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INTRODUCTION—FEFERMAN 1977

In Part I of Feferman's now famous paper, "Categorical Foundations and Foundations of Category Theory" (1977), he begins by noting Mac Lane's (1971) claims against set-theoretic foundations, namely: 1) that they are *inappropriate* for mathematics as practiced; this in light of the "increasingly dominant conception (by mathematicians) of mathematics as the study of abstract structures" and 2) that they are *inadequate* for the "full needs" of category theory (p. 149).

In consideration of claim 1), Feferman notes that this structuralist view "has been favored particularly by workers in category theory because of its successes in organizing substantial portions of algebra, topology, and analysis," and is perhaps "best expressed" (Ibid.) by Lawvere (1966). In that paper Lawvere formulates a first-order theory whose objects are conceived to be arbitrary categories and whose morphisms are functors between them; this theory is subsequently referred to as that expressed by the CCAF axioms (CCAF standing for the Category of all Categories as a Foundation of Mathematics).

Feferman notes Kreisel's objections to these proposals and, more generally, to the program for categorical foundations, "among which are arguments for what is achieved in set-theoretical foundations that is not achieved in other schemes (present or projected)." Instead of elaborating on these, however, he brings to the fore what he considers to be a "very simple objection," "which is otherwise neutral on the question of 'proper foundations' for mathematics" (p. 150), namely:

> [*W*]*hen explaining* the general notion of structure and of particular kinds of structures such as groups, rings, categories, etc., we implicitly *presume as understood* the ideas of *operation* and *collection*; e.g., we say that a group consists of a collection of objects together with a binary operation satisfying such and such conditions. Next, when explaining the notion of *homomorphism* for groups or *functor* for categories, etc., we must again understand the concept of operation. (Ibid.)

Hence, he continues,

> . . . we must make use of the unstructured notions of operation and collection to explain the structural notions to be studied. The *logical* and *psychological priority* . . . of the notions of operation and collection is thus evident. . . It follows that a theory whose objects are supposed to be highly structured and which does not explicitly reveal assumptions about operations and collections cannot claim to constitute a foundation for mathematics, simply because those assumptions are unexamined. It is evidently begging the question to treat collections (and operations between them) as a category, which is supposed to be one of the objects of the universe of the theory to be formulated. (Ibid.)

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Feferman then states that

> [t]he foundations of mathematics must still be pursued in a direct examination of the notions of operation and collection. There are at present only two (more or less) coherent and comprehensive approaches to these, based respectively on the Platonist and constructivist viewpoints. (p. 151)

The Platonist viewpoint is taken as having been fully elaborated by extensional set theory. The constructive, intensional, approach is held, by contrast, as "still undergoing development." In any case, the consequence of these two incompatible accounts appears to be that "multiple foundations" (Ibid.) may be necessary. Moreover, while other foundations may still be put forth, Feferman does not believe that these considerations affect his claim that, in any case, some theory of operations and collections must be prior to the explanation of the basic categorical notions and their application to particular classes of structures.

He next considers two "rejoinders" by Mac Lane.[1] The first is that recent work in topos theory shows "that the program of categorical foundations has made considerable progress (as formulated in the Elementary Theory of the Category of all Sets (ETCS) and certain of its extensions)... and this makes the discussion above out of date and beside the point" (pp. 151–152). That is, this shows that "set-theoretical foundations and categorical foundations are entirely equivalent and hence one cannot assign any logical priority to the former" (p. 152). The second is "that questions of psychological priority are 'exceedingly fuzzy' and subjective. Further, mathematicians are well known to have very different intuitions, and these may strongly be affected by training." (Ibid.)

Feferman next offers the following responses to these two rejoinders:

> i) my use of 'logical priority' refers not to relative strength of formal theories but to the order of definition of concepts, in cases where certain of these *must* be defined before others. For example, the concept of vector space is logically prior to that of linear transformation...;
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> ii) 'psychological priority' has to do with natural order of understanding. This is admittedly 'fuzzy' but not always 'exceedingly' so;
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> iii) the general concepts of operations and collection have logical priority with respect to structural notions (such as 'group', 'category', etc.) because the latter are defined in terms of the former but not conversely. I realize that workers in category theory are so at home in their subject that they find it more natural to think in categorical rather than set-theoretical terms, but I would liken this to not needing to hear, once one has learned to compose music;
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> iv) the preceding has to do with an order between concepts according to which some of them appear more basic than others. There is in consequence an order between theories of these concepts;
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> v) since topoi are just special kinds of categories the objections here to a program for categorical foundations of mathematics apply all the more to foundations via theories of topoi. ...the axioms [of ETCS and its extensions] were clearly obtained by tracing out just what was needed to secure the translations of [axiomatic theories of sets] into the language of topoi. This applies particularly to the replacement scheme and bears out my contention of the priority set-theoretical concepts. (pp. 152–154)

[1] It is unclear whether these claims by Mac Lane are published, Feferman here states that they arose "in correspondence."

Finally, in his last response, Feferman sums up Part I of his paper as follows,

> vi) ... let me repeat that I am *not* arguing for accepting current set-theoretical foundations of mathematics. Rather, it is that on the Platonist view of mathematics something like present systems of set theory must be prior to any categorical foundations. More generally, on any view of abstract mathematics priority must lie with notions of operation and collection. (p. 154)

In making the move to Part II of his paper, Feferman turns to consider Mac Lane's claim 2) that set-theoretical foundations are *inadequate* for the full needs of category theory, because we cannot form either a) the category of *all* structures of a given kind or b) the category of *all* functors between any two categories. It is in this light that he next considers the two "restricted" set-theoretical foundations for category theory that had then been proposed, again in Mac Lane (1971), namely, that of i) using Grothendieck universes and ii) reducing to the BG (Bernays/Godel) theory via the distinction between 'small' and 'large' categories. He argues that neither of these proposals resolve the inadequacies in "non-relative or non-partial ways," and, in any case, the "restrictions employed seem mathematically unnatural and irrelevant." (p. 155) Yet, he does acknowledge that "there is no urgent or compelling reason to pursue foundations of unrestricted category theory, since the schemes i), ii) (or their variants and refinements) serve to secure all practical purposes." (Ibid.) It is in their place, and for what Feferman calls "aesthetic reasons," that the remainder of Part II of his (1977) paper is devoted to offering a new, nonextensional, theory *T* of operations and collections as a foundation of unrestricted category theory.

As explained in Feferman's paper, the (1977) proposal was subsequently abandoned. What he concentrates on here, following a relatively brief discussion of more recent claims on behalf of categorical foundations, is the return to an alternative approach in an enriched stratified theory of sets and classes, that he had pursued earlier in an unpublished paper from 1974. This, he claims, illustrates just how far one can go in meeting what he considers to be four basic requirements for the foundations of unrestricted category theory within a suitable axiomatic system S:

(R1) Form the category of all structures of a given kind, such as the category of all groups, the category of all topological spaces, and the category of all categories.
(R2) Form the category of all functors between any two given categories.
(R3) Establish the existence of basic structures such as the natural numbers and carry out all the standard set-theoretical operations (pairing, unions, intersections, complements, powers, products, quotients, etc.)
(R4) Establish the consistency of S relative to a currently accepted system of set theory.

Previously, Feferman's stratified system S* succeeded in meeting (R1) and (R2) in full and most of (R3), with the consistency of S* established relative to an extension of ZFC by the assumption of two strongly inaccessible cardinals. What was missing from (R3) was how to deal with two problems: 1) carrying out the passage from a class A and equivalence relation E on A to the class A/E of all equivalence classes x/E for x in A, and 2) carrying out the formation of Cartesian products $\Pi B_x[x \in A]$. Both of these are essentially unstratified steps. The new contribution here is a way of taking care of Problem 1 via a system S† obtained by a simple modification of S*. But Problem 2 is still an obstacle to this particular approach. Since dealing with products (and limits more generally) is essential to certain basic results in category theory such as Freyd's Adjoint Functor Theorem, that means more work must be done, either by a further improvement of a stratified approach or by coming up with something quite different to satisfy all of (R1)–(R4). That, then, is "what remains to be done."

In his 1977 paper, Feferman raised the following problem confronting claims that category theory (CT) provides an independent foundation for mathematics: it takes for granted central foundational notions of 'collection' and 'operation' rather than explaining these, as any *bona fide* foundational framework must. In his own contribution of 2003, Hellman reinforced this point by asking how category and topos theories (TT) get started, arguing that they presuppose the set-theoretically explicable notion of 'satisfaction', or, equivalently, a theory of relations, in order to articulate what a mathematical structure is. As a friendly amendment, Hellman (2003) sketched a Theory of Large Domains, which, without employing set membership, characterizes cumulative hierarchies of strongly inaccessible height and then goes on to construct within them commonly studied mathematical spaces or structures and categories thereof. The combination of mereology and plural quantification is there deployed so that there is not even the appearance of reliance on set theory.

McLarty (2004, 2005) and Awodey (2004), in differing ways, defended the autonomy of CT and TT, without such a theory of large domains. McLarty claiming that particular theories of categorical set theory or a theory of category of categories already suffice as foundational frameworks, Awodey claiming that the "top-down" approach of CT and TT obviate the need for an independent theory of relations. In his current contribution, Hellman explains (again) why he regards these replies to his 2003 as unsatisfactory, and then offers a more modest proposal than the Theory of Large Domains, one which eschews modal logic and also the "bottom-up" approach of explicitly constructing typical mathematical structures, mappings among them, and categories thereof, in favor of the sort of "top-down" approach that Awodey favors.

Again, the combination of mereology and plural quantification suffices and dispels even the semblance of dependence on set theory. The upshot, Hellman concludes, is to sustain a pluralism of foundations along lines actually foreseen by Feferman (1977), something that should be welcomed as a way of resolving this long-standing debate.

Landry (2011) argued that in rationally reconstructing Hilbert's organizational use of the axiomatic method, we can construct a *pure algebraic* version of category-theoretic mathematical structuralism. More specifically, in reply to Shapiro (2005), she showed that we can be structuralists *all the way down*; we do not have to appeal to some assertory background theory to guarantee the truth of our axioms.

In this paper Landry explores the ways in which Feferman's (1977) arguments have been used (and misused) in the philosophical literature to argue both for and against category-theoretic structuralism. Having navigated the philosophical landscape, her aim is to directly reply to Feferman (1977) by tying together three threads: a) Hilbert's distinction between the genetic and axiomatic method; b) Awodey's distinction between top-down, algebraic, and bottom-up, assertory, ways of working; and c) her distinction between the organizational versus constitutive use of the term 'foundation'. Thus, whereas previously, when speaking of the foundational role of category theory, she was content use the terms 'language' or 'framework', her intent here is to retake the term 'foundation' and argue that one can be a category-theoretic structuralist *all the way up*.

The aim of Marquis's paper is to answer Feferman's argument by acknowledging it. Indeed, he accepts that one has to start from collections and operations, at least logically. But one does not have to go straight to ZF(C). It is entirely reasonable to suppose that, on any abstract conception of mathematical objects, a different theoretical framework can be developed, a framework in which categories play an indispensable role.

In his paper, Marquis argues that for the latter to be possible, one has to modify three fundamental aspects of the foundational landscape: the underlying logic, the notion of set and the notion of category itself. Marquis discusses why, in general, these modifications have to be made and sketches three ways in which they are being developed, with a special emphasis on Makkai's general approach. Finally, he makes a series of remarks raising doubts on the psychological aspect of Feferman's argument.

Working as he does in categorical foundations for mathematics, McLarty was intrigued by the sweep of Feferman's claim that one cannot even think of such things either logically or psychologically, or at least nearly cannot. McLarty has replied to that in the past but remains puzzled as to how Feferman supposed categorical foundationalists took up an impossible, or nearly impossible, way of thinking in the first place? Here he addresses Feferman's explanation, as offered in Berkeley, that categorical foundations are a "deceptive ideological shell game." In this paper, he argues, to the contrary, that categorical foundations formalize currently standard approaches in actual mathematics.

McLarty notes that Kreisel's project of foundations as a deep analysis of mathematical practice and Mac Lane's interest in truths actually used to organize practice, are distinct but no way contradictory. He stresses the practical need for an articulate organization of principles in mathematics and looks at categorical foundations in that way. He also notes that Feferman shares one thing with categorical foundationalists, as against logicists or those advocating ZF set theory, which is that Feferman, like categorical foundationalists, takes transformations to be equally fundamental with collections and does not reduce them to a special case of collections.