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# A NOTE ON THE CENTRAL LIMIT THEOREM FOR GEODESIC RANDOM WALKS

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In this article we use a theorem of T.G. Kurtz to prove a central limit theorem for geodesic random walks.

## 0. Introduction

In [1] Jorgensen defines the concept of a random walk in a general Riemannian manifold and investigates the behavior in the limit of a sequence of such random walks. His approach to the problem in question is based upon the use of semigroup methods due to Trotter and Stone. The aim of this note is to give a new proof of some of Jorgensen's results using instead the following theorem of Kurtz and several ideas in [2].

**THEOREM 0.1.** *For* $N = 1, 2, \ldots$ *let* $T_N$ *be a linear contraction on a Banach space* $L_N$ , *let* $\varepsilon_N$ *be a positive number and put* $A_N = \left(T_N - I\right)/\varepsilon_N$ . *Assume that* $\lim_{N\to\infty} \varepsilon_N = 0$ . *Let* $\{T(t)\}$ *be a strongly continuous contraction semigroup on a Banach space* $L$ *with generator* $A$ *and let* $D$ *be a core for* $A$ . *Let* $\pi_N : L \to L_N$ *be bounded linear operators such that* $\sup_N \|\pi_N\| < \infty$ . *Then the following are equivalent:*

*(a) for each* $f \in L$ , $\left\|T_N^{[t/\varepsilon_N]}\pi_N f - \pi_N T(t) f\right\| \to 0$ *for all*

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*$t \geq 0$ ;*

*(b) for each $f \in D$ there exists $f_N \in L_N$ such that $\|f_N - \pi_N f\| \to 0$ and $\|A_N f_N - \pi_N A f\| \to 0$ .*

## 1. A limit theorem

Let $M$ be a Riemann manifold of dimension $n$ . The Brownian motion in $M$ is a stochastic process $\left\{X_t^x : t \geq 0\right\}$ with continuous sample paths such that $X_0^x = x$ and

$$f\left(X_t^x\right) - f(x) + \int_0^t \frac{a}{2} (\Delta f)\left(X_s^x\right) ds \ , \quad a > 0 \ ,$$

is a martingale for every $f \in C_0^\infty(M)$ . (See [2]).

Let $x \in M$ , $T_x(M)$ be the tangent space of $M$ at $x$ , $S_x(M)$ the unit sphere of $T_x(M)$ , $S(M)$ the bundle of tangent unit spheres. Let $\xi_x \in S_x(M)$ and $x\left(\cdot, \xi_x\right)$ be the unit speed geodesic starting from $x$ in the direction $\xi_x$ . Let $\{\tau(k) : k = 0, 1, \ldots\}$ be identically independantly distributed in $\mathbb{R}^+$ with distribution $P$ . Assume $E\left(\tau(k)\right) = 1$ , $E\left(\tau^2(k)\right) = a$ , $E\left(\tau^3(k)\right) < \infty$ . Let $\{Z(k) : k = 0, 1, \ldots\}$ be the random variables in $S(M)$ such that

$$Z(0) = \left(x_0, \xi_0\right), \ldots, Z(k) = \left(x_k, \xi_k\right)$$

where

$$x_k = x_{k-1}\left(\tau(k-1), \xi_{k-1}\right) \ ,$$

$$\xi_k = \xi_{x_k} \ ,$$

and the conditional distribution of $\xi_k$ given $\left\{\left(x_0, \xi_0\right), \ldots, \left(x_{k-1}, \xi_{k-1}\right)\right\}$ is the uniform distribution on $S_{x_k}$ denoted by $\mu$ . The sequence $\left\{x_0, x_1, \ldots\right\}$ is called a geodesic random walk with identically distributed steps. (See [1] and [2].)

THEOREM 1.1. *For* $N = 1, 2, \ldots$ *let* $X_0^N = x$ *and*

$$X_k^N = X_{k-1}^N\left(N^{-\frac{1}{2}}\tau(k-1),\ \xi_{k-1}\right) .$$

*Assume the Ricci curvature of* $M$ *is bounded from below. Then for every* $f$ *in* $C(M)$ , $\lim_{N\to\infty} E\left[f\left(X_{[Nt]}^N\right)\right] = E\left[f\left(X_{t/n}^x\right)\right]$ .

Proof. In the notations of Theorem 0.1, let $L_N = B(S(M))$ , $D = C_K^2(M)$ . Let $P\cdot(x, \xi)$ be the parallel transport along the geodesic $X(\cdot, \xi)$ . For $N = 1, 2, \ldots$ let $\left\{Z_N(k) = \left(X_k^N, \xi_k\right) : k = 0, 1, \ldots\right\}$ and for $f \in L_N$ define

$$T_N f(x, \xi) = \iint f\left(x\left(\tau_1 N^{-\frac{1}{2}}, \xi\right), P_{N^{-\frac{1}{2}}\tau_1}(x, \xi)(\eta)\right) dP\mu(d\eta) ,$$

$$A_N = N\left(T_N - I\right) .$$

(Note that $T_N$ is the linear contraction associated to the Markov chain $Z_N$ in $S(M)$ .)

For $f$ in $C_K^2(S(M))$ , $\xi \in S_x(M)$ , $\eta \in S_x(M)$ let

$$u(t, x, \xi, \eta) = f\left(X(t, \xi), P_t(x, \xi)(\eta)\right) .$$

Then clearly

$$\frac{\partial u}{\partial t}(0, x, \xi, \eta) = \xi^i \frac{\partial u}{\partial x_i} - \Gamma_{j,k}^i \xi^j \eta^k \frac{\partial u}{\partial \xi_i} . \tag{1.1}$$

Let $f \in D$ , $h = D_\xi f$ , $k = (a/2)D_{\xi\xi} f$ , $f_N = f + N^{-\frac{1}{2}}h + N^{-1}k$ and $A = -(a/2n)\Delta f$ . We show that $\|A_N f_N - Af\| \to 0$ as $N \to \infty$ . Using (1.1) we have the following Taylor expansions for $f$ and $h$ (we replace $\tau(1)$ by $\tau_1$ and denote $\partial u/\partial t$ by $D_\xi^\eta f$ and $D_\xi^\xi$ by $D_\xi$ ),

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$$(1.2) \qquad \left|f\left(x\left(N^{-\frac{1}{2}}\tau_1\right)\right)-f(x)-N^{-\frac{1}{2}}\tau_1 D_\xi f(x)-N^{-1}\tau_1^2 D_\xi D_\xi f(x)\right| < KN^{-3/2}\tau_1^3 ,$$

$$(1.3) \qquad \left|h\left(x\left(N^{-\frac{1}{2}}\tau_1, \xi\right), P_{N^{-\frac{1}{2}}\tau_1}(x, \xi)(\eta)\right)-h(x, \eta)-N^{-\frac{1}{2}}\tau_1 D_\xi^\eta f(x, \eta)\right| < KN^{-1}\tau_1^2 .$$

For $g \in C(S(M))$ let $B$ be defined by

$$(1.4) \qquad Bg(x, \xi) = \int_{S_x(M)} g(x, \eta)\mu(d\eta) - g(x, \xi) .$$

By (1.2), (1.3) and the definition of $T_N$ there exists a constant $K$ depending on $M$ only such that

$$(1.5) \qquad N\left|T_N f(x, \xi)-f(x)-N^{-\frac{1}{2}}D_\xi f(x)-(a/2)N^{-1}D_\xi D_\xi f(x)\right| \le N^{-\frac{1}{2}}KE\left(\tau_1^3\right) ,$$

$$(1.6) \qquad \overline{N}\left|T_N h(x, \xi)-h(x, \xi)-Bh(x, \xi)-N^{-\frac{1}{2}}\int D_\xi^\eta h(x, \eta)\mu(d\eta)\right| \le N^{-\frac{1}{2}}aK ,$$

$$(1.7) \qquad \left|T_N k(x, \xi)-k(x, \xi)-Bk(x, \xi)\right| < N^{-\frac{1}{2}}(a/2)K .$$

Using then a triangle inequality

$$\left|A_N f_N(x, \xi)-Af(x)\right| \le N^{-\frac{1}{2}}K\left(E\left|\tau_1^3\right|+(3/2)a\right) + \left|Bh(x, \xi)+D_\xi f(x)\right|N^{\frac{1}{2}} + \left|Bk(x, \xi)+(a/2)D_{\xi\xi}f(x) + \int D_\xi^\eta f(x, \eta)\xi(d\eta)-Af(x)\right| .$$

To conclude the proof we have to show that

$$(1.8) \qquad Bh(x, \xi) + D_\xi f = 0 ,$$

$$(1.9) \qquad \int D_\xi^\eta h(x, \eta)\xi(d\eta) = 0 ,$$

$$(1.10) \qquad Bk(x, \xi) = Af(x) - (a/2)D_\xi D_\xi f(x) .$$

Since $\int \eta^i \xi\left(d\eta^i\right) = 0$ ,

$$Bh(x, \xi)_i = \int \eta^i \frac{\partial f}{\partial x_i} \xi\left(d\eta^i\right) - \xi^i \frac{\partial f}{\partial x_i} = -\xi^i \frac{\partial f}{\partial x_i} ,$$

and (1.8) follows. (1.9) follows from the fact that

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$$D_\xi^\eta h(x,\ \eta) = \xi^i \eta^k \frac{\partial^2 f}{\partial x_i \partial x_k} - \Gamma^i_{j,k} \xi^j \eta^k \frac{\partial f}{\partial x_k}$$

and

$$\int D_\xi^\eta h(x,\ \eta)\xi(d\eta) = \left(\xi^i \frac{\partial^2 f}{\partial x_i \partial x_k}(x) - \Gamma^k_{j,k} \xi^j \frac{\partial f}{\partial x_k}\right) \int \eta^k \mu(d\eta^k) = 0 \ .$$

For (1.10) see [2], p. 209. Since clearly $\|f_N - f\| \to 0$ Theorem 0.1 *(a)* is satisfied and Theorem 1.1 follows easily.

## References

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