, \qquad G(u,m) := \int_0^1 \frac{t^{u-1}\,\mathrm{d}t}{1+(m-1)t}.$$

Alternatively,

$$G(u,m) = \frac{B_{u,1-u}(1-1/m)}{(m-1)^u},$$

where $B_{u,v}$ denotes the incomplete beta function with parameters u and v, that is, for every x in $[0, 1)$,

$$B_{u,v}(x) := \int_0^x t^{u-1}(1-t)^{v-1}\,\mathrm{d}t.$$

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Lemma 5. *Assume that $u > 1$. Then*

$$mG(u, m) \le \frac{1}{u-1}.$$

The order of this upper bound is exact when m is large, since the function $(m-1)G(u, m)$ increases as m increases and converges to $1/(u-1)$ as $m \to \infty$.

Lemma 6. $G(1, m) = (\log m)/(m-1)$.

Lemma 7. *For any u, $(m-1)^u G(u, m)$ is an increasing function of $m \ge 1$.*

Lemma 8. *Assume that $u < 1$. As $m \to \infty$, $(m-1)^u G(u, m)$ converges to $c_u := \pi/\sin(\pi u)$. Thus, on the one hand, for any $m > 1$, $(m-1)^u G(u, m) \le c_u$. On the other hand, for any $m \ge 2$, $(m-1)^u G(u, m) \ge 1/2u$. For every $u < 1$, bounds of c_u are $c_u \ge \pi$ and*

$$\frac{1}{2u(1-u)} \le c_u \le \frac{1}{u(1-u)}.$$

Corollary 7. *Let $k < c$ and $\ell(k, c) := c/k(c-k)$. For every m,*

$$(m-1)^{k/c}\, \mathrm{E}_k^{c,m}(1/Z) \le \ell(k, c).$$

The order of this upper bound is exact, since

$$\lim_{m\to\infty} (m-1)^{k/c}\, \mathrm{E}_k^{c,m}(1/Z) \ge \tfrac{1}{2}\ell(k, c).$$

3.2. Proofs

Lemmas 6, 7, and 8 follow from the definitions.

Proof of Proposition 4. (i) For any x with $|x| < 1$ and any positive real number y,

$$\frac{1}{(1-x)^y} = \sum_{r\ge 0} x^r \frac{\Gamma(y+r)}{\Gamma(y)\Gamma(r+1)}.$$

Setting $y = Z/c$ and integrating yields

$$\mathrm{E}_k^{c,m}\left(\frac{1}{(1-x)^{Z/c}}\right) = \sum_{r\ge 0} \mathrm{E}_k^{c,m}\left(\frac{\Gamma(r+Z/c)}{\Gamma(Z/c)}\right)\frac{x^r}{\Gamma(r+1)}.$$

However, we also have

$$\mathrm{E}_k^{c,m}\left(\frac{1}{(1-x)^{Z/c}}\right) = g_{c,m}\left(\frac{1}{(1-x)^{1/c}}\right)^k = \frac{1}{(1-mx)^{k/c}}.$$

Using the expansion of $1/(1-mx)^{k/c}$ given above and equating the coefficients of the two series yields the result for any nonnegative integer r.

(ii) For any positive real numbers y and r with $y > r$,

$$\frac{\Gamma(r)\Gamma(y-r)}{\Gamma(y)} = \int_0^1 t^{y-r-1}(1-t)^{r-1}\,\mathrm{d}t.$$

Setting $y = Z/c$ and performing the integration yields

$$\Gamma(r)\,\mathrm{E}_k^{c,m}\left(\frac{\Gamma(Z/c - r)}{\Gamma(Z/c)}\right) = \int_0^1 g_{c,m}(t^{1/c})^k \frac{(1-t)^{r-1}}{t^{r+1}}\,\mathrm{d}t.$$

The change of variable $s := g_{c,m}(t^{1/c})^c = g_m(t)$ yields

$$\Gamma(r)\,\mathrm{E}_k^{c,m}\left(\frac{\Gamma(Z/c - r)}{\Gamma(Z/c)}\right) = \int_0^1 s^{k/c-r-1} \frac{(1-s)^{r-1}}{m^r}\,\mathrm{d}s,$$

which is the desired formula.

Proof of Lemma 4. Write $\mathrm{E}_k^{c,m}(1/Z)$ as the integral of $g_{c,m}(t)^k/t$ over $(0, 1)$. Use the change of variable $t' := g_{c,m}(t)^c$. This yields the first expression for G in the lemma. To obtain the expression for G in terms of the incomplete beta function, use the change of variable $t' := (m-1)t/(1+(m-1)t)$ in the first expression for G.

Proof of Lemma 5. In the first expression for G in Lemma 4, use the fact that $1 + (m-1)t$ lies between mt and m. Thus, $G(u, m)$ lies between the integral of t^{u-2}/m and the integral of t^{u-1}/m, that is, between $1/(u-1)m$ and $1/um$.

Proof of Corollary 7. Lemma 4 and Lemma 8 imply that the left-hand side of the inequality is indeed bounded by c_u/c, where $u := k/c$. Therefore, we can use the bound of c_u by $1/u(1-u)$ in Lemma 8. This yields the bound for every finite value of m. The limit as $m \to \infty$ is $c_u/c \geq 1/2uc(1-u) = \ell(k, c)/2$.

4. From $\mathcal{A}_c$ and $\mathcal{A}'_c$ to the general case

In this section, we show that every square-integrable branching process belongs either to the set $\mathcal{A}_c$ or to the set $\mathcal{A}'_c$, for a suitable value of c; we prove Theorem 3; and we describe the best possible values for the constants, c, of Theorem 2 in some specific examples.

4.1. Comparisons

Our next proposition is related to Theorem 3 and motivates Definition 4, below.

Proposition 6. *If $c_1 \leq c_2$ and $m \geq 1$ then $g_{c_1,m} \leq g_{c_2,m}$. If $c_1 < c_2$ and $m > 1$ then the distribution of $L_{c_2,m}$ belongs to $\mathcal{A}_{c_2}$ but not to $\mathcal{A}_{c_1}$ and the distribution of $L_{c_1,m}$ belongs to $\mathcal{A}'_{c_1}$ but not to $\mathcal{A}'_{c_2}$. Thus, $(\mathcal{A}_c)_c$ is a strictly increasing sequence and $(\mathcal{A}'_c)_c$ is a strictly decreasing sequence.*

Definition 4. For any square-integrable distribution ξ on $[1, \infty)$, let

$$A(\xi) := \inf\{c > 0 \colon \xi \in \mathcal{A}_c\}, \qquad A'(\xi) := \sup\{c > 0 \colon \xi \in \mathcal{A}'_c\}.$$

4.2. Examples

We now study some specific transformations and examples. Proposition 7 follows from the definitions.

Proposition 7. *For every ξ, we have $A'(\xi) \leq A(\xi)$. The inequality is strict except in two cases: when $A(\xi) = A'(\xi) = 0$, in which case ξ is a Dirac measure at $m \geq 1$, and when $A(\xi) = A'(\xi) = c$ is positive, in which case ξ is the distribution of a random variable $L_{c,m}$.*

Proposition 8. (i) *If the laws of the independent random variables* $1 + X$ *and* $1 + X'$ *belong to* $\mathcal{A}'_c$, *then the law of* $1 + X + X'$ *also belongs to* $\mathcal{A}'_c$. *This statement is false if* $\mathcal{A}'_c$ *is replaced by* $\mathcal{A}_c$.

(ii) *If the law of* $1 + X$ *belongs to* $\mathcal{A}_c$ *and* b *is positive, then the law of* $1 + bX$ *belongs to* $\mathcal{A}_{cb}$. *A similar statement holds if* $\mathcal{A}_c$ *and* $\mathcal{A}_{cb}$ *are respectively replaced by* $\mathcal{A}'_c$ *and* $\mathcal{A}'_{cb}$.

(iii) *If the law of* L *belongs to* $\mathcal{A}_c$ *and* L' *stochastically dominates* L, *then the law of* L' *also belongs to* $\mathcal{A}_c$. *For instance, if* b *is nonnegative then the law of* $L + b$ *belongs to* $\mathcal{A}_c$. *A similar statement holds if* $\mathcal{A}_c$ *is replaced by* $\mathcal{A}'_c$.

If ξ is a Dirac measure then $A'(\xi) = A(\xi) = 0$. Other common cases are as follows.

Proposition 9. (i) *If* ξ *is uniform on* $\{1, \dots, n\}$ *then* $A(\xi) < 1$; *more precisely,* $2n^{A(\xi)} = n+1$.

(ii) *If* ξ *is uniform on* $\{1, n\}$ *then* $A'(\xi) = (n - 1)/(n + 1)$ *and* $2^{A(\xi)+1} = n + 1$.

(iii) *If* ξ *is the law of* $1 + X$, *where* X *is* binomial(n, x)*-distributed, then* $A(\xi) < 1$; *more precisely,* $(1 + xn)(1 - x)^{nA(\xi)} = 1$.

(iv) *If* ξ *is the law of* $1 + X$, *where* X *is Poisson-distributed with mean* x, *then* $A(\xi) < 1$; *more precisely,* $\mathrm{e}^{xA(\xi)} = 1 + x$.

In the notation of Section 1.5, cases (i) to (iv) of Proposition 9 are such that $A(\xi) = 1/\gamma(\xi) > A'(\xi)$. To check that the three values $A(\xi)$, $1/\gamma(\xi)$, and $A'(\xi)$ can indeed be different, assume that $\xi := (1 - p)\delta_1 + (\delta_2 + \delta_3)p/2$ with $p \in (0, 1)$. Then $\gamma(\xi) := -\log(1 - p)/\log(1 + 3p)$. For $p = \frac{1}{5}$, we can check that the function $t \mapsto \mathrm{E}(t^L)/g_{1/\gamma,m}(t)$ has positive derivative at $t = 0$ and $t = 1$. Thus, some values of this function are greater than 1 and some are smaller than 1. This implies that $A(\xi) > 1/\gamma(\xi) > A'(\xi)$.

4.3. The Bernoulli case

Definition 5. If $\xi = (1 - x)\delta_1 + x\delta_2$ then we write $\alpha(x)$ for $A(\xi)$ and $\alpha_1(x)$ for $A'(\xi)$.

Proposition 10. *For any* x *in* $(0, 1)$, *we have* $\alpha_1(x) < \alpha(x) < 1$, *since*

$$\alpha_1(x) = \frac{1 - x}{1 + x}, \qquad (1 - x)^{\alpha(x)}(1 + x) = 1.$$

Thus, α *and* α_1 *decrease in* $(0, 1]$, *from* $\alpha(0^+) = \alpha_1(0^+) = 1$ *to* $\alpha(1) = \alpha_1(1) = 0$. *Both are discontinuous at* 0, *since* $\alpha(0) = \alpha_1(0) = 0$.

Note that $\alpha_1(x) \le 1 - x \le \alpha(x)$.

4.4. Proofs

Proof of Proposition 6. To prove this result, simply compare the logarithmic derivatives.

Proof of Theorem 3. Both results stem from the expansion of $g_{c,m}$ near 1, which reads as follows: when $t = o(1)$,

$$g_{c,m}(1 - t) = 1 - mt + \tfrac{1}{2}c(c + 1)m(m - 1)t^2 + o(t^2).$$

We also have

$$\mathrm{E}((1 - t)^L) = 1 - \mathrm{E}(L)t + \tfrac{1}{2}\,\mathrm{E}(L(L - 1))t^2 + o(t^2).$$

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A comparison of the second-order terms of these expansions yields the conditions on the variance of L for L to belong to $\mathcal{A}_c$ and, respectively, $\mathcal{A}'_c$.

To show that any square-integrable distribution belongs to $\mathcal{A}_c$ for suitable values of c, we first choose values of d and $s < 1$ sufficiently large to ensure that $\mathrm{E}(t^L) \le g_{d,m}(t)$ for every $t \ge s$. According to the expansion above, this is possible for any d such that

$$d(d+1)m(m-1) > \mathrm{E}(L(L-1)), \qquad m := \mathrm{E}(L).$$

We then choose a value of $c > d$ sufficiently large that $1/m^{1/c} \ge g_{d,m}(s)/s$. Thus, $\mathrm{E}(t^L) \le g_{d,m}(t) \le g_{c,m}(t)$ for every $t \ge s$ and, since $\mathrm{E}(t^L)/t$ is a nondecreasing function of t,

$$\mathrm{E}(t^L) \le t\,\mathrm{E}(s^L)/s \le t g_{d,m}(s)/s \le t/m^{1/c} \le g_{c,m}(t)$$

for any $t \le s$. The proof for the comparison with distributions in $\mathcal{A}'_c$ is similar.

Proof of Proposition 8. Part (i) follows from the fact that

$$g_{c,m}(t) g_{c,m'}(t) \ge t g_{c,mm'}(t).$$

We leave the verification of this as an exercice for the reader. Parts (ii) and (iii) are clear, and their proofs thus omitted.

5. Higher moments

Assume that ξ_i belongs to $\mathcal{A}_c$ for every $i \le n$, and let $m := M_n$. Then

$$M_n^r\,\mathrm{E}_k(1/Z_n^r) \le m^r\,\mathrm{E}_k^{c,m}(1/Z^r).$$

Expansions of $g_{c,m}(t)$ in the limit as $m \to \infty$ show that the distribution of Z/m with respect to $\mathrm{P}_1^{c,m}$ converges to the distribution of W with respect to a measure, P_1^c, such that

$$\mathrm{E}_1^c(\mathrm{e}^{-tW}) = (1+ct)^{-1/c}.$$

The distribution of W is gamma$(c, 1/c)$, that is, its density with respect to the Lebesgue measure, $\mathrm{d}w$, is

$$w^{c-1}\mathrm{e}^{-w/c}\,\mathbf{1}_{\{w\ge 0\}}\,/c^c\Gamma(c).$$

Furthermore, $g_{c,m}(t) \le g_c(t) := \mathrm{E}_1^c(t^W) = (1 - c\log t)^{-1/c}$. Hence,

$$M_n^r\,\mathrm{E}_k\biggl(\frac{1}{Z_n^r}\biggr) \le \mathrm{E}_k^c\biggl(\frac{1}{W^r}\biggr) = \frac{\Gamma(k/c - r)}{c^r\Gamma(k/c)}.$$

This inequality holds for every positive value of r and c and every positive integer k such that $k > cr$. These results prove the following proposition.

Proposition 11. (i) *Consider a c such that ξ_i belongs to $\mathcal{A}_c$ for every $i \le n$. For every positive real number r and every k such that $k > rc$, we have*

$$M_n^r\,\mathrm{E}_k\biggl(\frac{1}{Z_n^r}\biggr) \le \frac{1}{[k,c]_r}, \quad \textit{where } [k,c]_r := \frac{c^r\Gamma(k/c)}{\Gamma(k/c - r)}.$$

When r is an integer, $[k,c]_r = \prod_{i=1}^r (k - ic)$.

(ii) *Conversely, consider a c such that ξ_i belongs to $\mathcal{A}'_c$ for every i. Assume that $M_n \to \infty$. Then, for every k and every positive real number $r \ge k/c$,*

$$\lim_{n\to\infty} M_n^r\,\mathrm{E}_k(1/Z_n^r) = \infty.$$

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6. Bernoulli branching processes

6.1. Preliminaries

We first set some notation, to be able to deal with inhomogeneous processes.

Definition 6. The efficiency of a Bernoulli branching process is the sequence $\mathcal{X} := (x_i)_{i\geq 1}$ such that $\xi_i = (1-x_i)\delta_1 + x_i\delta_2$. Let $\mathbb{L}$ and $\mathbb{L}^*$ denote the sets of efficiencies such that x_i belongs to [0, 1] and, respectively, (0, 1] for every $i \geq 1$. For any $\mathcal{X}$ in $\mathbb{L}$, let $s(\mathcal{X}) := (x_{i+1})_{i\geq 1}$ denote the shifted sequence.

Definition 7. For any $k \geq 1$ and any efficiency $\mathcal{X}$, let

$$B_k(\mathcal{X}) := \lim_{n\to\infty} \mathrm{E}_k(1/Z_n) \prod_{i=1}^{n} (1 + x_i).$$

In the homogeneous case, in which $x_i = x$ for every $i \geq 1$, we write $B_k(x)$ for $B_k(\mathcal{X})$.

By convexity, the limit which defines $B_k(\mathcal{X})$ is also a supremum over $n \geq 0$; thus, $B_k(\mathcal{X}) \geq 1/k$. The functional B_k also describes $\mathrm{E}_k(1/Z_n)$ for finite values of n, since replacing every x_i, $i \geq n+1$, by 0 freezes the branching process at the value Z_n. Thus, uniform upper bounds of B_k on $\mathbb{L}$ yield upper bounds of $\mathrm{E}_k(1/Z_n)$ for finite values of n.

6.2. Results

The following result holds uniformly over the set $\mathbb{L}$ and is a consequence of the fact that $A(\xi) < 1$ for every Bernoulli distribution ξ; see Proposition 10.

Proposition 12. *For every efficiency $\mathcal{X}$ in $\mathbb{L}$ and every $k \geq 1$,*

$$\frac{1}{k} \leq B_k(\mathcal{X}) \leq \frac{1}{k-1}.$$

Thus, for every $n \geq 0$,

$$\frac{1}{k} \leq \mathrm{E}_k\left(\frac{1}{Z_n}\right) \prod_{i=1}^{n} (1 + x_i) \leq \frac{1}{k-1}.$$

The sequence $(B_k)_{k\geq 1}$ satisfies recursion relations, which we state in Proposition 13, that characterize it fully; see Proposition 14.

Proposition 13. *For every $k \geq 1$, the function B_k is measurable on $\mathbb{L}$. Furthermore, for every $k \geq 1$ and $\mathcal{X}$ in $\mathbb{L}$,*

$$B_k(\mathcal{X}) = (1 + x_1) \sum_i \binom{k}{i} x_1^i (1 - x_1)^{k-i} B_{k+i}(s(\mathcal{X})). \tag{1}$$

Proposition 14. *Let $(F_k)_{k\geq 1}$ denote a sequence of functionals defined on $\mathbb{L}^*$. Assume that, as $k \to \infty$, $kF_k(\mathcal{X}) \to 1$ uniformly over $\mathcal{X}$ in $\mathbb{L}^*$, and that $(F_k)_{k\geq 1}$ solves (1) on $\mathbb{L}^*$ for every $k \geq 1$. Then $F_k = B_k$ on $\mathbb{L}^*$ for every $k \geq 1$.*

The sequence $(B_k)_{k\geq 1}$ is entirely determined on $\mathbb{L}^*$ by the recursion (1) and by the bounds $1/k \leq B_k \leq 1/(k-1)$.

Note finally that the recursion (1) is just a special case of the following result. For any branching process of reproduction law $\Xi = (\xi_i)_{i\geq 1}$ and any $k \geq 1$, introduce

$$H_k(\Xi) := \lim_{n\to\infty} M_n \,\mathrm{E}_k(1/Z_n)$$

and the shifted mechanism $s(\Xi) := (\xi_{i+1})_{i\geq 1}$. Let ξ_1^{*k} denote the k-fold convolution of the measure ξ_1 with itself. Then

$$H_k(\Xi) = \mu_1 \sum_{i\geq k} \xi_1^{*k}(i) H_i(s(\Xi)).$$

6.3. The homogeneous case

We start with a version of the relation (1) in the homogeneous case.

Proposition 15. *For every x in* $(0, 1)$,

$$B_k(x) = (1+x) \sum_i \binom{k}{i} x^i (1-x)^{k-i} B_{k+i}(x).$$

The recursion whose left-hand side is $B_k(x)$ involves the whole set of values $B_k(x)$, $B_{k+1}(x)$, $\dots$, $B_{2k}(x)$. Thus, this system of equations does not directly yield the value of each $B_k(x)$. The exception is the case $k = 1$.

Corollary 8. *For every $x \neq 0$, $B_1(x) = B_2(x)(1+x)/x$.*

Our main result in this section is Proposition 16.

Proposition 16. *Let $\alpha_1(x) := (1-x)/(1+x)$ and $\alpha_2(x) := 1-x$. For every $k \geq 1$,*

$$\frac{1}{k-\alpha_1(x)} \leq B_k(x) \leq \frac{1}{k-\alpha_2(x)}.$$

Thus, $B_k(0^+) = 1/(k-1)$ and $B_k(1^-) = 1/k$.

For $k = 1$, Proposition 16 states that $B_1(x)$ is at least $1/(1-\alpha'(x))$. A tighter bound, namely

$$\frac{(1+x)^2}{1+3x} \leq x B_1(x) \leq 1,$$

obtains if we use Corollary 8 first and then Proposition 16. The lower bound is always greater than $\frac{8}{9} = 0.8889^-$. In our numerical simulations in Section 6.9, below, some values of $\lambda B_1(\lambda)$ are found to be as small as $B_* = 0.9274 \pm 0.0002$.

We could iterate the procedure, obtaining yet tighter upper and lower bounds for $\mathrm{E}_1^x(1/Z_n)$ or for any $\mathrm{E}_k^x(1/Z_n)$ with $k \geq 1$, to any prescribed accuracy. We develop this idea in Section 6.7.

We end this section with a conjecture.

Conjecture 1. *Every function $x \mapsto B_k(x)$ is decreasing in x on* $(0, 1]$.

A problem for further research is to prove this and find a natural explanation of the fact that $B_k(0^+)$ and $B_k(0)$ are not equal.

6.4. Proofs

Proof of Proposition 13. Since each $\mathrm{E}_k(1/Z_n)$ is measurable with respect to $(x_i)_{i\leq n}$, $B_k(x)$ is the limit of a measurable, nondecreasing sequence; hence, B_k is measurable. Regarding the recursion relation, we consider the conditioning on Z_1 of the Bernoulli branching process starting from $Z_0 = k$. On the event $\{Z_1 = k + i\}$, $(Z_{n+1})_{n\geq 0}$ follows the law of a Galton–Watson branching process of efficiency $s(x)$ starting from $k + i$. Hence, (1) follows from the fact that the distribution of $Z_1 - k$ is binomial(k, x_1).

Proof of Proposition 14. We show that the conditions in the statement of Proposition 14 define $(B_k)_k$ uniquely. The existence follows from the construction of each B_k. A proof of the uniqueness is as follows. Assume that the sequences of functionals (B'_k) and (B''_k) are solutions. In particular, $B'_k/B''_k \to 1$ uniformly on $\mathbb{L}^*$ as $k \to \infty$. Fix an ε. For every sufficiently large k and every x in $\mathbb{L}^*$,

$$B'_k(x) \leq (1 + \varepsilon)B''_k(x).$$

Since (B'_k) and (B''_k) both solve the recursion relation (1), a recursion over the decreasing values of k shows that $B'_k(x) \leq (1 + \varepsilon)B''_k(x)$ for every $k \geq 1$ and every x in $\mathbb{L}^*$. This recursion uses as a crucial tool the fact that no x_k is 0. Now, since ε is arbitrary, we have $B'_k \leq B''_k$ on $\mathbb{L}^*$ for every $k \geq 1$. By exchanging the roles of the two sequences, we see that $B'_k = B''_k$ on $\mathbb{L}^*$ for every $k \geq 1$.

6.5. Outline of the proof of Proposition 16

We start from relations between the functions B_k in Proposition 15, which read

$$B_k(x) = (1 + x)\,\mathrm{E}^x_k(B_{Z_1}(x)), \qquad k \geq 1.$$

With respect to the probability P^x_k, Z_1 is distributed like the sum of k independent, identically distributed random variables of distribution $(1-x)\delta_1 + x\delta_2$. The next lemma follows from the fact that $kB_k(x) \to 1$ as $k \to \infty$.

Lemma 9. *Assume that* $\liminf_{k\to\infty} k\varphi(k) \geq 1$ *and that, for every* $k \geq 1$,

$$(1 + x)\,\mathrm{E}^x_k(\varphi(Z_1)) \leq \varphi(k).$$

Then $B_k(x) \leq \varphi(k)$ *for every* $k \geq 1$. *Conversely, if* $\limsup_{k\to\infty} k\psi(k) \leq 1$ *and, for every* $k \geq 1$,

$$(1 + x)\,\mathrm{E}^x_k(\psi(Z_1)) \geq \psi(k),$$

then $B_k(x) \geq \psi(k)$ *for every* $k \geq 1$.

Definition 8. For every $k \geq 1$, let c_k denote the unique solution in $(0, 1)$ to the equation

$$(1 + x)\,\mathrm{E}^x_k\left(\frac{1}{Z_1 - c_k}\right) = \frac{1}{k - c_k}.$$

The next lemma becomes obvious if an equivalent definition of c_k, given below in part (ii) of Lemma 13, is used.

Lemma 10. *If* $c_k \leq c$ *for every* $k \geq 1$ *then* $B_k(x) \leq 1/(k - c)$ *for every* $k \geq 1$. *Conversely, if* $c_k \geq c$ *for every* $k \geq 1$ *then* $B_k(x) \geq 1/(k - c)$ *for every* $k \geq 1$.

Lemma 10 asserts that $B_k(x) \leq 1/(k - c)$ for $c = \sup\{c_k : k \geq 1\}$ and, by Lemma 11, this supremum is $c_1 = 1 - x$; Proposition 16 thus follows.

Lemma 11. *For every $k \geq 1$, $c_k \leq c_1 = 1 - x$.*

The following result shows that the technique above cannot yield a better value of $\alpha_1(x)$ than $\alpha_1(x) = (1-x)/(1+x)$.

Lemma 12. *As $k \to \infty$, $c_k \to (1-x)/(1+x)$. Furthermore, for $k \geq 2$, $c_k + xc_{k-1} \geq 1 - x$.*

Finally, we use the characterizations below to evaluate c_k.

Lemma 13. *For every $k \geq 2$, the following inequalities are equivalent to each other and to the fact that $c \geq c_k$:*

(i) $(1+x)\,\mathrm{E}_k^x(1/(Z_1 - c)) \leq 1/(k-c)$,

(ii) $k(1+x)\,\mathrm{E}_{k-1}^x(1/(Z_1 + 2 - c)) \geq 1$,

(iii) $k(k-1)x(1+x)\,\mathrm{E}_{k-2}^x(1/(Z_1 + 4 - c)) \leq xk - 1 + c$.

The reversed inequalities, (i′), (ii′), and (iii′), are equivalent to each other and to the fact that $c \leq c_k$.

6.6. Technical steps of the proof of Proposition 16

We prove Lemmas 11 and 12, assuming Lemma 13 to hold for the moment. By Jensen's inequality, the expectation of the inverse is greater than the inverse of the expectation. Thus, inequality (ii′) of Lemma 13 implies that

$$k(1+x) \leq (k-1)(1+x) + 2 - c.$$

This reads $c \leq 1 - x$. Since $c_1 = 1 - x$, Lemma 11 follows. Furthermore, we can and do restrict to $c \leq 1 - x$ in the reasoning below.

To prove Lemma 12, we first note that inequality (ii) involves the expected value of a concave function of $u := 1/(Z_1 - c_{k-1})$, namely the function $u \mapsto u/(1+bu)$ with $b := c_{k-1} + 2 - c$. The expected value of a concave function is at most the value of the function at the expected value of its argument. From the definition of c_{k-1}, inequality (ii) implies that

$$k(1+x) \geq (k - 1 - c_{k-1})(1+x) + c_{k-1} + 2 - c.$$

This is equivalent to $c \geq 1 - x - xc_{k-1}$. Hence, for any $k \geq 2$,

$$1 - x - xc_{k-1} \leq c_k \leq 1 - x. \tag{2}$$

This is enough to show that $c_k \geq (1-x)^2$ for every $k \geq 1$. Thus, we can and do further restrict to $c \geq (1-x)^2$ in the reasoning below.

In the second step of the proof of Lemma 12, we use inequality (iii′) in the way we used inequality (ii). That is, we note that inequality (iii′) involves the expected value of a concave function of $1/(Z_1 - c_{k-2})$ and apply Jensen's inequality once again. From the definition of c_{k-2}, inequality (iii′) implies that

$$k(k-1)x(1+x) \geq [(1+x)(k - 2 - c_{k-2}) + c_{k-2} + 4 - c](xk - 1 + c).$$

After some simplifications, this reads $A_1 k + A_0 \geq 0$, with

$$A_1 := (1-x)^2 - c + x^2 c_{k-2}, \qquad A_0 := (1-c)(2(1-x) - c - xc_{k-2}).$$

Since $c \geq (1-x)^2$ and $c_{k-2} \geq (1-x)^2$, simple bounds show that $A_0 \leq 1$. Hence, inequality (iii′) implies that $A_1 \geq -1/k$. Finally, for every $k \geq 3$, we have

$$(1-x)^2 \leq c_k \leq (1-x)^2 + x^2 c_{k-2} + 1/k. \tag{3}$$

We use the a-priori bounds of (2) and (3) as follows. On the one hand, the upper bound of c_k in (3) implies that

$$\limsup c_k \leq (1-x)^2 + x^2 \limsup c_k.$$

On the other hand, the lower bound of c_k in (2) implies that

$$1 - x - x \limsup c_k \leq \liminf c_k.$$

Hence, $\limsup c_k = \liminf c_k = (1-x)/(1+x)$. This proves Lemma 12.

Lemma 13 is a consequence of the following trick. Part (i) involves

$$\frac{k-c}{Z_1-c} = 1 - \frac{Z_1-k}{Z_1-c} =: 1-v.$$

By exchangeability, $\mathrm{E}_k^x(v)$ is k times the expected value of $(L_1-1)/(Z_1-c)$, where L_1 denotes the number of descendants of the first individual in the initial population. The event $\{L_1 - 1 \neq 0\}$ is equivalent to $\{L_1 = 2\}$ and has probability x. Thus, for every $k \geq 2$,

$$(k-c)\,\mathrm{E}_k^x\left(\frac{1}{Z_1-c}\right) = 1 - kx\,\mathrm{E}_{k-1}^x\left(\frac{1}{Z_1+2-c}\right).$$

With the convention that $Z_1 = 0$ P_0^x-almost surely, this relation holds for $k = 1$ as well. This translates inequality (i) or (i′) into inequality (ii) or (ii′). The translation of inequality (ii) or (ii′) into inequality (iii) or (iii′) uses the same technique, starting from $1/(Z_1+2-c)$. This concludes the proof of Proposition 16.

6.7. Algorithm

The following algorithm yields approximate values of B_k on $\mathbb{L}^*$, to any prescribed accuracy.

- Fix $n \geq 1$.
- For every $k \geq n+1$ and x, let $B_{k,n}^0(x) := 1/k$ and $B_{k,n}^1(x) := 1/(k-1)$.
- Find the unique sequence $(B_{k,n}^1)_{k \leq n}$ that solves the system of equations (1) for $k \leq n$ when every $B_k(s(x))$, $k \geq n+1$, is replaced with $B_{k,n}^1(s(x))$, that is, with the value $1/(k-1)$.
- Likewise, find the unique sequence $(B_{k,n}^0(x))_{k \leq n}$ that solves the system of equations (1) for $k \leq n$ when every $B_k(s(x))$, $k \geq n+1$, is replaced with $B_{k,n}^1(x)$, that is, with the value $1/k$.
- Then, for every $k \geq 1$ and every x,

$$B_{k,n}^0(x) \leq B_k(x) \leq B_{k,n}^1(x) \leq (1+1/n)B_{k,n}^0(x).$$

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6.8. Comments on the algorithm

Neither $(B^0_{k,n})_{k\geq 1}$ nor $(B^1_{k,n})_{k\geq 1}$ solves the full system of equations (1). For any fixed values of k and x, $(B^0_{k,n}(x))_{n\geq 1}$ is a nondecreasing sequence that converges to $B_k(x)$ as $n \to \infty$. Likewise, $(B^1_{k,n}(x))_{n\geq 1}$ is a nonincreasing sequence that converges to $B_k(x)$ as $n \to \infty$. Increasing values of n yield more and more accurate approximations of each $B_k(x)$, and the relative error is of order at most $1/n$.

In the Bernoulli case, we can use initial values better than $B^0_{k,n}(x)$ and $B^1_{k,n}(x)$, namely, for every $k \geq n+1$ and x,

$$b^0_{k,n}(x) := \frac{1}{k - \alpha_1(x)}, \qquad b^1_{k,n}(x) := \frac{1}{k - \alpha_2(x)}.$$

The relative error, which was at most $1 + 1/n$ in the first version of the algorithm, is now at most

$$1 + \frac{\alpha_2(x) - \alpha_1(x)}{n + 1 - \alpha_2(x)} \leq 1 + \frac{3 - 2\sqrt{2}}{n + x}.$$

Numerically, this is at most $1 + 0.172/n$, for every x.

6.9. Simulations in the homogeneous case

Using the algorithm above with $n := 1000$ prompts the following refinements. Define $B(x) := B_1(x)x = B_2(x)(1 + x)$. Simulations show that B decreases on $(0, x_*)$ from $B(0^+) = 1$ to $B(x_*) =: B_*$ and increases on $(x_*, 1]$ from B_* to $B(1) = 1$, where

$$x_* = 0.38 \pm 0.01, \qquad B_* = 0.9274 \pm 0.0002.$$

This implies that, for every positive x,

$$B_*/x \leq B_1(x) \leq 1/x.$$

Simulations show that B_2 and, hence, B_1 decrease on (0, 1].

7. Size-biased offspring

When computing harmonic means, it may prove convenient to use size-biased distributions, defined as follows. Assume that L and L_i are independent, identically distributed positive, integrable random variables and that L' is an independent, size-biased copy of L, that is, for every t in [0, 1],

$$\mathrm{E}(t^{L'}) := \mathrm{E}(Lt^L)/\,\mathrm{E}(L).$$

Then, for any nonnegative integer k,

$$\mathrm{E}(L)\,\mathrm{E}\left(\frac{1}{L_1 + \cdots + L_k + L'}\right) = \frac{1}{k+1}.$$

For more details on this and many other relations between size-biased distributions and branching mechanisms, see Chapter 10 of Lyons and Peres (2006).

Can we use this in our branching setting? Assume first that $1 \leq L \leq c + 1$ almost surely, for a given integer c. Since $L' \leq c + 1 \leq L_{k+1} + \cdots + L_{k+c+1}$ almost surely, this implies that

$$\mathrm{E}(L)\,\mathrm{E}_k\left(\frac{1}{Z_1}\right) \leq \frac{1}{k - c}$$

for every $k \geq c+1$. More generally, the inequality above holds if $\mathrm{E}(t^{L'}) \geq \mathrm{E}(t^L)^{c+1}$ for every t in [0, 1] and a given positive real number c.

In our setting, this line of reasoning suffers from two drawbacks. First, it appears that to be able to iterate this inequality over n generations, we must assume that $k \geq nc$. Second, the inequality $\mathrm{E}(t^{L'}) \geq \mathrm{E}(t^L)^{c+1}$ implies that $c \geq A(\xi)$, where ξ denotes the law of L (the proof of this is easy and, thus, omitted). In other words, $k \geq c$ implies that ξ belongs to $\mathcal{A}_c$.

8. The case $k \leq c$

This section is a brief description of the behaviour of $\mathrm{E}_k(1/Z_n)$ when the hypotheses of Theorem 2 are false. Consider, for the sake of simplicity, a homogeneous branching process and let

$$p_i := \xi(i) = \mathrm{P}(L = i) = \mathrm{P}(Z_1 = i \mid Z_0 = 1).$$

Our first remark is that, for every $n \geq 0$ and $k \geq 1$,

$$\mathrm{E}_k(1/Z_n) \geq r_k^n/k, \qquad r_k := \max\{p_1^k, 1/\mu\}.$$

The bound $1/\mu$ is due to the convexity. The bound p_1^k is due to the fact that the probability of the event $\{Z_n = k\}$ is p_1^k.

The parameters μ and p_1, through r_k, indeed describe the asymptotics of $\mathrm{E}_k(1/Z_n)$, as follows. For the sake of simplicity, we exclude the degenerate case $p_1^k\mu = 1$, in which polynomial corrections appear. For every $k \geq 1$, there exists a finite, positive real number h_k such that

$$\lim_{n\to\infty} \frac{\mathrm{E}_k(1/Z_n)}{r_k^n} = h_k.$$

In the Bernoulli case, $p_1 < 1/\mu$ and, hence, $r_k = 1/\mu$ for every $k \geq 1$, and the $r_k = p_1^k$ regime is nonexistent.

The limits h_k satisfy the following relations. Assume, for instance, that we wish to compute h_1 and that $\mu p_1 > 1$, whence $r_1 = p_1$. By conditioning on the value of Z_1, we obtain a relation between $\mathrm{E}_1(1/Z_{n+1})$ and the sequence $(\mathrm{E}_k(1/Z_n))_{k\geq 1}$. Letting n go to infinity yields

$$h_1 = 1 + \sum_{k\geq 2} \frac{H_k p_k}{p_1}, \qquad H_k := \sum_{n\geq 0} \frac{\mathrm{E}_k(1/Z_n)}{p_1^n}.$$

The term $\mathrm{E}_2(1/Z_n)/p_1^n$ behaves like $(r_2/p_1)^n$, that is, like $1/(\mu p_1)^n$ if $\mu p_1^2 > 1$ and like p_1^n if $\mu p_1^2 < 1$. Since both quantities are summable, H_2 is finite. Since $H_k \leq H_2$ for every $k \geq 2$, h_1 is finite as well.

When $\mu p_1 < 1$, $r_k = 1/\mu$ for every $k \geq 1$ and the same reasoning as above yields

$$h_1 = \mu \sum_{k\geq 1} p_k h_k.$$

Since $\mu p_1 < 1$, this gives h_1 as a linear combination of the h_k, $k \geq 2$. In turn, for every $k \geq 2$, a one-step recursion similar to the one we used before shows that h_k can be written

$$h_k = \mu \sum_{i\geq k} h_i \xi^{*k}(i).$$

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Since the coefficient of h_k on the right-hand side is $\mu p_1^k < 1$, this relation implies that h_k is a linear combination of the sequence $(h_i)_{i\geq k+1}$, and that this linear combination has nonnegative coefficients. Since $h_i \leq h_2$ for every $i \geq k+1$, the series converges. However, it does not seem easy to recover information about the coefficients h_1 or $h_k,\ k \geq 2$, from these relations.

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