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THE STABILITY OF BISHOP'S WARFARE STRATEGY

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Abstract

The paper analyses the dynamics of a duopoly output game involving a warfare strategy proposed by Robert Bishop. Necessary and sufficient conditions are obtained for the stability of a duopoly warfare game.

1. Introduction

Theocharis [7] was one of the first to investigate the stability of the Cournot oligopoly problem. He took a linear demand curve and assumed constant marginal costs c_i, $i = 1, \ldots, n$, for each of n sellers. Each seller assumes that his competitors' quantities of output will remain unaltered in the next business period. Then every seller attempts to maximize his profit during the next business period by producing what he believes will be the optimal quantity of goods. Based on these assumptions, Theocharis showed that stability occurs only in the duopoly case. For three sellers he concluded that finite oscillations about the equilibrium position occur and for more than three sellers there is always instability.

Theocharis' paper produced initial objections and improvements from Fisher [2] and from McManus and Quandt [4]. Thereafter various generalizations and extensions were developed.

Both the Fisher and the McManus and Quandt papers showed that either a more general output adjustment process or variable marginal cost functions will increase the stability of the process. Fisher assumed a linear demand curve and, for the discrete time case, worked from the equation

$$x_i(t+1) - x_i(t) = k_i(x_i^*(t+1) - x_i(t)),$$

where $x_i(t)$ is the quantity produced during the tth business period, $x_i^*(t+1)$ is the quantity which the Cournot strategy suggests should be produced during the

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$(t+1)$th business period, and k_i is the speed of adjustment or "confidence" which firm i has in its theoretical result $x_i^*(t+1)$. For this discrete case, it is always possible to find speeds of adjustment such that stable equilibrium occurs.

The variable cost functions, used by both Fisher and McManus and Quandt, were of the form

$$C_i = g_i + c_i x_i + \tfrac{1}{2} d x_i^2, \quad \text{firms } i = 1, \ldots, n,$$

where g_i, c_i and d are constants. The restriction $d_i = d$ for all $i = 1, \ldots, n$ means that the slope of the marginal cost curves is assumed to be the same for all the n firms.

Bishop [1] suggested possible alternative types of warfare when competing duopolists persist in making mutually incompatible demands upon each other. He discussed and illustrated a method by which each duopolist may convey to the other, without verbal communication, both his demand for a certain collusive profit and the threat of the warfare he would be prepared to wage, if necessary, to enforce that demand. Central to the strategy was a reaction schedule with three segments successively reflecting, first the collusive equilibrium proposed, secondly a warring response if that offer is refused, and finally a limiting influence on the severity of the warfare. Bishop considered in detail a specific case of constant and equal costs for the duopolists and concluded that stable equilibrium would always eventually occur in the discrete time situation he studied. However, Bishop did not explain *why* the duopolists would eventually reach the equilibrium state suggested, and neither was he concerned with *how* this equilibrium would be reached.

Osborne [5] considered the comparative statics of a number of duopoly output variation games with perfect information. Among the strategies considered was a variant of Bishop's threefold warfare proposal. Osborne's approach differed from Bishop's in that he sought the solution which is the best attainable by simultaneous pursuit of Bishop's strategies rather than an eventual equilibrium point.

In his paper, Bishop [1] also used the constant and equal cost structure to discuss a more general single warfare hypothesis. At each time period, t, a duopolist, i, assumes that his rival, j, will maintain an unchanged output in the next period. Duopolist i selects his new output to maximize

$$\pi_i(t+1) - a_i \pi_j(t+1), \quad i, j = 1, 2, \quad i \neq j,$$

where $\pi_i(t+1)$ is firm i's profit at time $(t+1)$ and each a_i is constant.

The present paper investigates the significance of the a_i's and considers the dynamics of a duopoly model involving Bishop's warfare strategy. It incorporates the non-instantaneous speeds of adjustment, k_i, of Fisher [2] and extends his quadratic cost structure to allow marginal cost curves of duopolists to have

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different slopes. We will show that necessary conditions for asymptotic stability of the system are that the sum of the k_i's be bounded above and that an interaction term involving the a_i's be less than unity. It should be understood that when we are considering stability of the system, we are in fact referring to *asymptotic* stability.

2. Cost, demand and profit functions

Theocharis [7], Fisher [2] and others assumed a market demand curve of the form

$$p' = a - b\sum_{i=1}^{n} x_i, \tag{1}$$

where a and b are positive constants and x_i is the output of firm i. Osborne [5, 6] and Bishop [1] chose, instead of equation (1), the relationship

$$p = D - \sum_{i=1}^{n} x_i, \tag{2}$$

where D is a positive constant. It should be noted that equation (2) is no less general than equation (1); in fact, the latter reduces to the former by letting $p = p'/b$ and $D = a/b$. For simplicity we shall use (2).

In place of Bishop's constant and equal cost functions, we suppose that the cost function of firm i is more generally a quadratic function of firm i's output, that is,

$$C_i = g_i + c_i x_i + \tfrac{1}{2} d_i x_i^2, \quad i = 1, \ldots, n. \tag{3}$$

Note that, unlike Fisher [2], we have not restricted firms to having identically sloping marginal cost curves.

The profit of firm i is given by

$$\begin{aligned} \pi_i &= p x_i - C_i \\ &= -g_i + A_i x_i + B_i x_i^2 - x_i \sum_{\substack{j=1 \\ j \neq i}}^{n} x_j, \end{aligned} \tag{4}$$

where $A_i = D - c_i$ and $B_i = -(1 + \tfrac{1}{2} d_i)$.

We will assume that firm i's marginal profit is a decreasing function of its own output. This requires

$$d_i > -2, \quad i = 1, \ldots, n. \tag{5}$$

We can reasonably place the additional restriction that it should, at some time, be possible for the ith seller to make a non-negative profit by producing a positive output. This should certainly be so when $x_j = 0$, $j = 1, \ldots, n$, $j \neq i$, in which case

$\pi_i = -g_i + A_i x_i + B_i x_i^2$. As g_i, $-B_i$ and x_i are all non-negative, such non-negative profits can only be achieved if $A_i \geqslant 0$, that is,

$$c_i \leqslant D, \quad i = 1, \ldots, n. \tag{6}$$

Now for firm i, the marginal cost is $c_i + d_i x_i$. Assuming that marginal costs are always positive and remembering that $0 \leqslant x_i \leqslant D$, we obtain

$$c_i > 0 \tag{7}$$

and

$$d_i D > -c_i. \tag{8}$$

Equations (6)–(8) imply that

$$d_i > -1, \tag{9}$$

which states that marginal costs cannot decline as fast as price.

3. The warfare concept of a Bishop strategy for two firms

If two firms cannot agree on outputs, then they are likely to adopt warring attitudes. Bishop [1] considered the warfare strategy whereby each firm i selected $x_i(t+1)$ to maximize $\pi_i(t+1) - \pi_j(t+1)$, $i, j = 1, 2$, $i \neq j$, assuming that

$$x_j(t+1) = x_j(t),$$

where $x_i(t)$ and $\pi_i(t)$ refer to the production quantity and profit, respectively, of firm i at time period t. Osborne [5] incorporated a version of this strategy into his analysis. He considered the possible outcomes when duopolists adopted various strategies, one of which was Bishop's. However, we shall discuss the more general case, also considered by Bishop, where each firm i maximizes $\pi_i(t+1) - a_i \pi_j(t+1)$ on the assumption that each firm j holds its own production constant. The significance of the a_i will be discussed at length later in this section. When $a_i = 0$ we have the Cournot strategy. When $a_i = -1$ we have the case of joint profit maximization by firm i and when $a_i = 1$ we have the "rivalistic" case (see Fouraker and Siegel [3]).

From (6) we have, in the duopoly case,

$$\pi_i = -g_i + A_i x_i + B_i x_i^2 - x_1 x_2.$$

Firm i assumes that $x_j(t+1) = x_j(t)$ and maximizes $\pi_i(t+1) - a_i \pi_j(t+1)$ to give

$$x_i(t+1) = \{(1 - a_i) x_j(t) - A_i\}/(2B_i). \tag{10}$$

Before investigating the significance of the a_i's we consider the contract curve obtained from $\det(\partial \pi_i / \partial x_j) = 0$. This yields

$$A_1 A_2 + (2B_1 A_2 - A_1) x_1 + (2B_2 A_1 - A_2) x_2 + 4B_1 B_2 x_1 x_2 - 2B_1 x_1^2 - 2B_2 x_2^2 = 0. \tag{11}$$

This represents an hyperbola, an ellipse or a parabola according to whether $(B_1 B_2 - 1)$ is positive, negative or zero. Thus many possible cases may arise. For illustration, consider the case where the contract curve is a parabola and $A_2 < A_1 < 2A_2$. For this case, Fig. 1 shows part of the contract curve and the reaction functions of duopolists using the Bishop strategy, for various values of a_1 and a_2. That part of the contract curve passing through the points $(A_2, 0)$ and $(0, A_1)$ has been omitted.

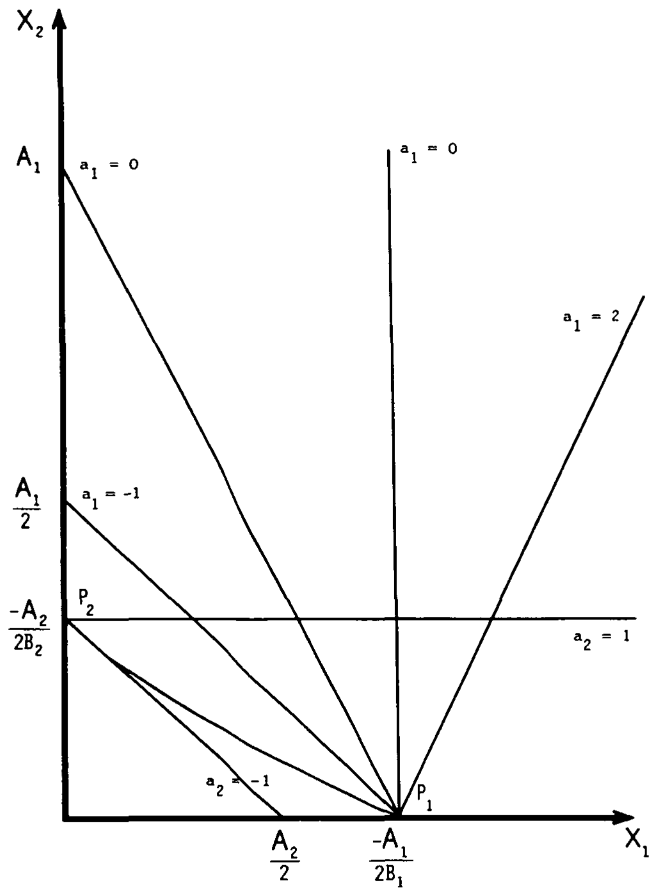

Fig. 1. This diagram shows the reaction functions of duopolists using the Bishop strategy for various values of a_1, a_2 (given $A_2 < A_1 < 2A_2$).

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From (10), the reaction function for firm i is

$$x_i = \{(1-a_i)\,x_j - A_i\}/(2B_i), \quad i,j=1,2, \quad i \neq j. \tag{12}$$

In Fig. 1, the lines through P_i ($x_i = -A_i/(2B_i)$, $x_j = 0$) indicate the reaction functions for firm i, for various values of a_i, $i = 1,2$. The curve through P_1 and P_2 is the contract curve. It indicates the shape of the curve which can arise when it is a parabola or an ellipse. In general, the point of intersection of the reaction functions of the two firms is found by solving (12), giving coordinates

$$x_i = \frac{2A_i B_j - (a_i - 1)\,A_j}{(a_1-1)(a_2-1)-4B_1 B_2} = R_i, \quad i,j=1,2, \quad i \neq j, \tag{13}$$

provided $(a_1-1)(a_2-1)-4B_1 B_2 \neq 0$.

The condition $(a_1-1)(a_2-1)-4B_1 B_2 = 0$ is equivalent to the reaction functions for firms i and j being parallel. The special case of these two curves being coincident will not be examined in this paper.

Consider now what increment in profit firm i believes that its new production rate will yield. This is given by

$$\pi_i(t+1)-\pi_i(t) = -[\{A_i+2B_i\,x_i(t)-x_j(t)\}^2 - \{a_i\,x_j(t)\}^2]/(4B_i). \tag{14}$$

Firm i also believes that the increment in firm j's profit will be

$$\pi_j(t+1)-\pi_j(t) = [A_i+(a_i-1)\,x_j(t)+2B_i\,x_i(t)]\,x_j(t)/(2B_i). \tag{15}$$

Thus firm i would expect the relative improvement in its own position to be

$$\begin{aligned} I_i &= \{\pi_i(t+1)-\pi_i(t)\} - \{\pi_j(t+1)-\pi_j(t)\} \\ &= -[\{A_i+2B_i\,x_i(t)\}^2 - \{(1-a_i)\,x_j(t)\}^2]/(4B_i). \end{aligned} \tag{16}$$

If $a_i = 0$, then $I_i = -[\{A_i+2B_i\,x_i(t)\}^2 - \{x_j(t)\}^2]/(4B_i)$ which may be positive, zero or negative. Firm i seeks to maximize its own profit, even if this means improving the position of the rival firm more than it does its own. This is precisely the Cournot strategy. Hence the Cournot strategy can be considered a special case of this Bishop warfare strategy.

If $a_i = 1$, then $I_i = -[A_i+2B_i\,x_i(t)]^2/(4B_i) \geqslant 0$. Firm i expects to improve its position relative to firm j, or at worst to maintain its relative position, as a result of changing its production schedule.

If $a_i = -1$, then $I_i = -[\{A_i+2B_i\,x_i(t)\}^2 - 4\{x_j(t)\}^2]/(4B_i)$. Firm i is seeking to maximize the combined profit $\pi_i(t+1)+\pi_j(t+1)$. This is essentially a strategy of cooperation rather than warfare. If both firms adopt this strategy ($a_1 = a_2 = -1$) then they are pursuing the same objective.

If $0 < a_i < 1$, firm i is concerned partly with the common good, and partly with its own self-interest. Taking $a_i < -1$ would be an irrational and self-defeating strategy.

For $a_i = 1 \pm r$, $r > 0$, the values of I_i are identical. Since $\pi_i(t+1)$ would be greater for $a_i = 1 - r$, firm i should not choose a value of a_i greater than unity. Bishop also reached this conclusion. However, a firm might still select $a_i = 1 + r$ if it is confident that it can withstand short-term losses in order to drive its opposition from the market.

If x_1 and x_2 are non-negative, then $I_1 \geqslant 0$, provided

$$|A_1 + 2B_1 x_1| \geqslant |(1-a_1) x_2|.$$

In the rivalistic case, that is $a_1 = 1$, we have $I_1 \geqslant 0$ regardless of the non-negative values of x_1 and x_2. The shaded area in Fig. 2 indicates where $I_1 \geqslant 0$ for $a_1 \neq 1$.

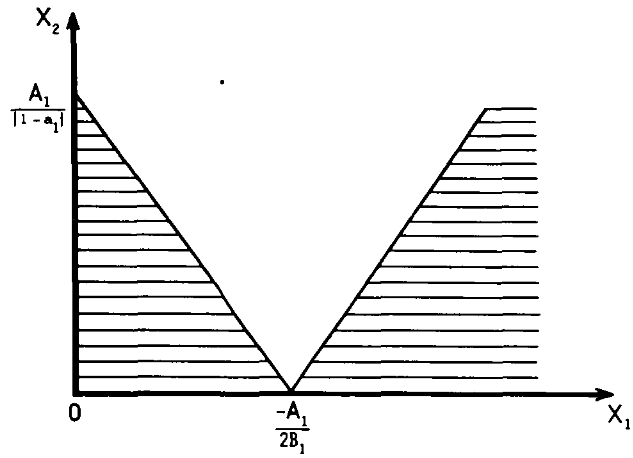

Fig. 2. The shaded regions in this diagram indicate where firm 1 expects to improve its position relative to firm 2 when it adopts a Bishop warfare strategy with $a_1 \neq 1$.

Suppose that firm i is interested only in improving its profit relative to its rival, and it is uncertain whether its rival will maintain unchanged production. If firm i uses a Bishop warfare strategy, then it should choose $a_i = 1$. However, if $a_i \neq 1$, firm i should choose a production x_i away from $-A_i/(2B_i)$ (see Fig. 2).

4. Evolution and stability of games involving the Bishop warfare strategy

We now investigate the situation where each firm i, $i = 1, 2$, repeatedly obtains a value $x_i^*(t+1)$ after maximizing $\pi_i(t+1) - a_i \pi_j(t+1)$, based on the assumption

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that its opposition maintains an unaltered production at time $(t+1)$. From (10),

$$x_i^*(t+1) = \{(1-a_i)\,x_j(t) - A_i\}/(2B_i). \tag{17}$$

Incorporating Fisher's speeds of adjustment, as outlined in the Introduction, we let each firm i adjust its production with speed k_i. Thus

$$x_i(t+1) = x_i(t) + k_i\{x_i^*(t+1) - x_i(t)\}, \tag{18}$$

where $k_i > 0$, $i = 1,2$. Substitution of (17) into (18) gives

$$x_i(t+1) = (1-k_i)\,x_i(t) + \frac{k_i(1-a_i)}{2B_i}\,x_j(t) - \frac{A_i\,k_i}{2B_i}. \tag{19}$$

Having chosen k_i and a_i, equation (19) determines the new output for firm i. If k_i and a_i remain constant, and firm i uses (19) to determine all subsequent outputs, then (19) may be thought of as specifying the *strategy* of firm i. If both firms pursue this strategy, it follows from (19) that the output of firm i, $i = 1,2$, satisfies the linear difference equation

$$\begin{aligned} x_i(t+2) - (2-k_1-k_2)\,x_i(t+1) + \{(1-k_1)(1-k_2) - k_1\,k_2\,E_1\,E_2\}\,x_i(t) \\ = (B_i\,E_i\,A_j - A_i\,B_j)\,k_1\,k_2/(2B_1\,B_2), \end{aligned} \tag{20}$$

where $E_i = (a_i - 1)/(2B_i)$. Hence E_i has the same sign as $1-a_i$. Equation (20) has the solution

$$\left.\begin{aligned} x_i(t) &= P\lambda_+^t + Q\lambda_-^t + R_i, \\ x_j(t) &= \frac{P(1-k_i-\lambda_+)}{k_i\,E_i}\lambda_+^t + \frac{Q(1-k_i+\lambda_-)}{k_i\,E_i}\lambda_-^t + R_j, \end{aligned}\right\} \tag{21}$$

where R_i, $i = 1,2$, are given by (13), P and Q are constants whose values depend on the initial outputs $x_1(1)$ and $x_2(1)$, and λ_+, λ_- are assumed to be distinct roots of the characteristic equation of (20), namely

$$\lambda^2 - (2-k_1-k_2)\,\lambda - \{k_1\,k_2\,E_1\,E_2 - (1-k_1)(1-k_2)\} = 0. \tag{22}$$

The form of the solution (21) will be different in the case $\lambda_+ = \lambda_-$. However, the following stability analysis still applies to this particular case.

It follows from (22) that

$$\lambda_\pm = \tfrac{1}{2}\{2-k_1-k_2\} \pm \tfrac{1}{2}\{(k_1-k_2)^2 + 4k_1\,k_2\,E_1\,E_2\}^{\frac{1}{2}}. \tag{23}$$

The roots λ_+, λ_- will be real provided the discriminant $\{(k_1-k_2)^2 + 4k_1\,k_2\,E_1\,E_2\}$ is non-negative, that is

$$(k_1-k_2)^2 + 4k_1\,k_2\,E_1\,E_2 \geqslant 0. \tag{24}$$

As $k_1, k_2 > 0$, (24) is satisfied for all $k_1, k_2 > 0$ if $E_1 E_2 \geqslant 0$. For convenience we put

$$\alpha = E_1 E_2 = \frac{(1-a_1)(1-a_2)}{4B_1 B_2}. \tag{25}$$

Hence if $\alpha \geqslant 0$, that is, either a_1 and a_2 are both greater than unity or both less than unity, then we have real roots for all $k_1, k_2 > 0$. To investigate inequality (24) for $\alpha < 0$, we temporarily introduce the change of variables

$$\left.\begin{aligned} X &= (\surd 2)^{-1}(k_1 + k_2), \\ Y &= (\surd 2)^{-1}(k_1 - k_2). \end{aligned}\right\} \tag{26}$$

Substituting (26) into (24), the inequality becomes

$$\alpha X^2 + (1-\alpha)\, Y^2 \geqslant 0. \tag{27}$$

For $\alpha < 0$, it is convenient to write

$$\alpha = -\gamma, \tag{28}$$

so that $\gamma > 0$, and (27) may be written

$$(1+\gamma)\, Y^2 - \gamma X^2 \geqslant 0, \tag{29}$$

or equivalently

$$(\surd(1+\gamma)\, Y - \surd(\gamma)\, X) . (\surd(1+\gamma)\, Y + \surd(\gamma)\, X) \geqslant 0. \tag{30}$$

In terms of the original variables k_1, k_2, inequality (30) becomes

$$\begin{aligned} &[(\surd(1+\gamma) - \surd(\gamma))\, k_1 - (\surd(1+\gamma) + \surd(\gamma))\, k_2] \\ &\qquad \times [(\surd(\gamma) + \surd(1+\gamma))\, k_1 + (\surd(\gamma) - \surd(1+\gamma))\, k_2] \geqslant 0. \end{aligned} \tag{31}$$

The boundary of the region defined by inequality (31) is given by the equations

$$k_2 = \left(\frac{\surd(1+\gamma) - \surd(\gamma)}{\surd(1+\gamma) + \surd(\gamma)}\right) k_1 \tag{32}$$

and

$$k_2 = \left(\frac{\surd(1+\gamma) + \surd(\gamma)}{\surd(1+\gamma) - \surd(\gamma)}\right) k_1. \tag{33}$$

As $\gamma > 0$, $\surd(1+\gamma) > \surd(\gamma)$. Hence $\surd(1+\gamma) + \surd(\gamma) > \surd(1+\gamma) - \surd(\gamma) > 0$. Equations (32) and (33) are thus of the form

$$k_2 = ck_1$$

and

$$k_2 = \frac{1}{c} k_1, \quad 0 < c < 1.$$

For $\alpha < 0$ (that is, $E_1 E_2 < 0$), Fig. 3 shows the regions in the $k_1 - k_2$ plane corresponding to real and complex roots, given $k_1, k_2 > 0$.

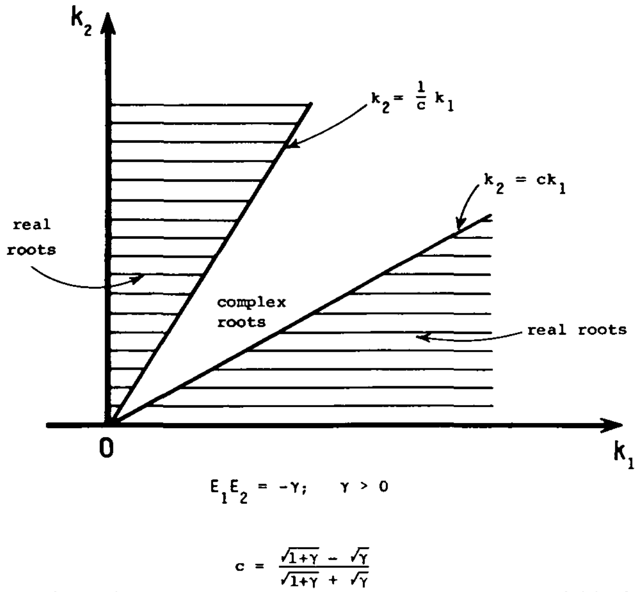

Fig. 3. This shows the regions in the k_1, k_2 plane corresponding to real (shaded) and complex roots of the characteristic equation, given $E_1 E_2 < 0$ and $k_1, k_2 > 0$.

Having found the regions corresponding to real or complex roots of equation (22), we now investigate the stability of solutions (21) to equation (20). Stable solutions will exist if, for both real and complex roots $\lambda_\pm$, $|\lambda_\pm| < 1$.

For the case of *real roots* $\lambda_\pm$, stable solutions exist provided

$$-1 < \tfrac{1}{2}[2-(k_1+k_2)+\{(k_1-k_2)^2+4\alpha k_1 k_2\}^{\frac{1}{2}}] < 1 \tag{34}$$

and

$$-1 < \tfrac{1}{2}[2-(k_1+k_2)-\{(k_1-k_2)^2+4\alpha k_1 k_2\}^{\frac{1}{2}}] < 1. \tag{35}$$

Inequalities (34) and (35) correspond to $|\lambda_+| < 1$ and $|\lambda_-| < 1$, respectively. They can be rearranged to give, respectively,

$$-4+(k_1+k_2) < \{(k_1-k_2)^2+4\alpha k_1 k_2\}^{\frac{1}{2}} < k_1+k_2 \tag{36}$$

and

$$-(k_1+k_2) < \{(k_1-k_2)^2+4\alpha k_1 k_2\}^{\frac{1}{2}} < 4-(k_1+k_2). \tag{37}$$

To simultaneously satisfy inequalities (36) and (37), we need

$$\max\{-(k_1+k_2),\ -4+(k_1+k_2)\} < \{(k_1-k_2)^2+4\alpha k_1 k_2\}^{\frac{1}{2}} < \min\{(k_1+k_2),\ 4-(k_1+k_2)\}. \tag{38}$$

Hence there are two cases to consider, namely

(i) $4-(k_1+k_2) \geqslant (k_1+k_2)$ or $k_1+k_2 \leqslant 2$

and

(ii) $4-(k_1+k_2) < k_1+k_2$ or $k_1+k_2 > 2$.

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First let us investigate case (i), when $k_1+k_2 \leqslant 2$ and thus

$$\min\{(k_1+k_2), 4-(k_1+k_2)\} = k_1+k_2$$

and

$$\max\{-(k_1+k_2), k_1+k_2-4\} = -(k_1+k_2)<0.$$

It follows that, when $k_1+k_2 \leqslant 2$, inequality (38) reduces to

$$0<\{(k_1-k_2)^2+4\alpha k_1 k_2\}^{\frac{1}{2}}<(k_1+k_2),$$

which may be further simplified to yield

$$4(\alpha-1)k_1 k_2<0. \tag{39}$$

It follows that for real roots ((24) holds) and when $k_1+k_2 \leqslant 2$, $k_1, k_2>0$, we have stable real solutions to equation (20) for $\alpha<1$.

Let us now turn our attention to case (ii), that is, when $k_1+k_2>2$. In this case

$$\min\{k_1+k_2, 4-(k_1+k_2)\} = 4-(k_1+k_2)$$

and

$$\max\{-(k_1+k_2), k_1+k_2-4\} = k_1+k_2-4,$$

and inequality (38) becomes

$$-4+(k_1+k_2)<\{(k_1-k_2)^2+4\alpha k_1 k_2\}^{\frac{1}{2}}<4-(k_1+k_2). \tag{40}$$

We note that inequality (40) can never be satisfied for any real k_1, k_2 if $k_1+k_2 \geqslant 4$. Thus for $k_1+k_2 \geqslant 4$ there can *never* be stable real solutions to the difference equation (20).

When $2<k_1+k_2<4$, inequality (40) reduces to

$$0<\{(k_1-k_2)^2+4\alpha k_1 k_2\}^{\frac{1}{2}}<4-(k_1+k_2),$$

which may be further simplified to give

$$(\alpha-1)k_1 k_2+2(k_1+k_2)<4. \tag{41}$$

Recalling that we are considering $2<k_1+k_2<4$, then in order for inequality (41) to be satisfied, we must have

$$(\alpha-1)k_1 k_2<0,$$

that is,

$$\alpha<1.$$

The boundary of the region defined by inequality (41) is given by the equation

$$(\alpha-1)k_1 k_2+2(k_1+k_2)=4, \tag{42}$$

which has two asymptotes, at $k_1=2/(1-\alpha)$ and $k_2=2/(1-\alpha)$.

We are now in a position to delineate the regions of the k_1-k_2 plane corresponding to real stable solutions of (20). For $\alpha>0$, $\alpha=0$ and $\alpha<0$, these regions are shown in Figs. 4(i), 4(ii) and 4(iii), respectively.

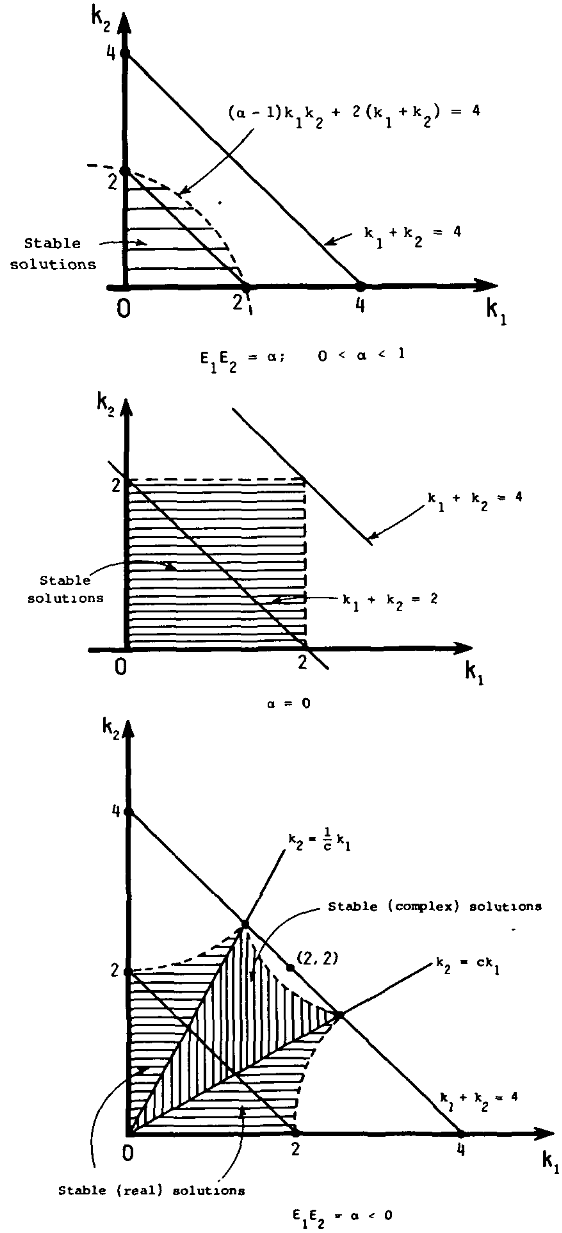

Fig. 4. This shows the regions in the k_1, k_2 plane corresponding to stable solutions of the difference equation (20) for $0<E_1E_2<1$ (4(i)), $E_1E_2=0$ (4(ii)) and $E_1E_2<0$ (4(iii)).

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For the case of *complex roots* $\lambda_{\pm}$, stable solutions again exist provided $|\lambda_{\pm}|<1$. As complex roots can only occur for $\alpha<0$, we again employ (28), that is,

$$\alpha=-\gamma, \quad \gamma>0.$$

In this case we may write

$$\lambda_{\pm}=\tfrac{1}{2}[2-(k_1+k_2)]\pm i\tfrac{1}{2}[4\gamma k_1 k_2-(k_1-k_2)^2]^{\frac{1}{2}}, \tag{43}$$

from which it follows that $|\lambda_{\pm}|<1$ is equivalent to

$$-1\leqslant(1+\gamma)k_1 k_2-(k_1+k_2)<0. \tag{44}$$

The boundary of the region defined by inequality (44) is given by the equations

$$\left.\begin{aligned} k_2&=\frac{k_1}{(1+\gamma)k_1-1}, \\ k_2&=\frac{k_1-1}{(1+\gamma)k_1-1}. \end{aligned}\right\} \tag{45}$$

The region of the k_1–k_2 plane corresponding to stable complex solutions is shown in Fig. 4(iii). It is of interest to note that the point $k_1=k_2=1$ is in a region corresponding to stable solutions only when $\gamma<1$ or, equivalently, $\alpha>-1$. Thus, the solution of the Cournot problem with immediate and complete adjustment is unstable when $\alpha<-1$.

One further constraint has yet to be discussed. An equilibrium point is of practical interest only if it lies in the first quadrant of the x_1–x_2 plane. This will be the case provided $R_i>0$, $i=1,2$. For asymptotically stable solutions to occur, it is necessary that $\alpha<1$. Hence necessary conditions for such stable solutions to occur in the first quadrant are $\alpha<1$ and $R_i>0$, $i=1,2$. The conditions are equivalent, respectively, to

$$(a_1-1)(a_2-1)<(2+d_1)(2+d_2) \tag{46}$$

and

$$a_i>1-(2+d_i)(A_i/A_j), \quad i,j=1,2, \quad i\neq j. \tag{47}$$

Inequality (47) places a lower limit on the aggressiveness a firm must show in order to reach a balance with the opposition, corresponding to a positive output of its own product. Inequality (46) places an upper limit on the aggressiveness of the firms if a stable situation is ever to be reached. We note that if

$$a_i<1+(2+d_i)(A_i/A_j), \quad i,j=1,2, \quad i\neq j,$$

then (46) is satisfied. Hence a sufficient condition for any equilibrium point to occur in the first quadrant is

$$|a_i-1|<(2+d_i)(A_i/A_j), \quad i,j=1,2, \quad i\neq j. \tag{48}$$

5. Summary

We have investigated Bishop's suggestion that a duopolist, i, might adopt the strategy of selecting $x_i(t+1)$ to maximize $\pi_i(t+1) - a_i \pi_j(t+1)$, assuming that his rival, j, maintains an unchanged output $x_j(t+1) = x_j(t)$. Dynamic solutions for a duopoly situation have been found. Necessary conditions for stable equilibrium points to occur are $0 \leqslant k_1 + k_2 \leqslant 4$ and $\alpha < 1$, where

$$\alpha = (a_1 - 1)(a_2 - 1)/(2+d_1)(2+d_2).$$

These inequalities place upper limits on the speeds of adjustments and the aggressiveness factors respectively.

Sufficient conditions to ensure stability of the system are, first in the case of $(a_1 - 1)$ and $(a_2 - 1)$ having the same sign,

$$0 \leqslant k_1 + k_2 \leqslant 2 \quad \text{and} \quad 0 \leqslant (a_1 - 1)(a_2 - 1) < (2+d_1)(2+d_2).$$

In the case of $(a_1 - 1)$ and $(a_2 - 1)$ having opposite signs (that is $a_i < 1$ and $a_j > 1$), a sufficient condition to ensure stability is $k_i < 2(1-\alpha)^{-1}$. Hence, as the more aggressive player increases his aggressiveness, this sufficient condition requires that the k_i decrease towards zero.

To ensure that any equilibrium points that occur, do so in the first quadrant, it is sufficient that $|a_i - 1| < (2+d_i)(A_i/A_j)$.

When Fisher considered the Cournot solution of the oligopoly problem, he found that sufficiently small speeds of adjustment could always be found to produce equilibrium, regardless of the number of sellers. In addition, increasing marginal cost functions were always a stabilizing factor. For the model we have considered, increasing marginal costs are a stabilizing factor, but the effect of the speeds of adjustment varies with different a_i's. However, these speeds of adjustment or confidence factors have to exceed unity to affect the stability of the system. The notion of the a_i's is unique to the Bishop strategy. If either firm is too vindictive (a_i too large) stability will not occur.

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