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ON SOME GELFAND-MAZUR LIKE THEOREMS IN p-NORMED ALGEBRAS

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This paper is dedicated to the memory of
the late Professor C.T. Rajagopal

The main theorem of this paper shows that a complex p-normed algebra which is a pre-Bezout domain is isomorphic to the field of complex numbers, if it is a generalized unique factorization domain. This theorem generalizes the previous result of the authors proved by them in their paper *Bull. Austral. Math. Soc.* 20 (1979), 247-252. Some applications are then given.

1. Introduction

A Gelfand-Mazur like theorem in a p-normed algebra is a theorem which asserts that a complex p-normed algebra is isomorphic to the field C of complex numbers, when the algebra satisfies some algebraic or analytical condition. The classical Gelfand-Mazur theorem is such an example. While every Banach algebra is an example of a p-normed algebra, not every p-normed algebra need be a Banach algebra. In fact, p-normed algebras need not be even locally convex. Żelazko, in a series of papers [*11*], [*9*] and [*10*] extended many results that are valid for commutative Banach algebras to the case of p-normed algebras which are commutative. Lack of

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local convexity makes these proofs completely different from the case of Banach algebras.

It is indicated below that by using the results of Żelazko, neither the algebra Γ of all entire functions, nor the algebra $P(x)$ of all complex polynomials can be made into a p-normed algebra. The latter is a principal ideal domain while the former is what we call a generalized unique factorization domain. Both of them are pre-Bezout domains. This leads us to the following conjecture.

CONJECTURE. *A complex p-normed algebra, which is a generalized unique factorization domain and a pre-Bezout domain, is isomorphic to the field C of complex numbers.*

In this paper we prove the above conjecture. This paper also derives some of the results proved in the paper of Srinivasan and Hu Shaing [8].

2.

We gather all the relevant definitions and theorems which we use in Section 3.

An *integral domain* A is a commutative ring with an identity element $1 \neq 0$ in which there are no divisors of zero. A *Bezout domain* is an integral domain A in which all finitely generated ideals are principal. This means given any two elements a and b of A the greatest common divisor d exists in A with

(2.1) $$d = ar + bs \text{ , for some } r \text{ and } s \text{ of } A \text{ .}$$

A *pre-Bezout domain* A is an integral domain in which property (2.1) holds for pairs of elements a and b which are co-prime. Every Bezout domain is a pre-Bezout domain, while the converse statement is not, in general, true. In a paper still to be published Mott and Zafrullah [5] have shown that, in a pre-Bezout domain, every irreducible element is indeed a prime element.

A principal ideal domain is also a Bezout domain. The algebra Γ of all entire functions is an example of a Bezout domain which is not a principal ideal domain.

The next definition is a generalization of a unique factorization domain.

DEFINITION 2.1. Let A be an integral domain. A is called a *generalized unique factorization domain* if, for each non-zero non-unit element $a \in A$, we can find a unique finite or infinite set of distinct primes $\{p_\alpha\}$, $\alpha \in S$, such that

$$(a) = \bigcap_{\alpha \in S} \left\{p_\alpha^{n_\alpha}\right\} \tag{2.2}$$

where $\{n_\alpha\}$ are unique positive integers corresponding to a, and S is an indexing set.

Every unique factorization domain is trivially a generalized unique factorization domain. However there are examples of generalized unique factorization domains which are not unique factorization domains. The algebra Γ of all entire functions is not a unique factorization domain. That Γ is a Bezout domain was proved by Helmer [2]. The statement that Γ is a generalized unique factorization domain follows from the following theorem of Helmer [2].

THEOREM 2.1 (Helmer [2], Theorem 6, p. 348). *Let Γ be the ring of all entire functions taken with the usual addition and multiplication. Every non-zero non-unit $f(z) \in \Gamma$ is expressible as either a finite or a countable infinite product of irreducible functions of Γ. The representation is unique except for the order of factors and units.*

While a unique factorization domain becomes a principal ideal domain if the Bezout domain condition is added to it, such is not the case for a generalized unique factorization domain. Γ is both a generalized unique factorization domain and a Bezout domain, but is not a principel ideal domain. It is the only natural example of a generalized unique factorization domain known to the authors which is not already a unique factorization domain.

DEFINITION 2.2. Let A be a complex linear algebra. A is called a *p-normed algebra* if there exists a functional $\| \ \|$ defined on A satisfying the following conditions:

(a) $\|x\| \geq 0$ for each $x \in A$ and $\|x\| = 0$ if and only if $x = 0$;

(b) $\|x+y\| \leq \|x\| + \|y\|$ for all x and y in A;

(c) $\|xy\| \leq \|x\|\|y\|$ for all x and y in A ;

(d) for $\alpha \in C$ and $x \in A$, $\|\alpha x\| = |\alpha|^p\|x\|$ for some fixed p such that $0 < p \leq 1$;

(e) A has an identity 1, with $\|1\| = 1$;

(f) the metric d defined on A by the relation $d(x, y) = \|x-y\|$ is complete.

REMARK 2.1. For the sake of convenience, we have assumed that our p-normed algebras are complete and that they have identity elements. Żelazko [*11*] considered the case of a p-normed algebra without an identity element. Every Banach algebra with an identity can be considered to be a p-normed algebra with $p = 1$. However, there are p-normed algebras which are not Banach algebras for p-normed algebras may not be even locally convex.

The concept of a topological zero divisor is analogous to the concept of the same in Banach algebras. The next theorem was proved by Żelazko for p-normed algebras.

THEOREM 2.2 (Żelazko [*10*]). *Let A be a complex p-normed algebra without any topological divisors of zero, other than the element* $0 \in A$. *Then A is isomorphic to the field C.*

The next theorem is a modification of Lemma 1.5.2 (p. 21) in Rickart [*6*]. But the result is stated here for the case of integral domains.

THEOREM 2.3. *Let A be a complex p-normed algebra which is an integral domain. Then the following conclusions hold:*

(1) *the necessary and sufficient condition for $h \in A$, $h \neq 0$, to be a topological divisor of zero is that the principal ideal (h) be not closed;*

(2) *if, for each $h \in A$, $h \neq 0$, (h) is closed, then A is isomorphic to the field C of complex numbers.*

THEOREM 2.4. *Let A be a complex p-normed algebra, which is also a division algebra. Then A is isomorphic to C.*

THEOREM 2.5 (Żelazko [9], [*10*]). *Let A be a complex p-normed algebra. Then*

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(1) *for each* $x \in A$, *if* x *satisfies* $\|x-1\| < 1$, *then* x *is invertible,*

(2) *every maximal ideal* M *is closed.*

Żelazko extended many of the results which are valid for commutative Banach algebras to complex p-normed commutative algebras. The maximal ideal theory which is usually called Gelfand's representation theory for commutative Banach algebras can be extended to that of commutative p-normed algebras.

Let A be a complex p-normed algebra, which is also commutative. Let Δ be the space of maximal ideals of A . Let

(2.3) $$K_s = \left(x \in A : \lim\|x^n\| = 0\right) .$$

Let the spectral radius for any $x \in A$ be defined by the relation

(2.4) $$\|x\|_s = \left(\sup\{|\lambda| : \lambda x \in K_s\}\right)^{-p} .$$

THEOREM 2.6 (Żelazko [9], [10]). *The spectrum* $\sigma(x)$ *of an element* $x \in A$ *is a compact subset of* C . *Further for each element* $x \in A$, *the following relation holds:*

(2.5) $$\sup_{M\in\Delta} |x(M)|^p = \|x\|_s .$$

The radical of A , $\operatorname{rad}(A)$, *is given precisely by*

(2.6) $$\operatorname{rad}(A) = \{x \in A : \|x\|_s = 0\} .$$

3.

In this section we prove many results that generalize the results stated in the paper of Srinivasan and Hu Shaing [8].

THEOREM 3.1 (Fundamental Theorem). *Let* A *be a complex* q*-normed algebra* $(0 < q \leq 1)$ *which is also a pre-Bezout domain. Let* p *be an irreducible element of* A . *For any non-negative integer* n , *the principal ideal* $\left(p^n\right)$ *is closed, or equivalently,* p^n *is not a topological divisor of zero.*

Proof. We first observe that, by a result of Mott and Zafrullah [5],

p is a prime, since A is a pre-Bezout domain. From now on the proof depends on induction on n.

CASE 1. For $n = 0$, the result is trivial.

CASE 2. Let $n = 1$. Consider the ideal (p). Let $\{pa_n\}$ be any sequence of elements in (p) converging to an element $a \in A$. To prove that $a \in (p)$, it suffices to show that p divides a. If p does not divide a, then p and a are co-prime. As A is a pre-Bezout domain, we can find r and s in A such that

(3.1) $$pr + as = 1 .$$

Since $\{pa_n\}$ converges to a, it follows that

(3.2) $$\|pa_n - a\| \to 0 \quad \text{as} \quad n \to \infty .$$

From (3.2) it follows that

(3.3) $$\|pa_n s - as\| \le \|s\| \|(pa_n - a)\| \to 0 .$$

The substitution of $as = 1 - pr$ in equation (3.3) yields

(3.4) $$\|pa_n s + pr - 1\| \to 0 \quad \text{as} \quad n \to \infty .$$

(3.4) implies that for sufficiently large n, say $n = n_0$,

(3.5) $$\|pa_{n_0} s + pr - 1\| < 1 .$$

It now follows from Theorem 2.5 (1) that $pa_{n_0} s + pr = p(a_{n_0} s + r)$ is invertible, thereby leading to the invertibility of p itself. The above contradicts the primality of p. Hence p must divide a. Consequently the ideal (p) is closed.

CASE 3. We now assume that for all i with $0 \le i \le m$, (p^i) is closed or equivalently that p^i is not a topological divisor of zero. We shall show that (p^{m+1}) is closed. Let $\{p^{m+1}a_n\}$ be any sequence in (p^{m+1}) converging to an element $a \in A$. We shall show that p^{m+1} divides a. If p^{m+1} does not divide a we can write $a = p^i h$, where h and p are relatively prime, and where obviously $0 \le i \le m$. As A

is a pre-Bezout domain, we can find s and t such that $ht + ps = 1$. Hence

$$p^i ht + p^{i+1} s = p^i \quad \text{or} \quad at = p^i - p^{i+1} s . \tag{3.6}$$

Since $\|p^{m+1}a_n - a\| \to 0$, it follows that

$$\|p^{m+1}a_n t - at\| \le \|t\| \left\| \left(p^{m+1}a_n - a\right) \right\| \to 0 . \tag{3.7}$$

Substituting for at, we obtain from (3.7) the relation

$$\left\| p^i \left(p^{m+1-i} a_n t + ps - 1\right) \right\| \to 0 \quad \text{as} \quad n \to \infty . \tag{3.8}$$

There are now two possibilities. Either, for some n_0,

$$\left\| p^{m+1-i} a_{n_0} t + ps - 1 \right\| < 1 , \tag{3.9}$$

or, for all n,

$$\left\| p^{m+1-i} a_n t + ps - 1 \right\| \ge 1 . \tag{3.10}$$

If equation (3.9) holds, it follows (from Theorem 2.5 (1)) that $p\left(p^{m-i} a_{n_0} t + s\right)$ is invertible, leading to the invertibility of p itself; thus (3.9) produces a contradiction. If (3.10) holds, then set

$$A_n = \left(p^{m+1-i} a_n t + ps - 1\right) \Big/ \left(\left\| p^{m+1-i} a_n t + ps - 1 \right\|^{1/q} \right) . \tag{3.11}$$

Then clearly $\|A_n\| = 1$ for each n. Now, using (3.10) and (3.8), we obtain

$$\begin{aligned} \left\| p^i A_n \right\| &= \left\| \left(p^i \left(p^{m+1-i} a_n t + ps - 1\right)\right) \Big/ \left(\left\| \left(p^{m+1-i} a_n t + ps - 1\right) \right\|^{1/q} \right) \right\| \\ &\le \left\| p^i \left(p^{m+1-i} a_n t + ps - 1\right) \right\| \to 0 . \end{aligned} \tag{3.12}$$

The relation (3.12) implies however that p^i $(0 \le i \le m)$ is a topological divisor of zero, thereby contradicting the inductive hypothesis. Thus both (3.9) and (3.10) lead to contradictions, on account of the assumption that p^{m+1} does not divide a. Hence p^{m+1} divides a. Thus $a \in \left(p^{m+1}\right)$.

This proves that (p^{m+1}) is closed. The proof by induction is complete.

THEOREM 3.2. *Let A be a complex p-normed algebra, which is a generalized unique factorization domain and a pre-Bezout domain. Then A is isomorphic to the complex field.*

Proof. By Theorem 2.3 (2), it suffices to show that for any $a \neq 0$, $a \in A$, (a) is closed. If a is a unit this is obvious. If a is a non-zero non-unit of A, then, by the definition of a generalized unique factorization domain, we can write

$$(3.13) \qquad (a) = \bigcap_{\alpha \in S} \left(p_\alpha^{n_\alpha}\right) .$$

As each $\left(p_\alpha^{n_\alpha}\right)$ is closed by Theorem 3.1 we see that (a) is closed. This completes the proof of the theorem.

PROPOSITION 3.3. *Let A be a complex p-normed algebra, which is also a principal ideal domain. Then A is isomorphic to the complex field C.*

Proof. We observe that a principal ideal domain is a unique factorization domain and hence a generalized unique factorization domain. Besides a principal ideal domain is a Bezout domain and hence a pre-Bezout domain. The result now follows from Theorem 3.2.

REMARK 3.1. Proposition 3.3 was proved for Banach algebras by Srinivasan and Hu Shaing in [8].

Let $A(D)$ denote the algebra of all complex valued functions which are continuous on the closed unit disc $\bar{D} = \{z/|z| \leq 1\}$ and which are analytic inside the open unit disc $D = \{z/|z| < 1\}$ taken under the usual algebraic operations. Hoffman [3] has shown that $A(D)$ is a Bezout domain ([3], p. 88). $A(D)$ can be turned into a p-normed algebra by defining, for $f(z) \in A(D)$,

$$(3.14) \qquad \|f\| = \sup_{|z| \leq 1} |f(z)|^p \quad (0 < p \leq 1) .$$

THEOREM 3.4. *Let $A(D)$ be the disc algebra mentioned above. Let Γ be the algebra of all entire functions. Let $C(x)$ be the algebra of all formal power series in one variable x over C. Then the following* *conclusions hold:*

(1) *neither* $C(x)$ *nor* Γ *can be made into* p*-normed algebras under any norm;*

(2) $A(D)$ *is not a generalized unique factorization domain.*

Proof. (1) $C(x)$ is a principal ideal domain. For $C(x)$ the result follows from Proposition 3.3. For Γ the result can be deduced from Theorem 3.2, since Γ is a generalized unique factorization domain which is also a Bezout domain.

(2) $A(D)$ is a p-normed algebra using the norm given in (3.14). If $A(D)$ were to be a generalized unique factorization domain, it would follow from Theorem 3.2 that it is isomorphic to C , since $A(D)$ is a Bezout domain. This shows that $A(D)$ cannot be a generalized unique factorization domain.

REMARK 3.2. By the 'Spectral Theorem' we shall mean the statement: 'The spectrum of any element in a complex p-normed algebra is a compact subset' (Theorem 2.6). Since Γ contains elements which have unbounded spectrum, the result that it cannot be made into a p-normed algebra could have been directly deduced from the 'Spectral Theorem'. However, the fact that the 'Spectral Theorem' does not imply Theorem 3.2 or Proposition 3.3 follows from the example of $C(x)$ in which every element has a bounded spectrum. The impossibility of making $C(x)$ into a p-normed algebra is a consequence of Theorem 3.2 (or Proposition 3.3), and it cannot be deduced from the 'Spectral Theorem'. That Theorem 3.2 is a true generalization of Proposition 3.3 follows from Γ . The impossibility of making it into a p-normed algebra follows from Theorem 3.2 and not from Proposition 3.3, since Γ is not a principal ideal domain. Iyer [4] has shown that Γ can however be made into a B_0-algebra (a locally convex complete metrizable algebra). Thus, while the two conditions of pre-Bezout domain and generalized unique factorization domain make a p-normed algebra trivial, such is not the case for B_0-algebras as is illustrated by Γ .

The proofs of the next two theorems are similar to the proofs given in the paper of Srinivasan [7], for the case of Banach algebras. We merely state these results for p-normed algebras.

THEOREM 3.5. *Let* A *be a complex* p*-normed algebra, which is also* *an integral domain. If* A *is locally finite, then* A *is isomorphic to* C .

THEOREM 3.6. *Let* C^n $(n \geq 2)$ *be the* n*-dimensional linear space of complex* n*-tuples over* C . C^n *cannot be simultaneously made into an integral domain and a* p*-normed algebra.*

REMARK 3.3. We cannot drop the requirement of 'integral domain' in Theorem 3.5 or in Theorem 3.6. In fact, C^n $(n \geq 2)$ can be made into a p-normed algebra, though not into an integral domain. This can be done as follows: if $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, we define multiplication by

$$x \cdot y = (x_1 y_1, \ldots, x_n y_n)$$

and

$$\|x\| = \max_{1 \leq i \leq n} |x_i|^p \quad (0 < p \leq 1) .$$

C^n $(n \geq 2)$ becomes a p-normed algebra, but obviously has divisors of zero.

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