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A Multiplicative Analogue of Schur's Tauberian Theorem

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Abstract. A theorem concerning the asymptotic behaviour of partial sums of the coefficients of products of Dirichlet series is proved using properties of regularly varying functions. This theorem is a multiplicative analogue of Schur's Tauberian theorem for power series.

A great workhorse of asymptotic enumeration is a theorem first given by Schur in [10] in 1918. It states:

Theorem 1 *Let* $\mathbf{S}(x) = \sum_{n\geq 0} s(n)x^n$ *and* $\mathbf{T}(x) = \sum_{n\geq 0} t(n)x^n$ *be two power series such that for some* $\rho \geq 0$

1. $\lim_{n\to\infty} \frac{t(n-1)}{t(n)} = \rho$,
2. $\mathbf{S}(x)$ *has radius of convergence greater than* ρ.

Let $r(n) = \sum_{i+j=n} s(i)t(j)$. *Then*

$$\lim_{n\to\infty} \frac{r(n)}{t(n)} = \mathbf{S}(\rho).$$

This theorem appears in [9] as Exercise 178 in Chapter 4 of Part I. With complex argument and complex coefficients it appears as Theorem 2 of [2] and Theorem 7.1 of [8].

A central thesis of Burris' book [4] is that there is a remarkably simple procedure to translate theorems in additive number theory into theorems in multiplicative number theory. However, Burris in [4] does not provide a true multiplicative analogue to Schur's Theorem under this translation, only an analogue weakened by an additional hypothesis; nor has a true multiplicative analogue been formulated elsewhere. One specialised version will be discussed later. The goal of this paper is to provide a true analogue of Schur's theorem under Burris' translation.

In this context the aforementioned translation procedure entails replacing the ratio test condition, $\lim_{n\to\infty} t(n-1)/t(n) = \rho$, with the regular variation condition, $\lim_{x\to\infty} T(xy)/T(x) = y^\alpha$ for $y > 0$, where $T(x) = \sum_{n\leq x} t(n)$ and T is eventually positive, and replacing power series with Dirichlet series. For this theorem the eventual positivity is not needed. Applying the translation we get the following statement:

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Theorem 2 *Given $\alpha \in \mathbb{R}$, let $\mathbf{S}(x) = \sum_{n\geq 1} s(n)n^{-x}$, $\mathbf{T}(x) = \sum_{n\geq 1} t(n)n^{-x}$ be two Dirichlet series with t real valued, and let $T(x) = \sum_{n\leq x} t(n)$. Suppose*

1. $\lim_{x\to\infty} \frac{T(xy)}{T(x)} = y^\alpha$ *for* $y > 0$,
2. $\mathbf{S}(x)$ *has abscissa of absolute convergence less than* α.

Let $r(n) = \sum_{i\cdot j=n} s(i)\cdot t(j)$[1] and $R(x) = \sum_{n\leq x} r(n)$. Then

$$\lim_{x\to\infty} \frac{R(x)}{T(x)} = \mathbf{S}(\alpha).$$

Burris's weakened analogue (Theorem 9.53, [4]) has the additional hypothesis $t(n) \geq 0$. We will use the following uniform convergence theorem for functions of regular variation along with some lemmas to prove a still more general theorem from which Theorem 2 follows as an immediate corollary.

***Theorem 3* (Uniform Convergence)** *If $f\colon [1,\infty) \to \mathbb{R}$ is measurable and eventually positive, and $\lim_{x\to\infty} f(xy)/f(x) = y^\alpha$ for $y > 0$, then $\lim_{x\to\infty} f(xy)/f(x) = y^\alpha$ uniformly for $y \in [a,b]$ with $0 < a < b < \infty$.*

This is a standard regular variation result. It appears as Theorem 1.3 of [5] and follows from Theorem 1.5.2 of [3].

Lemma 4 *If $\lim_{x\to\infty} f(xy)/f(x) = y^\alpha$ for $y > 0$ and $f\colon [1,\infty) \to \mathbb{R}$ is left or right continuous at every point, then f is eventually positive or eventually negative.*

Proof Let f satisfy the hypotheses; clearly f is eventually nonzero. Pick N large enough that $f(2x)/f(x) > 0$ and $f(3x)/f(x) > 0$ for $x \geq N$. Take $x, y \geq N$; since f is left or right continuous at y there is an interval $[a,b]$, $a \neq b$, containing y on which f always has the same sign. Choose positive integers k and ℓ such that $3^k x/2^\ell \in [a,b]$. This is possible since numbers of the form $3^k/2^\ell$ for positive integers k and ℓ are dense in $[1,\infty)$. Then

$$\frac{f(3^k x/2^\ell)}{f(x)} = \frac{f(3^k x/2^\ell)}{f(3^k x)}\frac{f(3^k x)}{f(x)} > 0.$$

So f is eventually positive or eventually negative. ∎

Lemma 5 *If $f\colon [1,\infty) \to \mathbb{R}$ is measurable, eventually positive, and bounded on any interval $[1,x)$, and $\lim_{x\to\infty} f(xy)/f(x) = y^\alpha$ for $y > 0$, then for any $\gamma < \alpha$ there exist constants M and C such that*

$$\frac{|f(x)|}{f(y)} \leq C(x/y)^\gamma, \quad \text{for } y \geq M \text{ and } 1 \leq x \leq y.$$

[1] That is, $\mathbf{R}(x) = \sum_{n\geq 1} r(n)n^{-x} = \mathbf{S}(x) * \mathbf{T}(x)$ where $*$ is the Dirichlet product.

Proof Choose $M_0 \geq 1$ such that, for $x \geq M_0$, $f(x) > 0$ holds as well as

$$\frac{f(x)}{f(2x)} < 2^{-\gamma}. \tag{1}$$

Now, for $\frac{1}{2} < u \leq 1$, $f(yu)/f(y)$ approaches u^α uniformly as $y \to \infty$. So pick $M \geq M_0$ such that for $y \geq M$ and $u \in (\frac{1}{2}, 1]$ we have

$$\frac{f(yu)}{f(y)} \leq u^\alpha + 1 \leq u^\gamma + 1. \tag{2}$$

Note that $f(x)$ is positive on $[M, \infty)$.

Take $y \geq M$ and $1 \leq x \leq y$. Suppose $x \geq M$. Then

$$\frac{|f(x)|}{f(y)} = \frac{f(x)}{f(y)} = \frac{f(x)}{f(2x)} \cdots \frac{f(2^{m-1}x)}{f(2^m x)} \frac{f(2^m x)}{f(y)},$$

where $2^m x \leq y < 2^{m+1}x$. Let $u = 2^m x/y$; then $u \in (\frac{1}{2}, 1]$. By (1) and (2)

$$\begin{aligned} \frac{|f(x)|}{f(y)} &\leq (2^{-\gamma})^m (u^\gamma + 1) \\ &= 2^{-\gamma m} u^\gamma + (2^{-\gamma})^m \\ &= (x/y)^\gamma + (2^{-\gamma})^m. \end{aligned}$$

Now $\log_2(y/x) - 1 < m \leq \log_2(y/x)$; so if $\gamma \geq 0$

$$\frac{|f(x)|}{f(y)} \leq (x/y)^\gamma + (2^{-\gamma})^{\log_2(y/x)-1} = (1 + 2^\gamma)(x/y)^\gamma,$$

and if $\gamma < 0$

$$\frac{|f(x)|}{f(y)} \leq (x/y)^\gamma + (2^{-\gamma})^{\log_2(y/x)} = 2(x/y)^\gamma.$$

Now suppose $x < M$. Since $f(x)$ is bounded on $[1, M)$ there exists an $M_1 \geq 1$ such that $|f(x)|/f(M) \leq M_1$ for $1 \leq x < M$. We know

$$\frac{|f(x)|}{f(y)} = \frac{|f(x)|}{f(M)} \frac{f(M)}{f(y)};$$

so if $\gamma \geq 0$

$$\frac{|f(x)|}{f(y)} \leq M_1(2^\gamma + 1)(M/y)^\gamma \leq M_1(2^\gamma + 1)M^\gamma (x/y)^\gamma,$$

and if $\gamma < 0$

$$\frac{|f(x)|}{f(y)} \leq 2M_1(M/y)^\gamma \leq 2M_1(x/y)^\gamma.$$

Hence $C = \max\big(2M_1, M_1(1+2^\gamma)M^\gamma\big)$ works in all cases. ■

For the following theorem we will use general Dirichlet series of a particular form; namely series $\sum_{n\geq 1} s(n)\sigma_n^{-x}$ where $\{\sigma_n\}$ is an increasing positive sequence of real numbers such that $\sigma_n \to \infty$ as $n \to \infty$. General Dirichlet series are discussed in detail in [6].

Note that the Dirichlet product [6, Chapter VIII] of two such series is also such a series, since if $\sum_{n\geq 1} s(n)\sigma_n^{-x}$ and $\sum_{n\geq 1} t(n)\tau_n^{-x}$ are two such series then their Dirichlet product is the series $\sum_{n\geq 1}\sum_{\sigma_i\tau_j=\rho_n} s(i)t(j)\rho_n^{-x}$ where $\{\rho_n\}$ is the ascending sequence formed by all the values of $\sigma_i\tau_j$; so $\rho_n \to \infty$ as $n \to \infty$.

Theorem 6 *Given $\alpha \in \mathbb{R}$, let $\mathbf{S}(x) = \sum_{n\geq 1} s(n)\sigma_n^{-x}$, $\mathbf{T}(x) = \sum_{n\geq 1} t(n)\tau_n^{-x}$ be two general Dirichlet series of the above form where s and t are complex-valued, and let $T(x) = \sum_{\tau_n\leq x} t(n)$. Suppose*

1. *$T = bT^* + U$ where $0 \neq b \in \mathbb{C}$, $\lim_{x\to\infty} U(x)/T^*(x) = 0$, and T^* is real valued, left or right continuous at every point, and bounded on any interval $[1, x)$,*
2. *$\lim_{x\to\infty} \frac{T^*(xy)}{T^*(x)} = y^\alpha$ for $y > 0$,*
3. *$\mathbf{S}(x)$ has abscissa of absolute convergence less than α.*

Let $\{\rho_n\}$ be the ascending sequence formed by all the values of $\sigma_i\tau_j$ and let $r(n) = \sum_{\sigma_i\tau_j=\rho_n} s(i)\cdot t(j)$ and $R(x) = \sum_{\rho_n\leq x} r(n)$. Then

$$\lim_{x\to\infty} \frac{R(x)}{T(x)} = \mathbf{S}(\alpha).$$

Proof By replacing b by $-b$ if necessary and by Lemma 4 we can assume T^* is eventually positive.

Notice that T^* is measurable, since if we take an open set V then for every $v \in (T^*)^{-1}(V)$ there is an interval I_v containing v such that $T^*(I_v) \subseteq V$. For every rational $v \in (T^*)^{-1}(V)$ let $B_v = \bigcup_{x:v\in I_x} I_x$ which is an interval. Then $(T^*)^{-1}(V) = \bigcup_{v\in\mathbb{Q}\cap(T^*)^{-1}(V)} B_v$; so $(T^*)^{-1}(V)$ is measurable.

Pick M_0 such that $|U(y)/T^*(y)| < |b|/2$ for $y \geq M_0$. Let us redefine $T^*(x)$ to be 1 on $[1, M_0]$ and $U(x)$ to be $T(x) - b$ on $[1, M_0]$. Then the hypotheses of the theorem still hold and T^* remains measurable and eventually positive. Further $U(x)/T^*(x)$ is bounded on $[1, \infty)$, say by $M_2/|b|$, since it is bounded on (M_0, ∞) by the choice of M_0, $U(x)/T^*(x) = T(x) - b$ on $[1, M_0]$, and T is bounded on $[1, M_0]$.

Let α_s be the abscissa of absolute convergence of $\mathbf{S}(x)$, then $\alpha_s < \alpha$ by assumption. Choose γ such that $\alpha_s < \gamma < \alpha$. By Lemma 5 there exist constants $M_1 \geq M_0$ and C such that

$$\frac{|T^*(x)|}{T^*(y)} \leq C(x/y)^\gamma \quad \text{for } y \geq M_1 \text{ and } 1 \leq x \leq y,$$

and $T^*(y) > 0$ for $y \geq M_1$. For $y \geq M_1$ and $1 \leq x \leq y$,

$$\begin{aligned}\frac{|T(x)|}{|T(y)|} &= \frac{|T^*(x)|}{T^*(y)} \frac{|1 + U(x)/bT^*(x)|}{|1 + U(y)/bT^*(y)|} \\ &\leq C(x/y)^\gamma 2(1 + M_2) \\ &= C'(x/y)^\gamma\end{aligned}$$

where $C' = 2C(1 + M_2)$. Also

$$\lim_{x\to\infty} \frac{T(xy)}{T(x)} = \lim_{x\to\infty} \frac{T^*(xy)}{T^*(x)} \frac{\big(1 + U(xy)/bT^*(xy)\big)}{\big(1 + U(x)/bT^*(x)\big)} = y^\alpha.$$

From the triangle inequality with $x \geq M_1$,

$$\left|\mathbf{S}(\alpha) - \frac{R(x)}{T(x)}\right| \leq \underbrace{\left|\mathbf{S}(\alpha) - \sum_{\sigma_n \leq x} s(n)\sigma_n^{-\alpha}\right|}_{\text{I}} + \underbrace{\left|\sum_{\sigma_n \leq x} s(n)\sigma_n^{-\alpha} - \frac{R(x)}{T(x)}\right|}_{\text{II}}.$$

Clearly term I goes to 0 as $x \to \infty$. Thus it is sufficient to show that term II vanishes as $x \to \infty$. Now

$$\begin{aligned}R(x) &= \sum_{\rho_n \leq x} \sum_{\sigma_i\tau_j = \rho_n} s(i)t(j) = \sum_{\sigma_i\tau_j \leq x} s(i)t(j) \\ &= \sum_{\sigma_i \leq x} s(i) \sum_{\tau_j \leq x/\sigma_i} t(j) = \sum_{\sigma_n \leq x} s(n)T(x/\sigma_n).\end{aligned}$$

So for any $M \geq M_1$ and any $x \geq M$,

$$\begin{aligned}&\left|\sum_{\sigma_n \leq x} s(n)\sigma_n^{-\alpha} - \frac{R(x)}{T(x)}\right| \\ &\qquad = \left|\sum_{\sigma_n \leq x} s(n)\sigma_n^{-\alpha} - \frac{1}{T(x)} \sum_{\sigma_n \leq x} s(n)T(x/\sigma_n)\right| \\ &\qquad = \left|\sum_{\sigma_n \leq x} s(\sigma_n)\left(\sigma_n^{-\alpha} - \frac{T(x/\sigma_n)}{T(x)}\right)\right| \\ &\qquad \leq \underbrace{\left|\sum_{\sigma_n \leq M} s(n)\left(\sigma_n^{-\alpha} - \frac{T(x/\sigma_n)}{T(x)}\right)\right|}_{\text{III}} + \underbrace{\left|\sum_{M < \sigma_n \leq x} s(n)\left(\sigma_n^{-\alpha} - \frac{T(x/\sigma_n)}{T(x)}\right)\right|}_{\text{IV}}.\end{aligned}$$

Term III goes to 0 as $x \to \infty$ since there are finitely many $\sigma_n \leq M$ and for any fixed n

$$\lim_{x\to\infty} \frac{T(x/\sigma_n)}{T(x)} = \sigma_n^{-\alpha}.$$

Thus it is sufficient to show that term IV goes to 0 as $M \to \infty$. For term IV,

$$\left| \sum_{M<\sigma_n\le x} s(n)\left(\sigma_n^{-\alpha} - \frac{T(x/\sigma_n)}{T(x)}\right) \right| \le \sum_{\sigma_n>M} |s(n)|\sigma_n^{-\alpha} + \sum_{M<\sigma_n\le x} |s(n)| \frac{|T(x/\sigma_n)|}{|T(x)|}$$

$$\le \sum_{\sigma_n>M} |s(n)|\sigma_n^{-\alpha} + C' \sum_{\sigma_n>M} |s(n)|\sigma_n^{-\gamma}$$

for $M \ge 1$. The sums on the right side go to 0 as $M \to \infty$ since they are tail ends of convergent series. This finishes the proof. ■

For the final corollary we need a definition of Knopfmacher.

***Definition 7* ([7], pp. 11–12)** An *arithmetical semigroup* G is a commutative semigroup with identity element 1, with a subset P such that every $a \in G$, $a \neq 1$ has a unique factorization up to ordering into elements of P, and with a real valued norm $|\cdot|$ satisfying

1. $|1| = 1$, $|p| > 1$ for $p \in P$,
2. $|ab| = |a|\,|b|$ for all $a, b \in G$, and
3. the number of elements $a \in G$ of norm $|a| \le x$ is finite for each real $x > 0$.

A specialised version of Theorem 6 appeared in Knopfmacher's book [7] as Lemma 3.6. Using notation close to Theorem 6 it states:

***Corollary 8* (Lemma 3.6, [7])** *Let G be an arithmetical semigroup. Let s and t be functions from G to $\mathbb{C}$. Let $\mathbf{S}(z) = \sum_{a\in G} s(a)|a|^{-z}$, and let $T(x) = \sum_{|a|\le x} t(a)$. Suppose*

1. $T(x) = Bx^{\alpha}(\log x)^r + O\big(x^{\beta}(\log x)^s\big)$ *where $\alpha > 0$, $0 \le \beta \le \alpha$, and r and s are nonnegative integers with the property that $\beta < \alpha$ if $r = 0$, while $s < r$ if $\beta = \alpha$;*
2. $\mathbf{S}(z)$ *is absolutely convergent for z with* $\operatorname{Re} z > \nu$ *where $\nu < \alpha$.*

Let $r(a) = \sum_{b\cdot c=a} s(b)\cdot t(c)$ and $R(x) = \sum_{|a|\le x} r(a)$. Then as $x \to \infty$,

$$R(x) = \big(B\mathbf{S}(\alpha) + o(1)\big)\, x^{\alpha}(\log x)^r.$$

Proof Suppose G is finite. Then $T(x)$ and $R(x)$ are eventually constant. If $B \neq 0$ then $T(x) = Bx^{\alpha}(\log x)^r + O\big(x^{\beta}(\log x)^s\big) \to \infty$ as $x \to \infty$ which is a contradiction. If $B = 0$ then the result holds, since $R(x)/x^{\alpha}(\log x)^r \to 0$ as $x \to \infty$.

Now suppose G is infinite. Let $\{\rho_n\}$ be the ascending sequence of values of $|a|$ for $a \in G$; note that $\rho_n \ge 1$ for all n and $\rho_n \to \infty$ by Definition 7. Let

$$r'(n) = \sum_{|a|=\rho_n} r(a), \quad s'(n) = \sum_{|a|=\rho_n} s(a), \quad \text{and} \quad t'(n) = \sum_{|a|=\rho_n} t(a).$$

Then $r'(n) = \sum_{\rho_i\rho_j=\rho_n} s'(i)\cdot t'(j)$, $R(x) = \sum_{\rho_n\le x} r'(n)$, and $T(x) = \sum_{\rho_n\le x} t'(n)$. Let $\mathbf{S}'(z) = \sum_{n\ge 1} s'(n)\rho_n^{-z}$. $\mathbf{S}'(z)$ can be obtained from $\mathbf{S}(z)$ by rearranging and collecting terms; thus they are equal whenever $\mathbf{S}(z)$ converges absolutely and the abscissa of absolute convergence of $\mathbf{S}'(z)$ is at most ν. Assume $B \neq 0$. Then by Theorem 6 we get

$$\begin{aligned}\mathbf{S}(\alpha) = \mathbf{S}'(\alpha) &= \lim_{x\to\infty} \frac{R(x)}{T(x)} \\ &= \lim_{x\to\infty} \frac{R(x)}{Bx^\alpha(\log x)^r + O\big(x^\beta(\log x)^s\big)} \\ &= \lim_{x\to\infty} \frac{R(x)}{Bx^\alpha(\log x)^r}.\end{aligned}$$

Therefore $R(x) = \big(B\mathbf{S}(\alpha) + o(1)\big)\, x^\alpha(\log x)^r$.

Now assume $B = 0$. This case is an asymptotic bound, not an asymptotic equality, and so is not a consequence of Theorem 6. Let α_s be the abscissa of absolute convergence of $\mathbf{S}(z)$. Take $\gamma \geq \beta$ such that $\alpha_s < \gamma < \alpha$ if $\beta < \alpha$ and $\gamma = \alpha = \beta$ otherwise. For some C and for $x \geq 1$ we have $|T(x)| \leq Cx^\gamma\big(1 + (\log x)^s\big)$ since $T(x)$ takes a finite number of values in any finite interval. Thus

$$\begin{aligned}\frac{|R(x)|}{x^\alpha(\log x)^r} &= \frac{|\sum_{\rho_k\le x} T(x/\rho_k)s(k)|}{x^\alpha(\log x)^r} \\ &\leq \frac{\sum_{\rho_k\le x} C(x/\rho_k)^\gamma\Big(1 + \big(\log(x/\rho_k)\big)^s\Big)\,|s(k)|}{x^\alpha(\log x)^r} \\ &\leq Cx^{\gamma-\alpha}\big((\log x)^{-r} + (\log x)^{s-r}\big)\sum_{\rho_k\le x} |s(k)|\rho_k^{-\gamma} \\ &\to 0\end{aligned}$$

as $x \to \infty$. Therefore in all cases $R(x) = \big(B\mathbf{S}(\alpha) + o(1)\big)\, x^\alpha(\log x)^r$. ■

Notice that the regular variation condition is much more general than Knopfmacher's condition. Knopfmacher also assumes G satisfies Axiom A [7, p. 90], namely that $\big|\{a \in G : |a| \leq x\}\big| = Ax^\delta + O(x^\nu)$ as $x \to \infty$ with $A > 0$, $0 \leq \nu < \delta$.

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