A MODEL OF THE REAL NUMBERS

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The model of the real numbers described below was suggested by the fact that each irrational number ρ determines a linear ordering of J^2, the additive group of ordered pairs of integers. To obtain the ordering, we define $(m, n) \leq (m', n')$ to mean that $(m'-m)\rho \leq n' - n$. This order is invariant with group translations, and hence is called a "group linear ordering". It is completely determined by the set of its "positive" elements, in this case, by the set of integer pairs (m, n) such that $(0, 0) \leq (m, n)$, or, equivalently, $m\rho < n$. The law of trichotomy for linear orderings dictates that only the zero of an ordered group can be both positive and negative. Hence, if ρ is rational, the above construction gives, not a group linear ordering, but only a group quasi-ordering, in which all of the integer pairs (m, n) with $m\rho = n$ are both positive and negative. Modifying the above definition of order to

$$(\alpha) \qquad (0, 0) \leq (m, n) \Longleftrightarrow \begin{array}{l} m\rho < n \ \text{ or} \\ m\rho = n \ \text{ with } \ m \geq 0 \end{array}$$

produces the same ordering as before when ρ is irrational and also produces a group linear ordering when ρ is rational.

The foregoing considerations suggest that it is possible to define postulationally, with no reference to any real numbers, a set, Σ, of linear orderings of J^2, and to define on Σ an order, an addition, and a multiplication so that Σ becomes a complete ordered field and hence a model of the real numbers. This is precisely how the model of the real numbers with which this paper deals is to be constructed.

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Hereafter let small greek letters α, β, γ, ... denote the elements of J^2, reserving θ for $(0, 0)$. Let small latin letters m, n, k, l, a, b, ... denote integers, and let us agree that m with or without subscripts and primes is always a positive integer and, in particular, never represents 0. We shall indicate binary relations over J^2 by capital latin letters R, S, T, ...

The α such that $\theta \; R \; \alpha$ are called R-positive, and the set of them is denoted by K_R^+. Note that this means that if R is reflexive, $\theta \in K_R^+$. But the term "positive" applied to an integer shall have its usual meaning. It is well known that when R is a partial ordering of a group, it is completely determined by K_R^+.

The integer pairs (k, n) which are positive according to the lexicographic ordering of J^2 form an additive semi-group which we shall refer to as K^+. Thus

$$K^+ = \{(m, n)\} \cup \{(0, m)\} \cup \{\theta\} .$$

Let us define a class $\Sigma = \{R, S, T, \ldots\}$ of binary relations over J^2. Seven postulates will be required. The first four simply say that the elements of Σ shall be group linear orderings of J^2, but this class is far too wide to serve as a model of the real numbers, and postulates 5, 6, and 7 restrict Σ, in ways to be explained below, so that it may become a complete ordered field.

DEFINITION 1

$R \in \Sigma$ if and only if R is a binary relation over J^2 with the properties:

1. $\theta \; R \; \theta$

2. $\theta \; R \; \alpha$ and $\theta \; R \; \beta \implies \theta \; R \; (\alpha + \beta)$

3. $\alpha \neq \theta \implies$ exactly one of $\theta \; R \; \alpha$ or $\theta \; R(-\alpha)$

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4. $\alpha \, R \, \beta \Longleftrightarrow \theta \; R(\beta - \alpha)$

5. $\theta \; R(0, 1)$

6. If (m, n) is incomparably small, then $\theta \; R(m, n)$

7. Integers n and n' exist such that

$$(1, n) R \, \theta \; R(1, n') .$$

To explain postulate 5, when two relations R and Q have the property $\alpha \, R \, \beta \Longleftrightarrow \beta \, Q \, \alpha$, we say that R is dual to Q. Clearly when R is dual to Q, then Q is dual to R, and the totality of binary relations on a given set may be classified into dual pairs. What postulate 5 does is to include in Σ just one relation from each dual pair.

Explanation of postulate 6 requires a discussion of non-archimedean orderings and incomparably small elements. A partial ordering R is called archimedean if whenever $a \, R \, b$ and a is non-zero there is an integer n such that $b \, R \, na$. Thus under a non-archimedean ordering R there are non-zero elements a, b such that $na \, R \, b$ for $n = 0, \pm 1, \pm 2, \ldots$. An element a with this property is called "incomparably small with respect to b". Since $0 . a \, R \, b$, b is necessarily R-positive. We say that "a is incomparably small" if there exists b, with respect to which a is incomparably small. Note that a is incomparably small if and only if every positive and negative integral multiple of a is incomparably small. The reader will have no difficulty in establishing that the orderings defined by (α) on page 239 are archimedean if ρ is irrational and non-archimedean if ρ is rational. In the latter case the elements (m, n) with $m\rho = n = 0$ are incomparably small when compared with, for example, $(0, 1)$. This observation is only a special case of

THEOREM 1. If R is a relation on J^2 satisfying postulates 1 to 5, and (m, n) is incomparably small, then (a, b) is incomparably small $\Longleftrightarrow mb = an$.

Let us first remark that there is no loss in generality in taking the incomparably small element in the special form (m, n), for (m, n) is incomparably small if and only if $(-m, -n)$ is.

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We shall use the following lemmas in the proof of theorem 1.

Lemma 1. $\alpha R \beta \Longleftrightarrow m\alpha R m\beta$. That $\alpha R \beta \Longrightarrow m\alpha R m\beta$ is obvious from postulates 4 and 2, using induction on m. The opposite implication is easily proved by assuming that $\alpha \not R \beta$.

Lemma 2. Let α be not of the form $(0, n)$. Then α is incomparably small $\Longleftrightarrow k\alpha R (0, 1)$ for each k.

We can take α in the form (m, n), since $\alpha \neq (0, n)$.

Let $\alpha = (m, n)$ be incomparably small and suppose that

$$h(m, n) R (p, q) \text{ for each } h. \quad \therefore \theta R (p, q).$$

Putting $h = p$, we obtain, using $m > 0$,

$$(mp, np) R (p, q) R (mp, mq),$$

$$\therefore \theta R (0, mq - np).$$

Hence, by postulate 5, $0 \leq mq - np$. Next we take $h = p + k[mq - np + 1]$ and obtain $(mp + k[mq - np + 1]m, np + k[mq - np + 1]n) R (p, q)$. Because $m > 0$, $(p, q) R m(p, q) R (mp, mq + 1)$

$$\therefore (k[mq - np + 1]m, k[mq - np + 1]n - [mq - np + 1]) R \theta,$$

$$[mq - np + 1] (km, kn - 1) R \theta$$

$$(km, kn - 1) R \theta$$

$$k(m, n) R (0, 1).$$

Since this is valid for any integer k, it follows that $k\alpha R (0, 1)$ for each k. The opposite implication is obvious.

To prove the theorem, let us first suppose that $mb = an$. We know that $k(m, n) R (0, 1)$ for each k. Hence

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$$ha\,(m,n)\;R\;(0,1)\;R\;(0,m) \quad \text{for each } h\,.$$

$$\therefore\; mh(a,b)\;R\;m(0,1) \quad \text{for each } h\,,$$

$$h(a,b)\;R\;(0,1) \quad \text{for each } h\,,$$

proving that (a,b) is incomparably small.

Conversely, suppose that $t = mb - an \neq 0$. Let h be an integer and calculate

$$h[m(a,b) - t(0,1)] = ha(m,n)\;R\;(0,1)\,,$$

since (m,n) is incomparably small. Hence $hm(a,b)\;R\;(ht+1)(0,1)$. Since $t \neq 0$, we can choose h so that $ht + 1 < 0$. Then $-(ht+1) > 0$, and $(0,1)\;R\;[-(ht+1)(0,1)]\;R\;[-hm(a,b)]$, proving that (a,b) is not incomparably small. The proof of theorem 1 is complete.

Theorem 1 establishes that if we display the elements of J^2 as the points in the cartesian plane which have integer co-ordinates, then, given R, all incomparably small elements, if there be any, lie on a straight line through the origin. The points on this line which lie on the same side of the origin are, by postulate 2, all R-positive together or all R-negative together. The effect of postulate 6 is to choose the incomparably small elements in the right half plane to be R-positive.

All points above the line through the incomparably small elements correspond to R-positive elements, while those below correspond to R-negative ones. More precisely:

THEOREM 2. If R is a relation on J^2 satisfying postulates 1 to 5, and if (m,n) is incomparably small while (p,q) is not, then $\theta\;R\;(p,q) \Longleftrightarrow mq - pn > 0$.

First suppose that $mq - pn > 0$. Then

$$\theta\;R\;(0, mq - pn - 1)\,.$$

Since $-p(m,n)\;R\;(0,1)$,

$$(0,-1)\;R\;(mp, mq - 1)$$

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$$\theta \ R \ m(p, q)$$

$$\theta \ R \ (p, q)$$

as we wish to show. The opposite implication is proved by showing that $mq - pn < 0$ implies $\theta \ R[-(p, q)]$.

Before discussing postulate 7, let us investigate the set $\bar{\Sigma}$ of orderings defined by postulates 1 to 6. After introducing some useful notation, we shall show that $\bar{\Sigma}$ is a linearly ordered complete lattice.

It is well known that R is completely determined by the set $K_R^+ = \{\alpha \mid \theta \ R \ \alpha\}$. Because $\theta \ R(a, b)$ if and only if $(-a, -b) \ R \ \theta$, we can even say that R is completely determined by the set $R^* = K_R^+ \cap K^+$, for R^* determines K_R^+. (For the definition of K^+, see p. 240.) It should be appreciated that, if R yields incomparably small elements, R^* contains just the ones with the first components greater than zero; that R^* is never void, since it always contains $\{(0, n) \mid n \geq 0\}$; and that R^* is simply the totality of the (m, n) which are in the relation $\theta \ R \ (m, n)$ together with $\{(0, n) \mid n \geq 0\}$.

DEFINITION 2

$$R \leq S \iff S^* \subset R^* .$$

Given R and $S \in \bar{\Sigma}$, either $R \leq S$ or $S \leq R$. To prove this, let us suppose that R^* contains an element (m, n) which is not in S^*. Then if $(m', n') \in S^*$, we have $(m, n) \ S \ \theta \ S \ (m', n')$.

Therefore

$$(mm', m'n) \ S \ \theta \ S \ (mm', mn') ,$$

$$\theta \ S \ (0, mn' - m'n) ,$$

$$0 \leq mn' - m'n ,$$

$$\theta \ R \ (0, mn' - m'n) .$$

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But $\theta \; R \; (mm', m'n)$.

Therefore $\theta \; R \; m(m', n')$

$\theta \; R \; (m', n')$

$(m', n') \in R^*$

$S^* \subset R^*$,

and this proves that given $R, S \in \bar{\Sigma}$, either $R^* \subset S^*$ or $S^* \subset R^*$ which is what we wish to prove.

Hence the relations in $\bar{\Sigma}$ are linearly ordered. Because they are linearly ordered, any finite set of them has as a least upper bound a relation which is in $\bar{\Sigma}$. We wish to show that any subset whatever of $\bar{\Sigma}$ has a least upper bound in $\bar{\Sigma}$. Let Ω be such a set. Consider $\bigcap_{Q \in \Omega} Q^*$; we show that there exists a member of $\bar{\Sigma}$, say R , such that $R^* = \bigcap_{Q \in \Omega} Q^*$. R is defined as follows: let

$$K_R^+ = (\bigcap_{Q \in \Omega} Q^*) \cup \{ -(m,n) \mid (m,n) \notin \bigcap_{Q \in \Omega} Q^* \}$$

and define

$$\alpha \; R \; \beta \Longleftrightarrow \beta - \alpha \in K_R^+ .$$

Taking the postulates in turn:

1. $\theta \in Q^*$ for $Q \in \Omega$. $\therefore \theta \in \bigcap_{Q \in \Omega} Q^*$. $\therefore \theta \in K_R^+$

$$\therefore \theta \; R \; \theta .$$

2. Let $\theta \; R \; \alpha$, $\theta \; R \; \beta$.

Let $R^{**} = \{ -(m,n) \mid (m,n) \;\; \bigcap_{Q \in \Omega} Q^* \}$.

$Q^{**} = \{ -(m,n) \mid (m,n) \notin Q \}$.

Observe that the following are equivalent:

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(i) $\gamma \in \bigcap_{Q \in \Omega} Q^*$

(ii) $\gamma \in Q^*$ for all $Q \in \Omega$

(iii) There is a $Q_o \in \Omega$ such that $\gamma \in Q^*$ for all $Q \geq Q_o$, $Q \in \Omega$.

Also note that

(iv) $\gamma \in R^{**}$

is equivalent to

(v) There is a $Q_o \in \Omega$ such that $\gamma \in Q^{**}$ for all $Q \geq Q_o$, $Q \in \Omega$.

Hence it follows that

$$\gamma \in K_R^+$$

is equivalent to

There is a $Q_o \in \Omega$ such that $\gamma \in K_Q^+$ for all $Q \geq Q_o$, $Q \in \Omega$.

Let $Q_1 \in \Omega$ such that $Q_1 \leq Q \in \Omega$ implies $\alpha \in K_Q^+$

Let $Q_2 \in \Omega$ such that $Q_2 \leq Q \in \Omega$ implies $\beta \in K_Q^+$

Let Q_o = the greater of Q_1, Q_2. Then $Q_o \leq Q \in \Omega$ implies $\alpha + \beta \in K_Q^+$.

Therefore $\alpha + \beta \in K_R^+$, as we wish to prove.

3, 4, 5. That R satisfies postulates 3, 4, and 5 is obvious from the definition of R in terms of K_R^+.

6. Suppose that (m, n) is incomparably small according to R. Since

$$0 < m = m(m'n+1) - m'mn,$$

theorem 2 implies

$$\theta \; R \; (m'm, \; m'n+1)$$

$$\therefore (m'm, m'n+1) \in \bigcap_{Q \in \Omega} Q^{*}$$

$$(m'm, m'n+1) \in Q^{*} \text{ for } Q \in \Omega$$

$$-m' \; (m,n) \; Q \; (0,1) \text{ for } Q \in \Omega \text{ and every } m'.$$

Now let us consider the positive multiples of (m, n). Let $Q \in \Omega$. Either

(i) $m'(m,n) \; Q \; (0,1)$ for every m'

or (ii) $(0,1) \; Q \; m'(m,n)$ for some m'.

In case (i), (m, n) is incomparably small according to Q. Since $Q \in \overline{\Sigma}$, it satisfies postulate 6.

$$\therefore \theta \; Q \; (m,n).$$

In case (ii), it follows from postulate 5 that $\theta \; Q \; m' \; (m,n)$. Hence by lemma 1 of the proof of theorem 1,

$$\theta \; Q \; (m,n).$$

Thus for $Q \in \Omega$, $\theta \; Q \; (m,n)$

$$\therefore (m,n) \in Q^{*} \text{ for } Q \in \Omega$$

$$\therefore \theta \; R \; (m,n).$$

This is postulate 6.

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Hence $R^* = \bigcap_{Q \in \Omega} Q^*$ is the star set of a relation, which we denote by R, satisfying postulates 1 to 6. That is, $R \in \bar{\Sigma}$. Clearly R is an upper bound to Ω. Suppose that S is an upper bound to Ω. Then $Q \leq S$ and $S^* \subset Q^*$ for all $Q \in \Omega$. Hence, $S^* \subset \bigcap_{Q \in \Omega} Q^* = R^*$ so that $R \leq S$. Hence R is the least upper bound to Ω.

We have proved

THEOREM 3. The relations on J^2 satisfying postulates 1 to 6 of definition 1 form a linearly ordered complete lattice.

Theorem 3 tells us that it will be impossible to define field operations over $\bar{\Sigma}$, for the real numbers form a conditionally complete, but not a complete, chain (linearly ordered lattice). In topological language, the real line is locally compact, but not compact. Our objective is to define a set Σ of relations on J^2 which can be made into a complete ordered field by means of appropriately defined operations. In connection with ordered groups, "complete" means "conditionally complete". We have framed axiom 7 in such a way as to exclude from Σ two elements of $\bar{\Sigma}$ which correspond to $+\infty$ and $-\infty$.

Having constructed a conditionally complete chain, we now turn our attention to defining field operations over it so that it becomes a complete ordered field.

If X, Y are understood to be elements of an order dense subfield of a complete ordered field, while A, B are field elements, then

$$A + B = \sup \{X \mid X \leq A\} + \sup \{Y \mid Y \leq B\} = \sup \{X + Y \mid X \leq A,\ Y \leq B\}$$

A similar multiplicative formula is valid when A, B, X and Y are all understood to be non-negative. It is this idea which we shall use to define the field operations over Σ.

Certain elements of Σ will figure largely in the definitions: let $T_{m,n} \in \Sigma$ be defined by

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$\theta\ T_{m,n}\ (p, q) \Longleftrightarrow 0 < mq - np$ or $0 = mq - np$ with $p \geq 0$.

(Cf. (α) and theorems 1 and 2.)

Since $T_{m,n} \leq T_{m',n'} \Longleftrightarrow T^*_{m',n'} \subset T^*_{m,n}$ $\Longleftrightarrow \theta\ T_{m,n}\ (m', n') \Longleftrightarrow 0 \leq mn' - m'n$, we have a useful rule for determining the order of two of these orderings:

$$T_{m,n} \leq T_{m',n'} \Longleftrightarrow 0 \leq mn' - m'n .$$

If Q, $R \in \Sigma$ with $Q < R$, then there is some (m, n) in Q^* but not in R^*. The elements $(m'm, m'n+1)$ obtained by varying m' are all in Q^* but not all in R^*, for, if they were, (m, n) would be incomparably small according to R, and hence in R^*. Therefore, for some value of m', $T_{m'm, m'n+1}$ is a proper subset of Q^* and a proper superset of R^*. That is, between any two distinct elements of Σ lies a $T_{m,n}$. Thus the $T_{m,n}$ are order dense in Σ, and any element of Σ is the supremum of an appropriately chosen set of $T_{m,n}$.

By postulate 7, integers n and n' exist such that $(1, n)\ R\ \theta\ R\ (1, n')$. Therefore $(m, mn)\ R\ \theta\ R\ (m, mn')$ for each m. That is, corresponding to each m, there are integers a and b such that $(m, a)\ R\ \theta\ R\ (m, b)$. This means that the sets

$$\{T_{m,n} \mid T_{m,n} < R\} \text{ and } \{T_{m,n} \mid T_{m,n} > R\}$$

are never void.

Since any set of integers bounded above (below) has a maximum (minimum), corresponding to each m is an n_m such that $(m, n_m)\ R\ \theta\ R\ (m, n_m + 1)$. The integers n_m will be used in the proofs below.

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We are now ready to define addition in Σ :

DEFINITION 3

Let $Q, R \in \Sigma$. The sum of Q and R, $Q + R$, is defined by

$$Q + R = \sup \{ T_{m, a+b} \mid T_{m,a} \leq Q , \ T_{m,b} \leq R \} .$$

According to this definition, Σ is closed with respect to addition. Because addition of integers is associative and commutative, this addition is too.

The reader may find it instructive to verify that

$$(Q + R)^{*} = \cap \{ T^{*}_{m, a+b} \mid (m, a) \, Q \, \theta , \ (m, b) \, R \, \theta , \ m = 1, 2, 3, \ldots \}$$

Our field must have a zero element. Applying definition 3,

$$Q + T_{1,0} = \sup \{ T_{m, a+b} \mid T_{m,a} \leq Q , \ T_{m,b} \leq T_{1,0} \} .$$

The values of b which occur corresponding to each m are simply the integers less than or equal to zero. Hence for each m, $a + b \leq a$, $T_{m, a+b} \leq T_{m,a}$, and $b = 0$ occurs. Thus $\sup \{ T_{m,a+b} \mid T_{m,a} \leq Q , \ T_{m,b} \leq T_{1,0} \} = \sup \{ T_{m,a} \mid T_{m,a} \leq Q \}$ or $Q + T_{1,0} = Q$. This fact justifies

DEFINITION 4

The zero of Σ, Z, is defined by:

$$Z = T_{1,0} .$$

It is easily verified that Z^{*} contains every (m, n) with $n \geq 0$ and no others.

DEFINITION 5

Let $R \in \Sigma$. The negative of R, $-R$, is defined by:

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$$-R = \sup \{ T_{m,-n} \mid \theta R(m,n) \} .$$

This definition is justified by the fact that $R + (-R) = Z$. To prove this we let n_m be the greatest integer n such that $T_{m,n} \leq R$ and n'_m the least integer n' such that $\theta R(m,n')$. For every m, $n_m - n'_m = 0$ or -1. If for some m, $n_m = n'_m$ then $T_{m,0} \in \{ T_{m,a+b} \mid T_{m,a} \leq R, \ T_{m,b} \leq -R \}$, while $a + b$ is always ≤ 0. But $T_{m,0} = Z$, and so in this case $Z = \sup \{ T_{m,a+b} \mid T_{m,a} \leq R, \ T_{m,b} \leq -R \}$. On the other hand, if for no m does $n_m = n'_m$, then $n_m - n'_m \equiv -1$ and $\sup \{ T_{m,a+b} \mid T_{m,a} \leq R, \ T_{m,b} \leq -R \} = \sup \{ T_{m,-1} \mid m = 1, 2, 3, \ldots \}$, the star set of which contains every (m,n) with $n \geq 0$ and no others, so that in this case also $Z = \sup \{ T_{m,a+b} \mid T_{m,a} \leq R, \ T_{m,b} \leq -R \}$. The proof that $R + (-R) = Z$ is complete.

Our definitions have the consequence that $Z \in \Sigma$ and that for all $R \in \Sigma$ we have $-R \in \Sigma$, $Z + R = R$, and $R + (-R) = Z$. Thus Σ has become an abelian group under addition.

Suppose that $Q, R, S \in \Sigma$ and that $Q \leq R$. Since

$$Q + S = \sup \{ T_{m,a+b} \mid T_{m,a} \leq Q, \ T_{m,b} \leq S \} ,$$

$$R + S = \sup \{ T_{m,a'+b} \mid T_{m,a'} \leq R, \ T_{m,b} \leq S \} ,$$

and since for each m the greatest a' occurring in the definition of $R + S$ is at least as large as any a occurring in the definition of $Q + S$, therefore

$$Q + S \leq R + S .$$

We have proved

THEOREM 4. The ordering of Σ defined by $Q \leq R \Longleftrightarrow R^* \subset S^*$ is invariant under group translations.

In the first stage of defining multiplication we confine ourselves to $\Sigma^+ = \{R \mid R \in \Sigma, Z \leq R\}$. Note that $Z \in \Sigma^+$.

DEFINITION 6

Let $Q, R \in \Sigma^+$.

The product of Q and R, QR, is defined by

$$QR = \sup \{T_{mm', nn'} \mid Z \leq T_{m,n} \leq Q, Z \leq T_{m',n'} \leq R\} .$$

It may be verified that if Q and $R \neq Z$,

$$(QR)^* = \cap \{T^*_{mm', nn'} \mid (m, 0)\ Q\ (m, n)\ Q\ \theta\ ;\ (m', 0)\ R\ (m, n)\ R\ \theta\ ;$$
$$m, m' = 1, 2, 3, \ldots\} .$$

The definition of multiplication in Σ is completed by the usual conventions of ring multiplication:

DEFINITION 7

If $Q, R \in \Sigma$,

$$(-Q)R = Q(-R) = -((-Q)(-R)) = -(QR) .$$

One of the quantities of definition 7 can always be evaluated using definition 6.

Because multiplication of integers is commutative and associative, this multiplication is too. Note that $RZ = Z$ for $R \in \Sigma$.

Clearly the product of two elements of Σ^+ is in Σ^+. This is a form of the ordered field postulate. In the presence of theorem 4 it is equivalent to another form with which the reader may be more familiar:

$$Q \leq R \text{ and } Z \leq S \Longrightarrow QS \leq RS .$$

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Let $R \in \Sigma^+$ and consider

$$RT_{1,1} = \sup \{ T_{mm',nn'} \mid T_{m,n} \leq R,\ T_{m',n'} \leq T_{1,1} \}.$$

For each m', the values taken by n' in the definition of $RT_{1,1}$ are $\leq m'$ and the value m' is taken. Hence

$$\sup \{ T_{mm',nn'} \mid T_{m,n} \leq R,\ T_{m',n'} \leq T_{1,1} \}$$

$$= \sup \{ T_{m,n} \mid T_{m,n} \leq R \} ,$$

proving that $R\,T_{1,1} = R$. This discussion justifies:

DEFINITION 8

The unity element of Σ, U, is defined by:

$$U = T_{1,1}$$

To define the reciprocals of non-zero elements, we at first confine ourselves to Σ^+.

DEFINITION 9

Let $R \in \Sigma$, $Z < R$. The reciprocal of R, R^{-1}, is defined by:

$$R^{-1} = \sup \{ T_{n,m} \mid \theta\ R\,(m,n)\} .$$

We now prove that $RR^{-1} = U$. We first show that $RR^{-1} \leq U$. Corresponding to each m, let n_m be an integer such that $(m, n_m)\ R\ \theta\ R\ (m, n_m + 1)$. Then

$$\theta\ R\,(-m'm, -m'n_m)$$

and

$$\theta\ R\,(mm', m(n_{m'}+1))$$

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$$\therefore \quad \theta\, R\,(0, mn_{m'} + m - m'n_m)\,.$$

$$\therefore \quad 0 \leq mn_{m'} + m - m'n_m\,.$$

Now $RR^{-1} = \sup \{ T_{mn', m'n} \mid T_{m,n} \leq R,\ \theta\, R\,(m', n') \}$. Take n in the form $n_m - h$ and n' in the form $n_{m'} + k$ where $0 \leq h$ and $1 \leq k$. Hence $mn' - m'n = mn_{m'} + km - m'n_m + hm' \geq mn_{m'} + m - m'n_m \geq 0$, by the result proved above. Thus $mn' \geq m'n$, and every $T_{mn', m'n}$ is $\leq T_{1,1}$.

We next show that among the $T_{mn', m'n}$ are ones which are arbitrarily close to $T_{1,1}$. Since $(m, n_m)\, R\, \theta\, R\,(m, n_m + 1)$,

$$T_{m, n_m} \leq R \text{ and } \theta\, R\,(m, n_m + 1)$$

$$T_{m(n_m + 1), mn_m} \in \{ T_{mn', m'n} \mid T_{mn} \leq R,\ \theta\, R\,(m', n') \}$$

$$T_{n_m + 1,\, n_m} \in \{ T_{mn', m'n} \mid T_{m,n} \leq R,\ \theta\, R\,(m'n') \}$$

for $m = 1, 2, 3, \ldots$.

It is not difficult to see that since $Z < R$, the n_m increase without limit as m increases, and consequently

$$T_{1,1} = \sup \{ T_{mn', m'n} \mid T_{m,n} \leq R,\ \theta\, R\,(m'n') \}\,.$$

The proof that $Z < R$ implies $RR^{-1} = U$ is complete.

If $R \neq Z$, one of R^{-1}, $(-R)^{-1}$ is defined. We use this fact to extend the definition of reciprocal to all non-zero elements of Σ:

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DEFINITION 10

Let $R \in \Sigma$, $R \neq Z$. The reciprocal of R, R^{-1}, is defined by

$$R^{-1} = -((-R)^{-1}) .$$

Using definitions 9 and 10, we see that even when $R < Z$, we have $RR^{-1} = U$.

We now have $U \in \Sigma$ and for all $R \neq Z$ in Σ, $R^{-1} \in \Sigma$, $RU = R$, $RR^{-1} = U$. Thus the non-zero elements of Σ have become an abelian group under multiplication.

Using the distributive law for integers in applying definitions 3 and 6 to $S(Q + R)$ and to $SQ + SR$, we see that they are the same elements of Σ.

Thus Σ together with the operations of addition and multiplication, the zero and unit elements, and the order relation of definition 2 is a complete ordered field, and is, therefore, a model of the real numbers.

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