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A REMARK ON A THEOREM OF LYAPUNOV

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Consider the linear ordinary differential equation

(1) $$x'(t) = Ax(t), \quad t \geq 0,$$

where $x \in E^n$, the n-dimensional Euclidean space and A is an $n \times n$ constant matrix. Using a matrix result of Sylvester and a stability result of Perron, Lyapunov [4] established the following theorem which is basic in the stability theory of ordinary differential equations:

THEOREM (Lyapunov). *The following three statements are equivalent:*

(I) *The spectrum $\sigma(A)$ of A lies in the negative half plane.*

(II) *Equation* (1) *is exponentially stable, i.e. there exist μ, $K > 0$ such that every solution $x(t)$ of* (1) *satisfies*

(2) $$\|x(t)\| \leq K \|x(t_0)\| e^{-\mu(t-t_0)}, \quad t \geq t_0 \geq 0$$

where $\| \ \|$ denotes the Euclidean norm.

(III) *There exists a positive definite symmetric matrix Q, i.e. $Q = Q^*$ and there exist $q_1, q_2 > 0$ such that*

(3) $$q_1\|\xi\|^2 \leq (Q\xi, \xi) \leq q_2\|\xi\|^2, \quad \xi \in E^n,$$

satisfying

(4) $$QA + A^*Q = -I,$$

where I is the identity matrix.

The purpose of this little note is to show that the above theorem remains valid if the underlying space E^n is replaced by an arbitrary Hilbert space $\mathscr{H}$, and A, Q are meant to be bounded linear operators on $\mathscr{H}$. We shall prove this result by establishing the sequence of implications (I) $\Rightarrow$ (II) $\Rightarrow$ (III) $\Rightarrow$ (I). The proof that (I) $\Rightarrow$ (II) is well known, in fact, it is valid under more general conditions (see, e.g., Massera and Schaffer [5] for Banach spaces and Almkvist [1] for Banach algebras). We present a proof of this for the sake of completeness. To show that (II) $\Rightarrow$ (III), we modify an ingenious proof due to Bellman [2] for the finite dimensional case. The proof that (III) $\Rightarrow$ (I) seems new. For a related extension of this result of Lyapunov to semigroups of operators on a Banach space with a continuous positive definite Hermitian form, we refer the reader to Datko [3]

(I) $\Rightarrow$ (II) Since $\sigma(A) \subseteq \{\lambda: \text{Re}\ \lambda < 0\}$, and $\sigma(A)$ is closed, there exists by the Spectral Mapping Theorem, a $\delta > 0$ such that $\sigma(e^A) \subseteq \{\lambda: |\lambda| \leq e^{-\delta} < 1\}$. Consequently,

the spectral radius $\nu(e^A)=\lim_{n\to\infty}\|e^{An}\|^{1/n}<e^{-\delta}$, and there exists a positive integer N such that $\|e^{AN}\|\le e^{-\delta}$. Denote $K_1=\sup_{0\le t\le N}\|e^{At}\|$. Let $t=kN+t_1$, where $0\le t_1<N$.

We observe

$$\begin{aligned}\|e^{At}\| &= \|e^{A(kN+t_1)}\|\le \|e^{kAN}\|\,\|e^{At_1}\|\\ &\le K_1\|e^{AN}\|^k\le K_1\,e^{-k\delta}\\ &= K_1\,e^{-(\delta/n)(kN+t_1)}\,e^{(\delta/N)t_1}\le Ke^{-\delta t},\end{aligned}$$

where $K=K_1\,e^{\delta}$, from which (II) readily follows.

(II) ⇒ *(III)* Consider the operator Q defined by

$$Q=\lim_{T\to\infty}\int_0^T e^{A^*t}\,e^{At}\,dt. \tag{5}$$

From (2), we have for $\xi\in\mathscr{H}$,

$$(e^{A^*t}\,e^{At}\xi,\xi)=\|e^{At}\,\xi\|^2\le K^2\|\xi\|^2\,e^{-2\mu t}, \tag{6}$$

hence the limit defined in (5) exists and defines a linear operator on $\mathscr{H}$. It is also a bounded operator with the operator norm $\le K^2/2\mu$. Clearly, $Q=Q^*$, and it remains to show that Q satisfies (3) and (4). Integrating (6), we have

$$\begin{aligned}(Q\xi,\xi) &= \left(\int_0^\infty e^{A^*t}\,e^{At}\,dt\,\xi,\xi\right)\\ &= \int_0^\infty (e^{A^*t}\,e^{At}\,\xi,\xi)\,dt=\int_0^\infty \|e^{At}\,\xi\|^2\,dt\\ &\le K^2\|\xi\|^2\int_0^\infty e^{-2\mu t}\,dt=\frac{K^2}{2\mu}\,\|\xi\|^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\xi\|^2 &= \|e^{-At}\,e^{At}\,\xi\|^2\le \|e^{-At}\|^2\|e^{At}\,\xi\|^2\\ &\le e^{2pt}(e^{At}\,e^{A^*t}\,\xi,\xi),\end{aligned} \tag{7}$$

where $p=\|A\|$ is the operator norm of A. From (7), we have

$$\frac{1}{2p}\|\xi\|^2\le\int_0^\infty e^{-2pt}\,\|\xi\|^2\,dt\le(Q\xi,\xi),$$

showing that Q satisfies (3). On the other hand, we have by a simple differentiation the following identity

$$\frac{d}{dt}(e^{A^*t}\,e^{At})=e^{A^*t}\,e^{At}\,A+A^*\,e^{A^*t}\,e^{At}. \tag{8}$$

Since $\lim_{t\to\infty}e^{A^*t}\,e^{At}$ exists and defines the null operator on account of (6), we integrate (8) from 0 to ∞ to obtain (4).

(III) ⇒ *(I)* Let $\lambda \in \sigma(A)$. Since $A-\lambda=Q^{-1}(QA-\lambda Q)$, $0\in\sigma(QA-\lambda Q)$. For a bounded linear operator T, we denote its numerical range by $W(T)=\{(T\xi,\xi):\|\xi\|=1\}$. By Hausdorff–Toeplitz Theorem [6], we have

$$0\in\overline{W(QA-\lambda Q)}\subseteq\overline{W(QA)}-\lambda\overline{W(Q)}.$$

Note that $\overline{W(QA)}=\{z:\operatorname{Re} z=-1\}$ and $W(Q)=\{z: z\geq q_1>0\}$, it follows that $\operatorname{Re}\lambda\leq {}^{-1}/_{q_1}<0$, proving (I).

References

1. G. Almkvist, *Stability of linear differential equations in Banach algebras*, Math. Scand. **14** (1964), 39–44.

2. R. Bellman, *Notes on matrix theory* X, *A problem in control*, Quart. Appl. Math. **14** (1957), 417–419.

3. R. Datko, *An extension of a theorem of A. M. Lyapunov to semi-groups of operators*, J. Math. Anal. Appl. **24** (1968), 290–295.

4. F. R. Gantmacher, *The theory of matrices*, English translation, (2 volumes), Chelsea, New York, 1959.

5. J. L. Massera and J. J. Schaffer, *Linear differential equations and functional analysis*, III. *Lyapunov's second method in the case of conditional stability*, Ann. of Math. **69** (1959), 535–574.

6. M. H. Stone, *Linear transformations in Hilbert space and their applications to analysis*, Amer. Math. Soc. Coll. **XV**, New York, (1932).

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