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Rationality of fields of invariants for some representations of $SL_2 \times SL_2$

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Abstract

We prove that the quotient by $SL_2 \times SL_2$ of the space of bidegree (a, b) curves on $\mathbb{P}^1 \times \mathbb{P}^1$ is rational when ab is even and $a \neq b$.

1. Introduction

The main objective of this article is to give a simple proof that the fields of invariants are rational for some irreducible representations of $SL_2 \times SL_2$. Such representations are realized as the spaces $V_{a,b} = H^0(\mathcal{O}_Q(a, b))$ of biforms of bidegree (a, b) on the surface $Q = \mathbb{P}^1 \times \mathbb{P}^1$. By symmetry we may restrict to the range $a \leqslant b$. In [She87], Shepherd-Barron proved that $\mathbb{P}V_{3,b}/SL_2 \times SL_2$ with b even is rational by analyzing transvectants for biforms. The case where $a = 1$ and b is even and at least 10 was also settled by him in another paper, [She88]. We shall prove the following result.

Theorem 1.1. *The quotient $|\mathcal{O}_Q(a, b)|/SL_2 \times SL_2$ is rational when $a < b$ and ab is even.*

Let V_d denote the SL_2-representation $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$. For most pairs (a, b) our proof will be based on the following simple idea: we identify $V_{a,b}$ with $V_a \otimes V_b = \mathrm{Hom}(V_a^\vee, V_b)$ and consider the natural fibration

$$\mathrm{Hom}(V_a^\vee, V_b) \dashrightarrow \mathbb{G}(a, \mathbb{P}V_b) \tag{1.1}$$

which associates to a linear map its image in $\mathbb{P}V_b$, where $\mathbb{G}(a, \mathbb{P}V_b)$ is the Grassmannian of a-planes in $\mathbb{P}V_b$. This is birationally a vector bundle on which the first factor of $SL_2 \times SL_2$ acts fiberwise and the second factor acts equivariantly. Starting from (1.1), we compare several fibrations and eventually reduce the problem to the rationality of $\mathbb{P}V_b/SL_2$, due to Katsylo and Bogomolov [BK85, Kat84a, Kat84b].

Although we have the fibration (1.1) for any $a \leqslant b$, difficulties arise in analyzing it for the following cases:

- when ab is odd, a Brauer–Severi scheme over $\mathbb{G}(a, \mathbb{P}V_b)/SL_2$ becomes birationally nontrivial;
- when $a = b$, $\mathbb{G}(a, \mathbb{P}V_b)$ is one point;
- when $a = 1$, GL_2 acts almost transitively on the fibers of (1.1);
- for a few other pairs (a, b), PGL_2 does not act almost freely on some of the relevant spaces.

The first two cases, excluded from Theorem 1.1, will be the subject of future study. For the third case (with b even), we just add a few supplements to the result of [She88], mainly using transvectants. To study the last case, we identify $\mathbb{P}V_{a,b}$ birationally with the space of parametrized

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rational curves of degree b in $\mathbb{P}^a$. We actually have $a=2$ in the relevant cases, and then the rationality can be proved by using the geometry of rational plane cubics and quartics.

We note that our argument utilizing the fibration (1.1) will apply more generally to a certain class of representations of product groups. In § 2.1 we formulate it in a couple of general forms (Propositions 2.4 and 2.5). We then apply it to $V_{a,b}$ in § 2.2, deducing Theorem 1.1 for $a>1$ and $b>4$. In §§ 3 and 4, we treat the remaining few cases in ad hoc ways as above.

Throughout this article we work over the complex numbers.

2. Fibration over the Grassmannian

In this section we prove Theorem 1.1 in the main case where $a>1$ and $b>4$. We first explain, in § 2.1, the method of proof in a general setting; we then apply it in § 2.2 to the present problem.

2.1 A general method

Let V and W be representations of algebraic groups G and H, respectively. We set

$$a=\dim \mathbb{P}V, \quad b=\dim \mathbb{P}W$$

and assume that $a\leqslant b$. The tensor product $V\otimes W$ is a representation of $G\times H$. We identify $V\otimes W$ with $\mathrm{Hom}(V^\vee, W)$ and consider the images of linear maps $V^\vee\to W$ that are injective. This defines a fibration

$$V\otimes W \dashrightarrow \mathbb{G}(a,\mathbb{P}W) \tag{2.1}$$

over the Grassmannian $\mathbb{G}(a,\mathbb{P}W)$ of a-planes in $\mathbb{P}W$. If we denote by $\mathcal{E}\to\mathbb{G}(a,\mathbb{P}W)$ the universal subbundle of rank $a+1$, then, by (2.1), $V\otimes W$ becomes $G\times H$-equivariantly birational to the vector bundle $V\otimes\mathcal{E}$ over $\mathbb{G}(a,\mathbb{P}W)$. Here G acts on V linearly and H acts on the bundle $\mathcal{E}$ equivariantly. Consequently, we have

$$\mathbb{P}(V\otimes W)/G\times H\sim\mathbb{P}(V\otimes\mathcal{E})/G\times H. \tag{2.2}$$

We shall present an approach to the rationality problem for $\mathbb{P}(V\otimes W)/G\times H$ that utilizes this description. Let $G_0\subset G$ (respectively, $H_0\subset H$) be the subgroup of elements which act trivially on $\mathbb{P}V$ (respectively, $\mathbb{P}W$). In particular, H_0 acts on the bundle $\mathcal{E}$ by certain scalar multiplications. We write $\overline{G}=G/G_0$ and $\overline{H}=H/H_0$.

Lemma 2.1. *Suppose that:*

(1) *$\overline{H}$ acts on $\mathbb{G}(a,\mathbb{P}W)$ almost freely;*

(2) *we have an H-linearized line bundle $\mathcal{L}$ over $\mathbb{G}(a,\mathbb{P}W)$ such that H_0 acts on $\mathcal{E}\otimes\mathcal{L}$ trivially.*

Then

$$\mathbb{P}(V\otimes W)/G\times H\sim(\mathbb{P}V^{\oplus a+1}/G)\times(\mathbb{G}(a,\mathbb{P}W)/H). \tag{2.3}$$

Proof. By assumption (2), the H-linearization of the bundle $\mathcal{E}'=\mathcal{E}\otimes\mathcal{L}$ descends to an $\overline{H}$-linearization. Then, by assumption (1), we may apply the no-name lemma to $\mathcal{E}'$, trivializing it as an $\overline{H}$-linearized vector bundle locally in the Zariski topology. Since $\mathbb{P}(V\otimes\mathcal{E})$ is canonically identified with $\mathbb{P}(V\otimes\mathcal{E}')$, we obtain the $G\times\overline{H}$-equivariant birational equivalence

$$\mathbb{P}(V\otimes\mathcal{E})=\mathbb{P}(V\otimes\mathcal{E}')\sim\mathbb{P}(V\otimes\mathbb{C}^{a+1})\times\mathbb{G}(a,\mathbb{P}W),$$

where both G and $\overline{H}$ act trivially on the factor $\mathbb{C}^{a+1}$. □

Note that any H-linearized line bundle $\mathcal{L}$ over $\mathbb{G}(a,\mathbb{P}W)$ is the tensor product of a power of the Plücker line bundle $\det\mathcal{E}^\vee$ and a 1-dimensional representation of H.

By (2.3), the rationality problem for $\mathbb{P}(V \otimes W)/G \times H$ can be decomposed into proving that $\mathbb{P}V^{\oplus a+1}/G$ is rational and that $\mathbb{G}(a, \mathbb{P}W)/H$ is stably rational of level no greater than $\dim(\mathbb{P}V^{\oplus a+1}/G)$. The latter two problems could be studied, for example, via the following reductions.

Lemma 2.2. *If $\overline{G}$ acts on $\mathbb{P}V^{\oplus a'}$ almost freely for some $a' \leqslant a$, we have*

$$\mathbb{P}V^{\oplus a+1}/G \sim \mathbb{C}^{(a+1)(a-a'+1)} \times (\mathbb{P}V^{\oplus a'}/G). \tag{2.4}$$

Proof. This is a consequence of the no-name lemma applied to the projection $\mathbb{P}V^{\oplus a+1} \dashrightarrow \mathbb{P}V^{\oplus a'}$ from some complementary summand $V^{\oplus a-a'+1}$, which is a $\overline{G}$-linearized vector bundle. $\square$

Lemma 2.3. *In addition to the assumptions* (1) *and* (2) *in Lemma 2.1, suppose that:*

(3) *$\overline{H}$ acts on $\mathbb{P}W$ almost freely.*

Then we have

$$\mathbb{C}^a \times (\mathbb{G}(a, \mathbb{P}W)/H) \sim \mathbb{C}^{a(b-a)} \times (\mathbb{P}W/H). \tag{2.5}$$

Proof. By the same argument as in the proof of Lemma 2.1, we see that $\mathbb{P}\mathcal{E}/H$ is birational to $\mathbb{P}^a \times (\mathbb{G}(a, \mathbb{P}W)/H)$. We regard $\mathbb{P}\mathcal{E}$ as the incidence variety

$$\mathbb{P}\mathcal{E} = \{(P, x) \in \mathbb{G}(a, \mathbb{P}W) \times \mathbb{P}W \mid x \in P\} \subset \mathbb{G}(a, \mathbb{P}W) \times \mathbb{P}W.$$

The fiber of the second projection $\pi\colon \mathbb{P}\mathcal{E} \to \mathbb{P}W$ over $x = [w] \in \mathbb{P}W$ is identified with $\mathbb{G}(a-1, \mathbb{P}(W/\mathbb{C}w))$. Therefore, if $\mathcal{F} \to \mathbb{P}W$ is the universal quotient bundle of rank b, $\mathbb{P}\mathcal{E}$ is identified with the relative Grassmannian $\mathbb{G}(a-1, \mathbb{P}\mathcal{F})$ over $\mathbb{P}W$ via π. Then $\mathbb{G}(a-1, \mathbb{P}\mathcal{F})$ is canonically isomorphic to $\mathbb{G}(a-1, \mathbb{P}\mathcal{F}')$ for the H-linearized bundle $\mathcal{F}' = \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}W}(1)$. Since H_0 acts on $\mathcal{F}$ and $\mathcal{O}_{\mathbb{P}W}(-1)$ by the same scalars, $\mathcal{F}'$ is $\overline{H}$-linearized. Now we can use the no-name lemma for $\mathcal{F}'$ to trivialize it as an $\overline{H}$-linearized vector bundle locally in the Zariski topology. Consequently, we obtain the $\overline{H}$-equivariant birational equivalence

$$\mathbb{P}\mathcal{E} \simeq \mathbb{G}(a-1, \mathbb{P}\mathcal{F}') \sim \mathbb{G}(a-1, \mathbb{P}^{b-1}) \times \mathbb{P}W,$$

where $\overline{H}$ acts on the factor $\mathbb{G}(a-1, \mathbb{P}^{b-1})$ trivially. $\square$

Comparing (2.3), (2.4) and (2.5) and noticing that $(a+1)(a-a'+1) > a$, we can summarize the above argument in the following proposition.

Proposition 2.4. *Let V and W be representations of G and H, respectively, such that $a = \dim \mathbb{P}V$ is smaller than $b = \dim \mathbb{P}W$. Assume that:*

(i) *we have an H-linearized line bundle $\mathcal{L}$ as in Lemma 2.1;*

(ii) *$\overline{G}$ acts on $\mathbb{P}V^{\oplus a'}$ almost freely for some $a' \leqslant a$;*

(iii) *$\overline{H}$ acts on $\mathbb{P}W$ and $\mathbb{G}(a, \mathbb{P}W)$ almost freely.*

Then, setting $N = (a+1)(a-a') + 1 + a(b-a)$, we have

$$\mathbb{P}(V \otimes W)/G \times H \sim \mathbb{C}^N \times (\mathbb{P}V^{\oplus a'}/G) \times (\mathbb{P}W/H).$$

In this way, the rationality problem for $\mathbb{P}(V \otimes W)/G \times H$ can be reduced, under several hypotheses, to results concerning stable rationality of $\mathbb{P}V^{\oplus a'}/G$ and $\mathbb{P}W/H$. We would like to mention that for invariant fields of linear representations, to prove stable rationality is rather easier than proving rationality in many cases.

For our application to $\mathrm{SL}_2 \times \mathrm{SL}_2$-representations, we also state a variant deduced from Lemmas 2.1 and 2.3, bypassing Lemma 2.2.

Proposition 2.5. *Let V and W satisfy the assumptions in Proposition 2.4, except* (ii). *Suppose instead that $\mathbb{P}V^{\oplus a+1}/G$ is rational of dimension $d \geqslant a$. Then, setting $M = d - a + a(b-a)$, we have*

$$\mathbb{P}(V \otimes W)/G \times H \sim \mathbb{C}^M \times (\mathbb{P}W/H).$$

Remark 2.6. When $a \geqslant b$, we can instead consider the kernels of linear maps $V^\vee \to W$ to obtain a fibration $V \otimes W \dashrightarrow \mathbb{G}(a-b-1, \mathbb{P}V^\vee)$. But if we identify $\mathbb{G}(a-b-1, \mathbb{P}V^\vee)$ with $\mathbb{G}(b, \mathbb{P}V)$ naturally, this coincides with the fibration $W \otimes V \dashrightarrow \mathbb{G}(b, \mathbb{P}V)$ as in (2.1).

2.2 Application to $V_{a,b}$

Let V_d denote the SL_2-representation $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$. We shall apply Proposition 2.5 to the $\mathrm{SL}_2 \times \mathrm{SL}_2$-representations $V_{a,b} = V_a \otimes V_b$ such that

$$1 < a < b, \quad b > 4, \quad ab \in 2\mathbb{Z}. \tag{2.6}$$

We have $G = H = \mathrm{SL}_2$, $G_0 = H_0 = \{\pm 1\}$ and $\overline{G} = \overline{H} = \mathrm{PGL}_2$. We first check the almost-freeness condition (iii) in Proposition 2.4.

Lemma 2.7. *Let $0 \leqslant a < b$ and $b > 4$. Then PGL_2 acts on $\mathbb{G}(a, \mathbb{P}V_b)$ almost freely.*

Proof. The case $a = 0$ is well known, so we assume $a > 0$. We first consider the case where $b - a \geqslant 4$. Observe that for a general point $x \in \mathbb{P}V_b$ and a general a-plane P through x, the orbit $\mathrm{PGL}_2 \cdot x$ does not intersect with P outside x. Indeed, if we consider the projection $\pi \colon \mathbb{P}V_b \backslash x \to \mathbb{P}^{b-1}$ from x, a general $(a-1)$-plane $P' \subset \mathbb{P}^{b-1}$ is disjoint from the 3-fold $\pi(\mathrm{PGL}_2 \cdot x \backslash x)$. Then our claim follows by taking the a-plane $P = \overline{\pi^{-1}(P')}$. Since $b > 4$, x is not fixed by any nontrivial $g \in \mathrm{PGL}_2$. Then g does not preserve P, because otherwise it would fix $x = P \cap (\mathrm{PGL}_2 \cdot x)$. This proves the lemma in the range $b - a \geqslant 4$. Since we have the dualities

$$\mathbb{G}(a, \mathbb{P}V_b) \simeq \mathbb{G}(a, \mathbb{P}V_b^\vee) \simeq \mathbb{G}(b-a-1, \mathbb{P}V_b),$$

the range $a \geqslant 3$ is also covered. For the remaining case of $(a, b) = (2, 5)$, $\mathbb{G}(2, \mathbb{P}V_5)$ is birationally identified with the quotient by PGL_3 of the space of morphisms $\mathbb{P}^1 \to \mathbb{P}^2$ of degree 5. Since a general rational plane quintic has its six nodes in a general position, it has no nontrivial stabilizer in PGL_3. This establishes our assertion for $\mathbb{G}(2, \mathbb{P}V_5)$. □

We now proceed according to the parity of b, assuming (2.6).

When b is even, the element $-1 \in \mathrm{SL}_2$ acts on V_b trivially so that the bundle $\mathcal{E}$ is already PGL_2-linearized. Moreover, the quotient $\mathbb{P}V_a^{\oplus a+1}/\mathrm{SL}_2$ is rational by [Kat84c] and has dimension $a^2 + 2a - 3 > a$. Hence the assumptions in Proposition 2.5 are satisfied, and we see that

$$\mathbb{P}V_{a,b}/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{C}^{a(b+1)-3} \times (\mathbb{P}V_b/\mathrm{SL}_2).$$

Then $\mathbb{P}V_b/\mathrm{SL}_2$ is rational by the results of Katsylo and Bogomolov [BK85, Kat84b].

When b is odd, the element $-1 \in \mathrm{SL}_2$ acts on V_b by multiplication by -1. Hence it acts on $\mathcal{E}$ also by multiplication by -1. In this case, since $\mathcal{E}$ has odd rank $a+1$ (remember that ab is even), $-1 \in \mathrm{SL}_2$ acts on the Plücker bundle $\mathcal{L} = \det \mathcal{E}^\vee$ by -1. Then we can twist $\mathcal{E}$ by $\mathcal{L}$ to cancel the action of $-1 \in \mathrm{SL}_2$. Thus condition (i) in Proposition 2.4 is satisfied. As in the case of even b, we then deduce that $\mathbb{P}V_{a,b}/\mathrm{SL}_2 \times \mathrm{SL}_2$ is birational to $\mathbb{C}^{a(b+1)-3} \times (\mathbb{P}V_b/\mathrm{SL}_2)$. Now $\mathbb{P}V_b/\mathrm{SL}_2$ is rational by [Kat84a].

In this way, Theorem 1.1 is proved for $a > 1$ and $b > 4$.

3. Rational space curves

In the rest of the article we study the cases excluded from (2.6) to complete the proof of Theorem 1.1. The cases of $(a,b)=(3,4)$ and $a=1$, $b=2n\geqslant 10$ were settled by Shepherd-Barron in [She87, She88]. (In [She88] he proved the rationality of $\mathbb{G}(1,\mathbb{P}V_b)/\mathrm{SL}_2$, which, by either (2.2) or (3.2), is birational to $\mathbb{P}V_{1,b}/\mathrm{SL}_2\times\mathrm{SL}_2$.) Hence the cases to be considered are

$$(a,b)=(2,3),(2,4),(1,4),(1,6),(1,8).$$

In this section we study the first three cases via geometric approaches. In § 3.1 we identify $|\mathcal{O}_Q(a,b)|$ birationally with the space of some parametrized rational space curves for any (a,b). Using that description, we study the cases $(a,b)=(2,3)$ and $(a,b)=(2,4)$ in §§ 3.2 and 3.3, respectively. The case $(a,b)=(1,4)$ is treated independently in § 3.4.

3.1 Rational space curves

Let $a,b>0$ be any positive integers. To a general curve C on $Q=\mathbb{P}^1\times\mathbb{P}^1$ of bidegree (a,b) we may associate a morphism $\phi_C\colon\mathbb{P}^1\to\mathbb{P}V_b=|\mathcal{O}_{\mathbb{P}^1}(b)|$ by regarding C as a family of b points on the second $\mathbb{P}^1$ factor parametrized by the first $\mathbb{P}^1$ factor.

LEMMA 3.1. *The curve $\phi_C(\mathbb{P}^1)$ has degree a, i.e. $\phi_C^*\mathcal{O}_{\mathbb{P}V_b}(1)\simeq\mathcal{O}_{\mathbb{P}^1}(a)$.*

Proof. By the Riemann–Hurwitz formula, the first projection $C\to\mathbb{P}^1$ has $r=2g_C-2+2b$ branch points where g_C is the genus of C. Substituting $g_C=(a-1)(b-1)$, we have $r=2a(b-1)$. These branch points on $\mathbb{P}^1$ correspond to the intersection of $\phi_C(\mathbb{P}^1)$ with the discriminant hypersurface D in $\mathbb{P}V_b$. Since D has degree $2(b-1)$, $\phi_C(\mathbb{P}^1)$ has degree a. □

Conversely, given a general morphism $\phi\colon\mathbb{P}^1\to\mathbb{P}V_b$ of degree a, we obtain a curve on $\mathbb{P}^1\times\mathbb{P}^1$ by pulling back the universal divisor on $\mathbb{P}V_b\times\mathbb{P}^1$. Reversing the above calculation, we see that C has bidegree (a,b).

Let $U_{a,b}$ be the space of morphisms $\mathbb{P}^1\to\mathbb{P}V_b$ of degree a, on which $\mathrm{PGL}_2\times\mathrm{PGL}_2$ acts as follows: the first PGL_2 factor acts on the source $\mathbb{P}^1$ of the morphisms, and the second PGL_2 factor acts on the target $\mathbb{P}V_b$ in the natural way. Then the above construction gives a $\mathrm{PGL}_2\times\mathrm{PGL}_2$-equivariant birational map

$$\mathbb{P}V_{a,b}=|\mathcal{O}_Q(a,b)|\dashrightarrow U_{a,b}. \tag{3.1}$$

We obtain, in particular, that

$$\mathbb{P}V_{a,b}/\mathrm{PGL}_2\times\mathrm{PGL}_2\sim U_{a,b}/\mathrm{PGL}_2\times\mathrm{PGL}_2.$$

If we denote by $R_{a,b}$ the space of rational curves of degree a in $\mathbb{P}V_b$, this may also be written as

$$\mathbb{P}V_{a,b}/\mathrm{PGL}_2\times\mathrm{PGL}_2\sim R_{a,b}/\mathrm{PGL}_2, \tag{3.2}$$

where PGL_2 acts on $R_{a,b}$ by its action on $\mathbb{P}V_b$. Since PGL_2 as the subgroup of $\mathrm{Aut}(\mathbb{P}V_b)\simeq\mathrm{PGL}_{b+1}$ is the stabilizer of a rational normal curve, we have

$$\mathbb{P}V_{a,b}/\mathrm{PGL}_2\times\mathrm{PGL}_2\sim(R_{a,b}\times R_{b,b})/\mathrm{PGL}_{b+1}.$$

Exchanging a and b, we also obtain

$$\mathbb{P}V_{a,b}/\mathrm{PGL}_2\times\mathrm{PGL}_2\sim R_{b,a}/\mathrm{PGL}_2\sim(R_{b,a}\times R_{a,a})/\mathrm{PGL}_{a+1}. \tag{3.3}$$

Remark 3.2. The map (3.1) and the description $\mathbb{P}V_{a,b}\sim\mathbb{P}(V_a\otimes\mathcal{E})$ in § 2 are connected by considering the linear span of $\phi_C(\mathbb{P}^1)$, which is generically a-dimensional and in which $\phi_C(\mathbb{P}^1)$ is a rational normal curve.

3.2 The case $(a, b) = (2, 3)$

By (3.3) it suffices to prove that $R_{3,2}/\mathrm{PGL}_2$ is rational, where $R_{3,2} \subset |\mathcal{O}_{\mathbb{P}^2}(3)|$ is the space of rational plane cubics and $\mathrm{PGL}_2 \subset \mathrm{PGL}_3$ is the stabilizer of some reference smooth conic Γ. We may take the homogeneous coordinates $[X, Y, Z]$ of $\mathbb{P}^2$ and normalize Γ so that it is defined by $XZ = Y^2$.

Every rational plane cubic has a unique singularity. We apply the slice method for the nodal map

$$\kappa : R_{3,2} \to \mathbb{P}^2, \quad C \mapsto \mathrm{Sing}\, C,$$

which is clearly PGL_2-equivariant. The group PGL_2 acts on $\mathbb{P}^2 - \Gamma$ transitively, and the stabilizer G of the point $p = [0, 1, 0]$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \ltimes \mathbb{C}^\times$, where $\mathbb{Z}/2\mathbb{Z}$ acts by $[X, Y, Z] \mapsto [Z, Y, X]$ and $\alpha \in \mathbb{C}^\times$ acts by $[X, Y, Z] \mapsto [\alpha^{-1}X, Y, \alpha Z]$. The fiber $\kappa^{-1}(p)$ is an open set of the linear system $\mathbb{P}V \subset |\mathcal{O}_{\mathbb{P}^2}(3)|$ of cubics that are singular at p. Hence we have

$$R_{3,2}/\mathrm{PGL}_2 \sim \mathbb{P}V/G.$$

The group G acts linearly on V, and we have the following G-decomposition:

$$V = \langle XYZ \rangle \oplus \langle X^2Z, Z^2X \rangle \oplus \langle X^2Y, YZ^2 \rangle \oplus \langle X^3, Z^3 \rangle.$$

Let $W = \langle X^2Z, Z^2X, X^2Y, YZ^2 \rangle$ and $W^\perp = \langle XYZ, X^3, Z^3 \rangle$, and consider the projection $\pi : \mathbb{P}V \dashrightarrow \mathbb{P}W$ from $W^\perp$. Then π is a G-linearized vector bundle. Since G acts on $\mathbb{P}W$ almost freely, by the no-name lemma we have

$$\mathbb{P}V/G \sim \mathbb{C}^3 \times (\mathbb{P}W/G).$$

The quotient $\mathbb{P}W/G$ is rational because it is 2-dimensional. This proves that $\mathbb{P}V_{2,3}/\mathrm{PGL}_2 \times \mathrm{PGL}_2$ is rational.

3.3 The case $(a, b) = (2, 4)$

By (3.3) it is sufficient to show that $R_{4,2}/\mathrm{PGL}_2$ is rational, where PGL_2 is the stabilizer in PGL_3 of some smooth conic. General rational plane quartics have three nodes. Let $S^3\mathbb{P}^2$ be the third symmetric product of $\mathbb{P}^2$, and consider the nodal map

$$\kappa : R_{4,2} \dashrightarrow S^3\mathbb{P}^2, \quad C \mapsto \mathrm{Sing}\, C. \tag{3.4}$$

General κ-fibers are open sets of sublinear systems of $|\mathcal{O}_{\mathbb{P}^2}(4)|$. Since PGL_2 acts linearly on $H^0(\mathcal{O}_{\mathbb{P}^2}(4))$, κ is birationally the projectivization of a PGL_2-linearized vector bundle. Since PGL_2 acts on $S^3\mathbb{P}^2$ almost freely, by the no-name lemma we have

$$R_{4,2}/\mathrm{PGL}_2 \sim \mathbb{P}^5 \times (S^3\mathbb{P}^2/\mathrm{PGL}_2).$$

Using the slice method (in the converse direction), we see that

$$S^3\mathbb{P}^2/\mathrm{PGL}_2 \sim (S^3\mathbb{P}^2 \times |\mathcal{O}_{\mathbb{P}^2}(2)|)/\mathrm{PGL}_3.$$

We then apply the slice method to the projection $S^3\mathbb{P}^2 \times |\mathcal{O}_{\mathbb{P}^2}(2)| \to S^3\mathbb{P}^2$. The group GL_3 acts on $S^3\mathbb{P}^2$ almost transitively, and the stabilizer G of

$$p_1 + p_2 + p_3 = [1, 0, 0] + [0, 1, 0] + [0, 0, 1]$$

is isomorphic to $\mathfrak{S}_3 \ltimes (\mathbb{C}^\times)^3$, where $\mathfrak{S}_3$ acts by the permutations of X, Y, Z and $(\mathbb{C}^\times)^3$ is the torus of diagonal matrices. Then we have

$$(S^3\mathbb{P}^2 \times |\mathcal{O}_{\mathbb{P}^2}(2)|)/\mathrm{PGL}_3 \sim |\mathcal{O}_{\mathbb{P}^2}(2)|/G \sim H^0(\mathcal{O}_{\mathbb{P}^2}(2))/G.$$

The G-representation $H^0(\mathcal{O}_{\mathbb{P}^2}(2))$ is decomposed as

$$H^0(\mathcal{O}_{\mathbb{P}^2}(2)) = \langle X^2, Y^2, Z^2 \rangle \oplus \langle XY, YZ, ZX \rangle.$$

We set $W = \langle X^2, Y^2, Z^2 \rangle$ and $W^\perp = \langle XY, YZ, ZX \rangle$. The group G acts on W almost transitively, so that we may apply the slice method to the projection $H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \to W$ from $W^\perp$. Hence, for the stabilizer $H \subset G$ of a general point of W, we have

$$H^0(\mathcal{O}_{\mathbb{P}^2}(2))/G \sim W^\perp/H.$$

Then $W^\perp/H$ is birational to $\mathbb{C}^\times \times (\mathbb{P}W^\perp/H)$, and $\mathbb{P}W^\perp/H$ is rational because it is 2-dimensional. This completes the proof that $\mathbb{P}V_{2,4}/\mathrm{PGL}_2 \times \mathrm{PGL}_2$ is rational.

3.4 The case $(a, b) = (1, 4)$

The quotient $\mathbb{P}V_{1,4}/\mathrm{PGL}_2 \times \mathrm{PGL}_2$ is birational to $\mathbb{G}(1, \mathbb{P}V_4)/\mathrm{PGL}_2$ by (3.2). Since $V_4 \simeq V_4^\vee$ as SL_2-representations, we have a PGL_2-equivariant isomorphism $\mathbb{G}(1, \mathbb{P}V_4) \simeq \mathbb{G}(1, \mathbb{P}V_4^\vee)$. By projecting the standard rational normal curve in $\mathbb{P}V_4^\vee$ from lines, we obtain a birational map

$$\mathbb{G}(1, \mathbb{P}V_4^\vee)/\mathrm{PGL}_2 \dashrightarrow R_{4,2}/\mathrm{PGL}_3.$$

Thus the problem is reduced to showing the rationality of $R_{4,2}/\mathrm{PGL}_3$.

We apply the slice method to the nodal map (3.4), which we now regard as a GL_3-equivariant map. We reuse the terms $p_1 + p_2 + p_3$ and G from § 3.3. Then, for the linear system $\mathbb{P}V$ of quartics singular at $p_1 + p_2 + p_3$, we have

$$R_{4,2}/\mathrm{PGL}_3 \sim \mathbb{P}V/G \sim V/G.$$

In terms of the coordinates $[X, Y, Z]$, the G-representation V is decomposed as

$$V = \langle X^2Y^2, Y^2Z^2, Z^2X^2 \rangle \oplus \langle X^2YZ, Y^2ZX, Z^2XY \rangle.$$

The rest of the proof is similar to the final step in § 3.3: we may use the slice method for the projection of V from either irreducible summand, and then resort to Castelnuovo's theorem to show that V/G is rational. Thus $\mathbb{P}V_{1,4}/\mathrm{PGL}_2 \times \mathrm{PGL}_2$ is rational.

4. Transvectants

In this section we treat the cases $(a, b) = (1, 6)$ and $(a, b) = (1, 8)$. We first recall in § 4.1 some basic facts about transvectants for biforms. In §§ 4.2 and 4.3 we study the two cases by applying the method of double fibration [BK85] to certain transvectants.

4.1 Transvectants for biforms

For two representations $V_{a,b}$ and $V_{a',b'}$ of $\mathrm{SL}_2 \times \mathrm{SL}_2$, their tensor product is

$$V_{a,b} \otimes V_{a',b'} = (V_a \boxtimes V_b) \otimes (V_{a'} \boxtimes V_{b'}) = (V_a \otimes V_{a'}) \boxtimes (V_b \otimes V_{b'}).$$

Applying the Clebsch–Gordan decomposition for SL_2,

$$V_d \otimes V_{d'} = \bigoplus_{r=0}^{d''} V_{d+d'-2r} \quad \text{where } d'' = \min\{d, d'\}, \tag{4.1}$$

we obtain the irreducible decomposition

$$V_{a,b} \otimes V_{a',b'} = \bigoplus_{r,s} V_{a+a'-2r, b+b'-2s},$$

where $0 \leqslant r \leqslant \min\{a, a'\}$ and $0 \leqslant s \leqslant \min\{b, b'\}$. By this decomposition we have an $\mathrm{SL}_2 \times \mathrm{SL}_2$-equivariant bilinear map

$$T^{(r,s)} : V_{a,b} \times V_{a',b'} \to V_{a+a'-2r, b+b'-2s},$$

unique up to scalar multiplication. Let $T^{(r)} : V_d \times V_{d'} \to V_{d+d'-2r}$ be the rth *transvectant*, i.e. an SL_2-bilinear map associated to (4.1). Then a standard argument in linear algebra shows that $T^{(r,s)}$ is given (up to a constant) by

$$T^{(r,s)}(P_1 \otimes P_2,\ P_1' \otimes P_2') = T^{(r)}(P_1, P_1') \otimes T^{(s)}(P_2, P_2'), \tag{4.2}$$

where $P_1 \in V_{a,0} = V_a$, $P_2 \in V_{0,b} = V_b$, $P_1' \in V_{a',0} = V_{a'}$ and $P_2' \in V_{0,b'} = V_{b'}$.

Let $[X, Y]$ be the homogeneous coordinate of $\mathbb{P}^1$. The transvectant $T^{(r)}$ is given explicitly by the following formula (cf. [Olv99]):

$$T^{(r)}(P, P') = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^r P}{\partial X^{r-i} \partial Y^i} \frac{\partial^r P'}{\partial X^i \partial Y^{r-i}}. \tag{4.3}$$

In particular, when $r = d' \leqslant d$, $T^{(d')}(P, P')$ is called the *apolar covariant* and is calculated by substituting $-\partial/\partial Y$ and $\partial/\partial X$ for X and Y, respectively, in P', applying that differential polynomial to P, and then multiplying it by $d'!$.

From (4.2) and (4.3) we may calculate the (r, s)th transvectant $T^{(r,s)}$ explicitly in terms of the bihomogeneous coordinate $([X_1, Y_1], [X_2, Y_2])$ of $\mathbb{P}^1 \times \mathbb{P}^1$. For example, when $a = a' = 1$ and $b \geqslant b'$, we have

$$T^{(1,s)}(X_1 \otimes P + Y_1 \otimes Q,\ X_1 \otimes P' + Y_1 \otimes Q') = T^{(s)}(P, Q') - T^{(s)}(Q, P'), \tag{4.4}$$

where $s \leqslant b'$, $P, Q \in V_{0,b} = V_b$ and $P', Q' \in V_{0,b'} = V_{b'}$.

4.2 The case $(a, b) = (1, 6)$

We shall apply the method of double fibration [BK85] to the bi-apolar covariant

$$T^{(1,2)} : V_{1,6} \times V_{1,2} \to V_{0,4}.$$

Note that $\dim V_{1,2} = \dim V_{0,4} + 1$. The image of $V_{1,6} \to \mathrm{Hom}(V_{1,2}, V_{0,4})$ given by $H \mapsto T^{(1,2)}(H, \bullet)$ is not contained in the degeneracy locus: for example, take H to be $X_1X_2^3Y_2^3 + Y_1(X_2^4Y_2^2 + X_2^2Y_2^4)$. Thus the $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-equivariant map

$$\varphi : \mathbb{P}V_{1,6} \dashrightarrow \mathbb{P}V_{1,2}, \quad \mathbb{C}H \mapsto \mathrm{Ker}(T^{(1,2)}(H, \bullet))$$

is well-defined. We note in passing that the φ-image of the above $X_1X_2^3Y_2^3 + Y_1(X_2^4Y_2^2 + X_2^2Y_2^4)$ defines a smooth curve on $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 4.1. *The group $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ acts transitively on the open locus U in $\mathbb{P}V_{1,2}$ of smooth curves. If we take $C \in U$ to be $X_1Y_2^2 + Y_1X_2^2 = 0$, its stabilizer G is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \ltimes \mathbb{C}^\times$, where $\mathbb{Z}/2\mathbb{Z}$ acts by $[X_i, Y_i] \mapsto [Y_i, X_i]$ and $\alpha \in \mathbb{C}^\times$ acts by $[X_1, Y_1] \mapsto [X_1, \alpha^2 Y_1]$, $[X_2, Y_2] \mapsto [X_2, \alpha Y_2]$.*

Proof. By the birational map (3.1), U is mapped isomorphically to the space of linear embeddings $\phi: \mathbb{P}^1 \to \mathbb{P}V_2$ such that $\phi(\mathbb{P}^1)$ is transverse to the diagonal conic $\Gamma \subset \mathbb{P}V_2$. The first assertion holds because the lines in $\mathbb{P}V_2$ transverse to Γ are all PGL_2-equivalent. The stabilizer in $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ of any $C \in U$ is mapped injectively by the projection to the second PGL_2, and its image is the stabilizer of the pencil $\phi_C(\mathbb{P}^1)$. Our second assertion follows from this observation and a little calculation. ◻

By this lemma we may apply the slice method to φ. The φ-fiber over $\mathbb{C}(X_1Y_2^2+Y_1X_2^2)$ is an open set of the projectivization of the linear space

$$V=\{H\in V_{1,6}\mid T^{(1,2)}(H,\ X_1Y_2^2+Y_1X_2^2)=0\}.$$

Then we have

$$\mathbb{P}V_{1,6}/\mathrm{PGL}_2\times\mathrm{PGL}_2\sim\mathbb{P}V/G,$$

where G is as described in the above lemma. The G-action on $\mathbb{P}V$ is induced from the linear G-action on V given by

$$\alpha\in\mathbb{C}^\times : P_1(X_1,Y_1)P_2(X_2,Y_2)\mapsto\alpha^{-4}P_1(X_1,\alpha^2Y_1)P_2(X_2,\alpha Y_2),$$

where $P_1\in V_{1,0}$ and $P_2\in V_{0,6}$.

We express elements of $V_{1,6}$ as X_1P+Y_1Q, $P=\sum_{i=0}^6\binom{6}{i}\alpha_iX_2^iY_2^{6-i}$ and $Q=\sum_{i=0}^6\binom{6}{i}\beta_iX_2^iY_2^{6-i}$. By direct calculation using (4.4) and (4.3), we see that V is defined by

$$\alpha_i=\beta_{i+2}\quad\text{for } 0\leqslant i\leqslant 4.$$

Then we have the G-decomposition $V=\bigoplus_{i=0}^4W_i$, where

$$\begin{aligned}
W_0&=\langle X_1X_2^2Y_2^4+Y_1X_2^4Y_2^2\rangle,\\
W_1&=\langle 10X_1X_2^3Y_2^3+3Y_1X_2^5Y_2,\ 3X_1X_2Y_2^5+10Y_1X_2^3Y_2^3\rangle,\\
W_2&=\langle 15X_1X_2^4Y_2^2+Y_1X_2^6,\ X_1Y_2^6+15Y_1X_2^2Y_2^4\rangle,\\
W_3&=\langle X_1X_2^5Y_2,\ Y_1X_2Y_2^5\rangle,\\
W_4&=\langle X_1X_2^6,\ Y_1Y_2^6\rangle.
\end{aligned}$$

For $i\geqslant 1$, the ith summand W_i is the induced representation of the weight-i scalar representation of $\mathbb{C}^\times$. The group G acts almost freely on $\mathbb{P}(W_1\oplus W_2)$. Therefore we may apply the no-name lemma to the projection $\mathbb{P}V\dashrightarrow\mathbb{P}(W_1\oplus W_2)$ from $W_0\oplus W_3\oplus W_4$, to get that

$$\mathbb{P}V/G\sim\mathbb{C}^5\times(\mathbb{P}(W_1\oplus W_2)/G).$$

Then $\mathbb{P}(W_1\oplus W_2)/G$ is 2-dimensional and hence rational. This finishes the proof that the quotient $\mathbb{P}V_{1,6}/\mathrm{PGL}_2\times\mathrm{PGL}_2$ is rational.

4.3 The case $(a,b)=(1,8)$

We want to show that the $(1,2)$th transvectant

$$T^{(1,2)}:V_{1,8}\times V_{1,4}\to V_{0,8}$$

determines a double fibration [BK85]. Note that $\dim V_{1,4}=\dim V_{0,8}+1$. The nondegeneracy condition is checked, for instance, by the following lemma.

Lemma 4.2. *Take $H=X_1X_2^2Y_2^6+Y_1X_2^6Y_2^2\in V_{1,8}$ and $H'=X_1Y_2^4+Y_1X_2^4\in V_{1,4}$. Then we have $T^{(1,2)}(H,H')=0$, and the linear maps $T^{(1,2)}(H,\bullet)\colon V_{1,4}\to V_{0,8}$ and $T^{(1,2)}(\bullet,H')\colon V_{1,8}\to V_{0,8}$ are both surjective.*

Proof. This is verified by a straightforward (but lengthy) calculation using (4.4) and (4.3). We leave it to the reader. □

Therefore, by [BK85], the $\mathrm{PGL}_2\times\mathrm{PGL}_2$-equivariant map

$$\mathbb{P}V_{1,8}\dashrightarrow\mathbb{P}V_{1,4},\quad \mathbb{C}H\mapsto\mathrm{Ker}(T^{(1,2)}(H,\bullet))$$

is well-defined, dominant, and birationally a projective space bundle. Explicitly, let

$$\mathcal{H} = \{(H, \mathbb{C}H') \in V_{1,8} \times \mathbb{P}V_{1,4} \mid T^{(1,2)}(H, H') = 0\}.$$

Then $\mathcal{H}$ is generically a sub-vector bundle of $V_{1,8} \times \mathbb{P}V_{1,4}$ invariant under the $\mathrm{SL}_2 \times \mathrm{SL}_2$-linearization. By the above lemma, $\mathcal{H}$ has generically the expected rank 9, and the restriction of the natural projection $\mathbb{P}\mathcal{H} \to \mathbb{P}V_{1,8}$ to the main component of $\mathbb{P}\mathcal{H}$ is birational. Since $\mathrm{SL}_2 \times \mathrm{PGL}_2$ acts linearly on $V_{1,8}$, $\mathcal{H}$ is in fact $\mathrm{SL}_2 \times \mathrm{PGL}_2$-linearized. On the other hand, consider the natural hyperplane bundle $\mathcal{O}_{\mathbb{P}V_{1,4}}(1)$ on $\mathbb{P}V_{1,4}$. The element $(-1, 1) \in \mathrm{SL}_2 \times \mathrm{PGL}_2$ acts on $\mathcal{O}_{\mathbb{P}V_{1,4}}(1)$ by -1, so that the bundle $\mathcal{H}' = \mathcal{H} \otimes \mathcal{O}_{\mathbb{P}V_{1,4}}(1)$ is $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-linearized. Then $\mathbb{P}\mathcal{H}'$ is canonically isomorphic to $\mathbb{P}\mathcal{H}$. The group $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ acts almost freely on $\mathbb{P}V_{1,4}$, as a general rational plane quartic has no nontrivial stabilizer in PGL_3 (cf. § 3.4). Hence we may apply the no-name lemma to $\mathcal{H}'$ and see that

$$\mathbb{P}\mathcal{H}/\mathrm{PGL}_2 \times \mathrm{PGL}_2 \sim \mathbb{P}\mathcal{H}'/\mathrm{PGL}_2 \times \mathrm{PGL}_2 \sim \mathbb{P}^8 \times (\mathbb{P}V_{1,4}/\mathrm{PGL}_2 \times \mathrm{PGL}_2).$$

In § 3.4 we proved that $\mathbb{P}V_{1,4}/\mathrm{PGL}_2 \times \mathrm{PGL}_2$ is rational. Therefore $\mathbb{P}V_{1,8}/\mathrm{PGL}_2 \times \mathrm{PGL}_2$ is rational.

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