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WHEN FLATS ARE TORSION FREE

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S. PAGE

Introduction. Given a complete Serre class τ this determines a torsion theory with τ the class of torsion modules. It also determines the torsion free modules. For the classical torsion in the category of abelian groups the torsion free modules are flat and visa-versa. Which rings are characterized by this property? More precisely: Which rings admit a torsion theory for which the concepts of torsion free and flat are equivalent? We also dispose of the cases when R admits a toısion theory for which torsion free is equivalent to injective and when projective is equivalent to torsion free.

To establish the notation, R will denote an associative ring with identity and ${}_R\mathcal{M}$ the category of unitary left R modules. Following Goldman, [2], by a torsion theory we will mean an idempotent kernel functor τ from ${}_R\mathcal{M}$ to ${}_R\mathcal{M}$ or equivalently an idempotent filter of left ideals $\mathcal{F}_\tau$. For τ a torsion theory and $M \in {}_R\mathcal{M}$ let $Q_\tau(M)$ be the corresponding quotient module. It is well known Q_τ is a functor, that $Q\tau(R)$ is a ring and there is a canonical ring homomorphism $R \to Q_\tau(R)$. If τ is a torsion theory such that Q_τ is exact and commutes with direct sums we say τ has property T.

2. **Injective projective torsion frees.** Suppose R is a semi-simple ring, i.e. a direct sum of simple left ideals. Then every module is injective so if we take the torsion theory corresponding to $\mathcal{F}=\{R\}$ injective and torsion free are equivalent. On the other hand, if torsion free is equivalent to injective, then every module is a submodule of a torsion free module, namely, its injective hull and so every module is torsion free, consequently $\mathcal{F}=\{R\}$ and every module over R is injective. This proves:

THEOREM. *Let R be a ring. Then R admits a torsion theory τ such that injective is equivalent to torsion free iff R is semi-simple.*

The case in which projective is equivalent to torsion free is almost as easy.

THEOREM. *Let R be a ring. Then R admits a torsion theory for which projective is equivalent to torsion free iff R is a left perfect, left hereditary and right coherent ring.*

Proof. If R is left perfect and left hereditary right coherent, let $\tau(M)=\{m/ m \in \ker f$ for all $f \in \operatorname{Hom}(M, R)\}$. Now if $\tau(m)=0$ then M imbeds in a product

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of copies of R and since R is left perfect and right coherent a product of copies of R is projective, [1]. R being hereditary forces M to be projective. If M is projective, M is a direct summand of a free module which is torsion free so M is torsion free, so projective is equivalent to torsion free.

Now suppose torsion free is equivalent to projective. Then $\tau(R)=0$ and all submodules of projective modules are projective, hence R is left hereditary. Since $\tau(R)=0$ and any direct product of torsion free modules is torsion free, any direct product of copies of R is projective so by [1] R is left perfect and right coherent.

3. **Torsion free equivalent to flat.** We now turn to the more interesting case of when torsion free and flat are equivalent. For this case we can only give a complete solution when the torsion theory τ has property T.

THEOREM *Let τ be a torsion theory with property T and such that R imbeds in $Q_\tau(R)$. Then torsion free is equivalent to flat if and only if* (i) w.gl.dim. $(R)\leq 1$ (ii) *$Q_\tau(R)$ is a Von Newman Regular ring*, (iii) *$Q_\tau(R)$ is left flat as an R module.*

Proof. If τ has torsion free equivalent to flat, then $Q_\tau(R)$ is left flat. Since τ has property T every $Q_\tau(R)$ module is torsion free, hence R-flat. But R-flat implies $Q_\tau(R)$ flat because $Q_\tau(R)\otimes_R Q_\tau(R)\cong Q_\tau(R)$ for $Q_\tau(R)\otimes_R Q_\tau(R)$ is also τ-torsion free; hence for M a $Q_\tau(R)$ module, $M\cong Q_\tau(R)\otimes_{Q_\tau(R)} M\cong Q_\tau(R)\otimes_R Q_\tau(R)\otimes_{Q_\tau(R)} M\cong Q_\tau(R)\otimes_R M$. So we have $Q_\tau(R)$ is Von Neuman regular. Since submodules of torsion free modules are torsion free, w.gl.dim $R\leq 1$.

Now suppose τ and R satisfy (i), (ii) and (iii). If A is any torsion free R-module, then A imbeds in $Q_\tau(A)$ which is $Q_\tau(R)$ flat. We claim $Q_\tau(R)$ flat implies R-flat. By (i) $Q_\tau(R)$ is left flat. If M is a $Q_\tau(R)$ flat module, then it is a filtered direct limit of $Q_\tau(R)$ projective modules which are R-flat, so M is R-flat. Since A imbeds in $Q(A)$, A must be flat by (ii). If A is R-flat, then the sequence $0\to R\otimes A\to Q\otimes A$ is exact, and therefore A imbeds in $Q_\tau(R)\cong Q\otimes A$, so is torsion free.

We now give some examples to show in the above (i), (ii), (iii) are independent.

(A) Let R be $K[x_1, x_2, \ldots, x_n]$, the ring of polynomials in n indeterminates over a field K. Then w.gl.dim $R=n$, so (i) is not satisfied if $n>1$. If τ is the torsion theory given by annihilation by any nonzero element of R, $Q_\tau(R)$ is the quotient field of R hence is Von Neuman regular and is also flat for τ has property T.

(B) Let K be a field and $\alpha: K\to K$ a ring map such that image $\alpha\neq K$. Let $R=K[x]$ with $xh=\alpha(h)x$. Now R is a left principal ideal domain and so l.gl. $R=1$. The left classical quotient ring is a division ring. Let τ be the torsion theory yielding this classical quotient ring. Now the classical quotient ring cannot be left R-flat for if it were, it would also be a right classical quotient ring, (the embedding of R into Q_τ is already an epimorphism) and R does not have the right ore condition:

(C) Let R be any non Von Neuman regular ring with $\tau(\mathscr{M})=0$ for all R-modules M. Then $Q_\tau(R)=R$.

COROLLARY. *Let R be a commutative integral domain. Then flat is equivalent to torsion free if and only if* w.gl.dim $R \leq 1$.

COROLLARY. *If R is a left and right semi-prime goldie ring with* w.gl dim $R \leq 1$, *then R is coherent (both left and right).*

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