MATHEMATICAL NOTES

A Review of Elementary Mathematics and Science.

On the Formula $\rho = h^2 u^2 (u + u_2)$.

This is one of the most important results in dynamics. Candidates for examination always expect a question on "Central Orbits," and they therefore learn the proof of this formula very thoroughly. There is no doubt, however, that very few students realise the physical meaning of the result. It is, at best, a magic formula which "gets out" examples.

The fact is that the proof usually given is one least conducive to the formation of some physical conception of the meaning of the formula. The process given in the text-books is one of elimination of the time between the equations

$$\ddot{r} - r\dot{\theta}^2 = -\rho; \quad r^2\dot{\theta} = h.$$

Now only the perfect mathematician could see through every step in a process of elimination. Most of us are satisfied, and with justice, to apply the processes of pure mathematics in an automatic fashion. The student, however, should be encouraged to appreciate the significance of each step in any argument he has occasion to use, and not to trust overmuch to mechanical manipulation.

The formula $\rho = h^2 u^2 (u + u_2)$ has an important physical meaning. It states that the central force is a measure of the amount of divergence of the actual path of the particle from the tangent at any point, *i.e.* from what the path would be if the force ceased to act. To make this clear, especially to physicists, who are doing applied mathematics for use in their physical studies, the following proof seems to be preferable.

Let the orbit have the polar equation

$$u = u(\theta).$$

Take a point (u, θ) on the orbit. The tangent has the polar equation

$$U = u\cos(\Theta - \theta) + u_1 \sin(\Theta - \theta),$$

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U, Θ being current coordinates on the tangent. After time δt, let the angle θ become $\theta + \delta\theta$. The inverses of the radii vectores to the curve and to the tangent are now

$$u + u_1 \delta\theta + u_2 (\delta\theta)^2/2$$

and $$u(1 - (\delta\theta)^2/2) + u_1 \delta\theta$$

to the second order of small quantities. Thus, the distance between the curve and the tangent in the direction $\theta + \delta\theta$ is, to the same approximation,

$$(u + u_2)(\delta\theta)^2/2u^2.$$

This represents the distance moved in the time δt under an acceleration ρ towards the centre of force, the initial velocity in this direction being zero.

Hence $$(u + u_2)(\delta\theta)^2/2u^2 = \rho(\delta t)^2/2,$$

i.e. $$\rho = (u + u_2)(d\theta/dt)^2/u^2.$$

But $$d\theta/dt = hu^2.$$

Thus we get $$\rho = h^2 u^2 (u + u_2).$$

Routh ("Dynamics of a Particle," p. 199) mentions the fact that $(u + u_2)$ indicates the convexity or concavity of a curve. It seems to me that the method of proof here given is as short as the one generally given, and has the advantages of being really intelligible to any student, and of indicating clearly the underlying dynamical principles.

S. Brodetsky.

Elementary Proof of the Formula $\frac{V^2}{R}$.

Let O be the centre of a regular polygon $ABCD$, round the perimeter of which a point P moves with uniform speed V.