THE CHORD LENGTH DISTRIBUTION FUNCTION FOR REGULAR POLYGONS

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Abstract

In this paper we obtain an elementary expression for the chord length distribution function of a regular polygon. The formula is derived using δ-formalism in Pleijel identity. In the particular cases of a regular triangle, a square, a regular pentagon, and a regular hexagon, our formula coincides with the results of Sulanke (1961), Gille (1988), Aharonyan and Ohanyan (2005), and Harutyunyan (2007), respectively.

Keywords: Chord length distribution function; Pleijel identity; regular polygons

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1. Introduction

Let $\mathbb{G}$ be the space of all lines g in the Euclidean plane $\mathbb{R}^2$, and let (p, φ), the polar coordinates of the foot of the perpendicular to g from the origin O, be the standard coordinates for a line $g \in \mathbb{G}$.

Let $\mu(\cdot)$ be the locally finite measure on $\mathbb{G}$, invariant with respect to the group of all Euclidean motions (translations and rotations). It is well known that the element of the measure up to a constant factor has the following form (see [1, p. XIII]):

$$\mu(\mathrm{d}g) = \mathrm{d}g = \mathrm{d}p \, \mathrm{d}\varphi,$$

where $\mathrm{d}p$ is the one-dimensional Lebesgue measure and $\mathrm{d}\varphi$ is the uniform measure on the unit circle.

For each bounded convex domain $\mathbb{D}$, we denote the set of lines that intersect $\mathbb{D}$ by

$$[\mathbb{D}] = \{g \in \mathbb{G} \colon g \cap \mathbb{D} \neq \varnothing\},$$

and we have (see [1, p. 195] and [2, p. 130])

$$\mu([\mathbb{D}]) = |\partial\mathbb{D}|,$$

where $\partial\mathbb{D}$ is the boundary of $\mathbb{D}$ and $|\partial\mathbb{D}|$ stands for the length of $\partial\mathbb{D}$.

A random line in $[\mathbb{D}]$ is one with distribution proportional to the restriction of μ to $[\mathbb{D}]$. Therefore,

$$\mathrm{P}(A) = \frac{\mu(A)}{|\partial\mathbb{D}|} \quad \text{for any Borel set } A \subset [\mathbb{D}].$$

Furthermore, let $A_{\mathbb{D}}^{y}$ be the set of lines that intersect $\mathbb{D}$, producing a chord $\chi(g) = g \cap \mathbb{D}$ of length less than y, i.e.

$$A_{\mathbb{D}}^{y} = \{g \in [\mathbb{D}] \colon |\chi(g)| \le y\}, \qquad y \in \mathbb{R}.$$

The distribution function for the length of a random chord χ of $\mathbb{D}$ is usually defined as

$$F(y) = \frac{1}{|\partial\mathbb{D}|} \mu(A_{\mathbb{D}}^{y}) = \frac{1}{|\partial\mathbb{D}|} \iint_{A_{\mathbb{D}}^{y}} \mathrm{d}\varphi \, \mathrm{d}p. \tag{1.1}$$

Therefore, to obtain the chord length distribution function for a bounded convex domain $\mathbb{D}$, we have to calculate the integral on the right-hand side of (1.1). Explicit formulae for the chord length distribution functions are known only for the cases of a disc, a regular triangle [11], a rectangle [6], a regular pentagon [3], and a regular hexagon [9].

The main result of this paper is an elementary expression for the chord length distribution function of a regular polygon. The formula is derived using δ-formalism in Pleijel identity. In the cases of regular triangles, squares, regular pentagons, and regular hexagons, our formula coincides with formulae available in the literature (see [3], [6], [9], and [11]).

The determination of the chord length distribution function has a long tradition of application to collections of bounded convex bodies forming structures in metals and ceramics. The series of formulae for chord length distribution functions may be of use in finding suitable models when empirical distribution functions are given (see [12, p. 116]). They also have an independent interest (see [4] and [5]). The results concerning certain infinite cylinders in which the bases of cylinders are a regular triangle or a rectangle (see [7] and [13]). Gille *et al.* [8] considered infinite cylinders with regular pentagonal and regular hexagonal bases.

2. The case of a convex polygon

Let $\mathbb{D}$ be a convex bounded polygon in the plane, and let $a_1, a_2, \ldots, a_n$ be the sides of $\mathbb{D}$. Then

$$[\mathbb{D}] = \bigcup_{i<j}([a_i] \cap [a_j]),$$

where $[a_i] \cap [a_j]$ is the set of lines hitting both sides a_i and a_j of $\mathbb{D}$.

We write

$$\begin{aligned}
F(y) &= \frac{1}{|\partial\mathbb{D}|} \sum_{i<j} \iint_{\{g \in [a_i] \cap [a_j] \colon |\chi(g)| \le y\}} \mathrm{d}\varphi \, \mathrm{d}p \\
&= \frac{1}{|\partial\mathbb{D}|} \left(\sum_{i<j}^{I} \iint_{\{g \in [a_i] \cap [a_j] \colon |\chi(g)| \le y\}} \mathrm{d}\varphi \, \mathrm{d}p + \sum_{i<j}^{II} \iint_{\{g \in [a_i] \cap [a_j] \colon |\chi(g)| \le y\}} \mathrm{d}\varphi \, \mathrm{d}p \right),
\end{aligned}$$

where the sum '$\sum^{I}$' is over all *nonparallel* pairs of segments $a_i, a_j \subset \partial\mathbb{D}$ and the sum '$\sum^{II}$' is over all *parallel* pairs of segments $a_i, a_j \subset \partial\mathbb{D}$.

In each of the integrals in the sum '$\sum_{i<j}^{I}$', one integration can be performed by passing to $(|\chi|, \varphi)$ coordinates. We have (see [2, p. 157])

$$\mathrm{d}g = \frac{\sin\alpha_1 \sin\alpha_2}{|\sin(\alpha_1 + \alpha_2)|} \, \mathrm{d}|\chi| \, \mathrm{d}\varphi, \tag{2.1}$$

where α_1 is the angle between a_i and $\chi(g) = g \cap \mathbb{D}$, and α_2 is the angle between a_j and $\chi(g)$ ($g \in [a_i] \cap [a_j]$), with α_1 and α_2 lying in one half-plane with respect to χ, inside of $\mathbb{D}$.

3. Pleijel's identity

Let $\mathbb{D}$ be a convex bounded polygon in the plane, and let $a_1, \ldots, a_n$ be the sides of $\mathbb{D}$. The so-called Pleijel identity for $\mathbb{D}$ is as follows (see [2, p. 156]):

$$\int_{[\mathbb{D}]} f(|\chi(g)|)\, \mathrm{d}g = \int_{\mathbb{G}} f'(|\chi|)|\chi|\cot\alpha_1 \cot\alpha_2\, \mathrm{d}g + \sum_{i=1}^{n}\int_0^{|a_i|} f(u)\, \mathrm{d}u, \qquad (3.1)$$

where $f(x)$ is a function with continuous first derivative $f'(x)$, α_1 and α_2 are the angles between $\partial\mathbb{D}$ and g at the endpoints of $\chi(g) = g \cap \mathbb{D}$ which lie in one half-plane with respect to χ, inside of $\mathbb{D}$, and $|a_i|$ is the length of a_i, $i = 1, \ldots, n$.

It was shown in [2, p. 156] that identity (3.1) is useful in calculating the chord length distribution function for the case of bounded convex polygons. If in (3.1) we formally set

$$f_y(u) = \begin{cases} 0 & \text{if } u \le y, \\ 1 & \text{if } u > y, \end{cases}$$

then the left-hand side integral in (3.1) will equal

$$\mu\{g \in [\mathbb{D}]: |\chi(g)| > y\},$$

i.e. the invariant measure of the set of chords of $\mathbb{D}$ whose length exceeds y. The derivative of $f_y(u)$ should be replaced by Dirac's δ-function concentrated at y (see [2, p. 156]). Therefore, we obtain

$$[1 - F(y)]|\partial\mathbb{D}| = \sum_{i<j}^{I}\iint_{[a_i]\cap[a_j]} \delta(|\chi| - y)|\chi|\cot\alpha_1 \cot\alpha_2\, \mathrm{d}g$$

$$+ \sum_{i<j}^{II}\iint_{[a_i]\cap[a_j]} \delta(|\chi| - y)|\chi|\cot\alpha_1 \cot\alpha_2\, \mathrm{d}g + \sum_{i=1}^{n}(|a_i| - y)^+,$$

where $x^+ = x$ if $x > 0$ and 0 otherwise.

For any continuous function f, we have (see [10])

$$\int_{\mathbb{R}^n} \delta(x - y)f(x)\, \mathrm{d}x = f(y). \qquad (3.2)$$

For each nonparallel pair a_i and a_j, we make the change of variables $(p, \varphi) \to (|\chi|, \varphi)$, and using (2.1) and (3.2), we obtain

$$\iint_{[a_i]\cap[a_j]} \delta(|\chi| - y)|\chi|\cot\alpha_1 \cot\alpha_2\, \mathrm{d}g = \frac{y}{\sin\gamma_{ij}}\int_{\Phi_{ij}(y)} \sin\varphi\,\sin(\gamma_{ij} - \varphi)\, \mathrm{d}\varphi,$$

where γ_{ij} is the angle between nonparallel sides a_i and a_j (or their continuations), φ is the angle between direction φ and the a_i direction, $i < j$, and

$$\Phi_{ij}(y) = \{\varphi: \text{a chord joining } a_i \text{ and } a_j \text{ exists with direction } \varphi \text{ and length } y\}.$$

Note that $\Phi_{ij}(y)$ is a subset in the space of directions in the plane; moreover, in the right-hand side integral, $\Phi_{ij}(y)$ corresponds to the set of angles, where the reference direction coincides

with the direction of segment a_i, $i < j$, and the reference origin coincides with the intersection of the lines containing a_i and a_j.

Moreover, for parallel sides a_i and a_j (i.e. $a_i, a_j \in \sum_{i<j}^{II}$), we have

$$\sum_{i<j}^{II} \iint_{[a_i] \cap [a_j]} \delta(|\chi| - y)|\chi| \cot \alpha_1 \cot \alpha_2 \, \mathrm{d}g$$

$$= -\sum_{i<j}^{II} I_{ij}(y) \int \delta(|\chi| - y)|\chi| h(\varphi_\chi) \tan^2 \varphi_\chi \frac{\mathrm{d}\varphi_\chi}{\mathrm{d}|\chi|} \, \mathrm{d}|\chi|$$

$$= -\sum_{i<j}^{II} I_{ij}(y) h(\varphi_y) \tan^2 \varphi_y \frac{b_{ij}}{\sqrt{y^2 - b_{ij}^2}},$$

where φ_y is either $\arccos(b_{ij}/y)$ or $2\pi - \arccos(b_{ij}/y)$, b_{ij} is the distance between the parallel segments a_i and a_j (i.e. the distance between the lines containing a_i and a_j), and the indicator $I_{ij}(y) = \mathbf{1}$(the length of the shortest chord hitting a_i and a_j is less than or equal to y which in turn is less than or equal to the length of the longest chord hitting a_i and a_j), where $\mathbf{1}(A)$ is the indicator function of the event A, i.e. $\mathbf{1}(A) = 1$ if A has occurred and 0 otherwise. Furthermore, $h(\varphi) \neq b_{ij}$ is the height of the maximal parallelogram with two sides equal to $\chi(\varphi) = g(\varphi) \cap \mathbb{D}$ and $g(\varphi) \in [a_i] \cap [a_j]$ ($g(\varphi)$ is a line with φ-direction), and the other two sides lying on the parallel sides a_i and a_j,

$$h(\varphi_\chi) = h\left(\arccos \frac{b_{ij}}{|\chi(\varphi)|} \right) + h\left(2\pi - \arccos \frac{b_{ij}}{|\chi(\varphi)|} \right).$$

Hence, $h(\cdot) = 0$ if the parallelogram is empty.

For the value of φ such that $|\chi(\varphi)| = y$, we have $h(\varphi_y) = h(\varphi_\chi)$. Therefore, we obtain

$$F(y) = 1 - \frac{1}{\sum_{i=1}^n |a_i|} \left(\sum_{i<j}^{I} \frac{y}{\sin \gamma_{ij}} \int_{\Phi_{ij}(y)} \sin \varphi \sin(\gamma_{ij} - \varphi) \, \mathrm{d}\varphi \right.$$

$$\left. - \sum_{i<j}^{II} I_{ij}(y) h(\varphi_y) \frac{\sqrt{y^2 - b_{ij}^2}}{b_{ij}} + \sum_{i=1}^n (|a_i| - y)^+ \right). \qquad (3.3)$$

In the case where $\partial \mathbb{D}$ contains no pairs of parallel sides, (3.3) coincides with the expression given in [2, p. 158].

It follows from (3.3) that to find the distribution function $F(y)$ we have to calculate integrals of the form

$$\frac{1}{\sin \gamma} \int_{\Phi(y)} \sin \varphi \sin(\gamma - \varphi) \, \mathrm{d}\varphi$$

for any two nonparallel segments a and b ($a \leq b$), where γ is the angle between a and b (or their continuations), and also calculate the second sum in (3.3) (for pairs of parallel sides). Here and below,

$$\Phi(y) = \{\varphi : \text{a chord joining } a \text{ and } b \text{ exists with direction } \varphi \text{ and length } y\}.$$

4. The case of a regular polygon

Consider a particular case of (3.3), where $\mathbb{D}$ is a regular polygon. We assume that edges of a regular polygon are ordered in the clockwise direction on $\partial\mathbb{D}$. We denote by γ_k the angle between a_1 and a_k (or their continuations), by d_k the length of the segment connecting the end of side a_1 with the beginning of a_k, and by h_k the distance between the beginning of segment a_k and the line containing the segment a_1.

Below we will use the following notation:

$$N = \begin{cases} n + \frac{1}{2} & \text{for a } (2n+1)\text{-gon,} \\ n & \text{for a } (2n)\text{-gon,} \end{cases} \quad \text{and} \quad \mathbf{1}(N) = \begin{cases} 0 & \text{if } N = n + \frac{1}{2}, \\ 1 & \text{if } N = n. \end{cases}$$

Consider a regular polygon. It is not difficult to verify that

$$\gamma_k = \frac{N - k + 1}{N}\pi, \quad k = 2, \dots, \lfloor N \rfloor + 1, \qquad \gamma_k = \gamma_{2(N+1)-k}, \quad k = \lfloor N \rfloor + 2, \dots, 2N,$$

$$d_2 = 0, \qquad d_3 = a, \qquad d_k = d_{k-2} + 2a\sin\frac{\gamma_{k-2}}{2}, \quad k = 4, \dots, \lfloor N \rfloor + 2,$$

$$d_k = d_{2(N+2)-k}, \quad k = \lfloor N \rfloor + 3, \dots, 2N, \qquad h_k = d_k\cos\frac{\gamma_k}{2},$$

where a is the length of the side of the polygon and $\lfloor N \rfloor$ is the floor function.

In this particular case, (3.3) can be rewritten as

$$F(y) = 1 - \frac{y}{a}\sum_{k=2}^{\lceil N \rceil}\frac{1}{\sin\gamma_k}\int_{\Phi_{1k}(y)}\sin\varphi\sin(\gamma_k - \varphi)\,\mathrm{d}\varphi - \frac{1}{a}(a - y)^+$$

$$+ \frac{1}{a}\mathbf{1}(N)\,\mathbf{1}(d_{n+1} \le y \le d_{n+2})\frac{\sqrt{y^2 - d_{n+1}^2}(a - \sqrt{y^2 - d_{n+1}^2})}{y}, \qquad (4.1)$$

where $\lceil N \rceil$ is the ceiling function.

It follows from (4.1) that in order to obtain an explicit form for the distribution function $F(y)$ of a regular polygon, we have to find the domains $\Phi_{1k}(y)$ and then calculate the integrals on the right-hand side of (4.1).

By definition we have

$$\Phi_{1k}(y) = \{\varphi\colon \text{a chord joining } a_1 \text{ and } a_k \text{ exists with direction } \varphi \text{ and length } y\}.$$

Below we use the notation
$$\theta_k = \arctan\left(\frac{a}{h_k} + \tan\frac{\gamma_k}{2}\right).$$

Note that if $k = 2$, we have $h_2 = 0$ and, therefore, $\theta_2 = \arctan(+\infty) = \pi/2$.

The line g with parameters (φ, p) intersects the sides a_1 and a_k if $\varphi \in (\gamma_k - \theta_k, \theta_k)$. Without loss of generality, we can assume that the reference direction (x-axis) coincides with the direction of the segment a_1 and the origin O coincides with the intersection of the lines containing a_1 and a_k. For the intersection point of the line $g = (\varphi, p)$ with the segment a_1, we have

$$x = \frac{p}{\cos\varphi} \quad \text{and} \quad y = 0.$$

Therefore,

$$g \cap a_1 = \begin{cases} x = \dfrac{p}{\cos \varphi}, \\ y = 0, \\ \dfrac{d_k}{2\sin(\gamma_k/2)} \cos \varphi \le p \le \left(a + \dfrac{d_k}{2\sin(\gamma_k/2)} \right) \cos \varphi. \end{cases}$$

For the intersection point of the line $g = (\varphi, p)$ with the segment a_k, we have

$$x \cos \varphi + y \sin \varphi = p \quad \text{and} \quad y = \tan \gamma_k x$$

or

$$x = \frac{p \cos \gamma_k}{\cos(\varphi - \gamma_k)} \quad \text{and} \quad y = \frac{p \sin \gamma_k}{\cos(\varphi - \gamma_k)}.$$

Hence, we obtain

$$g \cap a_k = \begin{cases} x = \dfrac{p \cos \gamma_k}{\cos(\varphi - \gamma_k)}, \\ y = \dfrac{p \sin \gamma_k}{\cos(\varphi - \gamma_k)}, \\ \dfrac{d_k}{2\sin(\gamma_k/2)} \cos(\varphi - \gamma_k) \le p \le \left(a + \dfrac{d_k}{2\sin(\gamma_k/2)} \right) \cos(\varphi - \gamma_k). \end{cases}$$

The line g which intersects the sides a_1 and a_k forms a chord χ of length

$$|\chi| = \frac{p \sin \gamma_k}{\cos \varphi \cos(\varphi - \gamma_k)}.$$

Therefore, we obtain the following system of relations:

$$\frac{d_k}{2\sin(\gamma_k/2)} \cos \varphi \le p \le \left(a + \frac{d_k}{2\sin(\gamma_k/2)} \right) \cos \varphi, \tag{4.2a}$$

$$\frac{d_k}{2\sin(\gamma_k/2)} \cos(\varphi - \gamma_k) \le p \le \left(a + \frac{d_k}{2\sin(\gamma_k/2)} \right) \cos(\varphi - \gamma_k), \tag{4.2b}$$

$$|\chi| = \frac{p \sin \gamma_k}{\cos \varphi \cos(\varphi - \gamma_k)}. \tag{4.2c}$$

If $\cos \varphi > \cos(\varphi - \gamma_k)$ then $\sin \varphi \sin \gamma_k < (1 - \cos \gamma_k) \cos \varphi$. Hence, $\tan \varphi \le \tan \gamma_k/2$. Therefore, $\varphi \in (\gamma_k - \theta_k, \gamma_k/2)$.

Since $\varphi \in (\gamma_k - \theta_k, \gamma_k/2)$, then (4.2a)–(4.2c) have the following form:

$$\frac{d_k}{2\sin(\gamma_k/2)} \cos \varphi \le p \le \left(a + \frac{d_k}{2\sin(\gamma_k/2)} \right) \cos(\varphi - \gamma_k),$$

$$|\chi| = \frac{p \sin \gamma_k}{\cos \varphi \cos(\varphi - \gamma_k)}.$$

Moreover, we obtain

$$\chi_{\min}(\varphi) = \frac{h_k}{\cos(\varphi - \gamma_k)}, \qquad \chi_{\max}(\varphi) = \frac{h_{k+1}}{\cos \varphi}.$$

The function $\chi_{\min}(\varphi)$ decreases in the domain $\varphi \in (\gamma_k - \theta_k, \gamma_k/2)$, and

$$\varphi_{\max} = \gamma_k - \theta_k, \qquad \varphi_{\min} = \frac{\gamma_k}{2}, \qquad \chi_{\min}(\varphi_{\max}) = d_{k+1}, \quad \text{and} \quad \chi_{\min}(\varphi_{\min}) = d_k.$$

If $\gamma_k < \theta_k$ then the function $\chi_{\max}(\varphi)$ decreases in the domain $\varphi \in (\gamma_k - \theta_k, 0)$ and increases in the domain $\varphi \in (0, \gamma_k/2)$. Hence, we have

$$\varphi_{\min} = 0, \qquad \chi_{\max}(\gamma_k - \theta_k) = d_{k+1}, \qquad \chi_{\max}(\varphi_{\min}) = h_{k+1}, \quad \text{and} \quad \chi_{\max}\left(\frac{\gamma_k}{2}\right) = d_{k+2}.$$

If $\gamma_k \geq \theta_k$ then the function $\chi_{\max}(\varphi)$ increases in the domain $\varphi \in (\gamma_k - \theta_k, \gamma_k/2)$. In this case,

$$\varphi_{\min} = \gamma_k - \theta_k, \qquad \varphi_{\max} = \frac{\gamma_k}{2}, \qquad \chi_{\max}(\varphi_{\min}) = d_{k+1}, \quad \text{and} \quad \chi_{\max}(\varphi_{\max}) = d_{k+2}.$$

Now let $\varphi \in (\gamma_k/2, \theta_k)$. In this case, (4.2a)–(4.2c) can be rewritten in the form

$$\frac{d_k}{2\sin(\gamma_k/2)}\cos(\varphi - \gamma_k) \leq p \leq \left(a + \frac{d_k}{2\sin(\gamma_k/2)}\right)\cos\varphi,$$

$$|\chi| = \frac{p\sin\gamma_k}{\cos\varphi\cos(\varphi - \gamma_k)}.$$

Moreover, we obtain

$$\chi_{\min}(\varphi) = \frac{h_k}{\cos\varphi}, \qquad \chi_{\max}(\varphi) = \frac{h_{k+1}}{\cos(\varphi - \gamma_k)}.$$

The function $\chi_{\min}(\varphi)$ increases in the domain $(\gamma_k/2, \theta_k)$ and

$$\varphi_{\max} = \theta_k, \qquad \varphi_{\min} = \frac{\gamma_k}{2}, \qquad \chi_{\min}(\varphi_{\max}) = d_{k+1}, \quad \text{and} \quad \chi_{\min}(\varphi_{\min}) = d_k.$$

If $\gamma_k < \theta_k$ then the function $\chi_{\max}(\varphi)$ decreases in the domain $(\gamma_k/2, \gamma_k)$ and increases in the domain (γ_k, θ_k). Hence, we obtain

$$\varphi_{\min} = \gamma_k, \qquad \chi_{\max}(\varphi_{\min}) = h_{k+1}, \qquad \chi_{\max}(\theta_k) = d_{k+1}, \quad \text{and} \quad \chi_{\max}\left(\frac{\gamma_k}{2}\right) = d_{k+2}.$$

If $\gamma_k \geq \theta_k$ then the function $\chi_{\max}(\varphi)$ decreases in the domain $(\gamma_k/2, \theta_k)$ and we have

$$\varphi_{\max} = \frac{\gamma_k}{2}, \qquad \varphi_{\min} = \theta_k, \qquad \chi_{\max}(\varphi_{\min}) = d_{k+1}, \quad \text{and} \quad \chi_{\max}(\varphi_{\max}) = d_{k+2}.$$

Therefore, in the case in which $\gamma_k < \theta_k$ we have

$$\Phi_{1k}(y) = \begin{cases} \left(\gamma_k - \arccos\dfrac{h_k}{y}, \arccos\dfrac{h_k}{y}\right) & \text{if } y \in [d_k, h_{k+1}), \\[2mm] \left(\gamma_k - \arccos\dfrac{h_k}{y}, -\arccos\dfrac{h_{k+1}}{y}\right) \\[2mm] \cup \left(\arccos\dfrac{h_{k+1}}{y}, \gamma_k - \arccos\dfrac{h_{k+1}}{y}\right) \\[2mm] \cup \left(\gamma_k + \arccos\dfrac{h_{k+1}}{y}, \arccos\dfrac{h_k}{y}\right) & \text{if } y \in [h_{k+1}, d_{k+1}), \\[2mm] \left(\arccos\dfrac{h_{k+1}}{y}, \gamma_k - \arccos\dfrac{h_{k+1}}{y}\right) & \text{if } y \in [d_{k+1}, d_{k+2}], \\[2mm] \varnothing & \text{if } y > d_{k+2} \text{ or } y < d_k, \end{cases}$$

and if $\gamma_k \geq \theta_k$, we have

$$
\Phi_{1k}(y) = \begin{cases}
\left(\gamma_k - \arccos \dfrac{h_k}{y}, \arccos \dfrac{h_k}{y} \right) & \text{if } y \in [d_k, d_{k+1}), \\[2ex]
\left(\arccos \dfrac{h_{k+1}}{y}, \gamma_k - \arccos \dfrac{h_{k+1}}{y} \right) & \text{if } y \in [d_{k+1}, d_{k+2}], \\[2ex]
\varnothing & \text{if } y > d_{k+2} \text{ or } y < d_k.
\end{cases}
$$

Now we can substitute the values of $\Phi_{1k}(y)$ into the right-hand side of (4.1) and calculate the corresponding integrals.

Therefore, the distribution function for a regular polygon has the following form:

$$
\begin{aligned}
F(y) = 1 - \frac{y}{a} \sum_{k=2}^{\lceil N \rceil} & \Bigg\{ \mathbf{1}(\gamma_k < \theta_k) \Bigg[\mathbf{1}(y \in [d_k, h_{k+1})) \left(\left[\frac{h_k}{y} \sqrt{1 - \frac{h_k^2}{y^2}} - \arccos \frac{h_k}{y} + \frac{\gamma_k}{2} \right] \cot \gamma_k \right. \\
& \left. + \frac{1}{2} - \frac{h_k^2}{y^2} \right) \\
& + \mathbf{1}(y \in [h_{k+1}, d_{k+1})) \\
& \quad \times \left(\left[\frac{h_k}{y} \sqrt{1 - \frac{h_k^2}{y^2}} - 2\frac{h_{k+1}}{y} \sqrt{1 - \frac{h_{k+1}^2}{y^2}} - \arccos \frac{h_k}{y} \right. \right. \\
& \quad \left. \left. + 2 \arccos \frac{h_{k+1}}{y} + \frac{\gamma_k}{2} \right] \cot \gamma_k + \frac{1}{2} - \frac{h_k^2}{y^2} \right) \\
& + \mathbf{1}(y \in [d_{k+1}, d_{k+2})) \\
& \quad \times \left(\left[\arccos \frac{h_{k+1}}{y} - \frac{h_{k+1}}{y} \sqrt{1 - \frac{h_{k+1}^2}{y^2}} - \frac{\gamma_k}{2} \right] \cot \gamma_k \right. \\
& \quad \left. - \frac{1}{2} + \frac{h_{k+1}^2}{y^2} \right) \Bigg] \\
& + \mathbf{1}(\gamma_k \geq \theta_k) \Bigg[\mathbf{1}(y \in [d_k, d_{k+1})) \\
& \quad \times \left(\left[\frac{h_k}{y} \sqrt{1 - \frac{h_k^2}{y^2}} - \arccos \frac{h_k}{y} + \frac{\gamma_k}{2} \right] \cot \gamma_k + \frac{1}{2} - \frac{h_k^2}{y^2} \right) \\
& + \mathbf{1}(y \in [d_{k+1}, d_{k+2})) \\
& \quad \times \left(\left[\arccos \frac{h_{k+1}}{y} - \frac{h_{k+1}}{y} \sqrt{1 - \frac{h_{k+1}^2}{y^2}} - \frac{\gamma_k}{2} \right] \cot \gamma_k \right. \\
& \quad \left. - \frac{1}{2} + \frac{h_{k+1}^2}{y^2} \right) \Bigg] \Bigg\} - \frac{1}{a}(a - y)^+ \\
& + \frac{1}{a} \mathbf{1}(N) \, \mathbf{1}(d_{n+1} \leq y \leq d_{n+2}) \frac{\sqrt{y^2 - d_{n+1}^2} \left(a - \sqrt{y^2 - d_{n+1}^2} \right)}{y}. \quad (4.2)
\end{aligned}
$$

It is not difficult to verify that this function is a continuous function, and, for a regular triangle, a square, a regular pentagon, and a regular hexagon, (4.2) coincides with those of [11], [6], [3], and [9], respectively.

From the above formulae, it is not difficult to specify a chord length density function $f(y) = F'(y)$ for regular polygons.

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