## A MAPPING PROBLEM AND $J_p$ -INDEX. II

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**1. Introduction.** In [9], indices for equivariant mappings have been defined in the case that the transformation groups are cyclic. Thus a question will naturally arise as to the generalization of [4, Theorem 2] or [8, § IV, Theorem 2.8]. In this paper we will generalize the above result when the transformation groups are of order  $p^a q^b$ , p, q are odd prime numbers. The method used here can be used directly for more general cyclic groups, say, of order  $n = p_1^{\alpha_1} \dots p_m^{\alpha_m}$ . However, the results are too complicated to be of interest.

**2.** The index of a Euclidean space. Let  $\alpha = p^{\alpha}$  and  $\beta = q^{b}$ , where p and q are odd prime numbers. Throughout this paper let  $S = \{1, s, \ldots, s^{\alpha\beta-1}\}$  act on  $R^{\alpha+\beta}$  and SO $(\alpha + \beta)$  as follows:

 $s(x_1, \ldots, x_{\alpha}, y_1, \ldots, y_{\beta}) = (x_2, \ldots, x_{\alpha}, x_1, y_2, \ldots, y_{\beta}, y_1).$ 

Let  $F(R^{\alpha+\beta}) = \{\bar{x} \mid \bar{x} \in R^{\alpha+\beta} \text{ and } s^i(\bar{x}) = \bar{x} \text{ for some } i, 0 < i < \alpha\beta\}$ . Then S acts on  $R_*^{\alpha+\beta} = R^{\alpha+\beta} - F(R^{\alpha+\beta})$  and on SO $(\alpha + \beta)$  properly discontinuously. It is clear that  $F(R^{\alpha+\beta}) = F_1(R^\alpha) \cup F_2(R^\beta)$ , where  $F_1 = F(R^\alpha) \times R^\beta$ ,  $F_2 = R^\alpha \times F(R^\beta)$  and F is the generalized (or fat) diagonal defined in [8, p. 411].

Let  $I^{\alpha+\beta}$  be the unit  $(\alpha + \beta)$ -cube; then we may assume that  $R^{\alpha+\beta} = \inf I^{\alpha+\beta}$ so that  $R_*^{\alpha+\beta} = \inf(I^{\alpha+\beta} - F(I^{\alpha+\beta})) \subset I^{\alpha+\beta} - F(I^{\alpha+\beta}) = I_*^{\alpha+\beta}$ . The inclusion is clearly an equivariant map so that  $\nu(R_*^{\alpha+\beta}) \leq \nu(I_*^{\alpha+\beta})$ , where  $\nu(X)$  is the  $J_p$ -index of X [9, (4.5)]. Now we have  $I_*^{\alpha+\beta} = I_*^{\alpha} \times I_*^{\beta}$ . Let K be a simplicial complex of I with the usual subdivision and  $K^{\alpha+\beta}$  the cell complex  $K \times \ldots \times K$  $(\alpha + \beta \text{ factors})$ . Let  $S_p = \{1, s^{\beta p^{\alpha-1}}, \ldots, s^{(p-1)\beta p^{\alpha-1}}\}$ . Let  $(K^{p^{\alpha-1}})_*^p$  be the subcomplex of  $K^{\alpha}$  which consists of all cells  $\sigma_1 \times \ldots \times \sigma_p$   $(\sigma_i \in K^{p^{\alpha-1}})$  with no vertex of K common to all these  $\sigma_i$ . It may be shown that  $|(K^{p^{\alpha-1}})_*^p|$  is a deformation retract of  $I_*^{\alpha}$  as mentioned in [10, Theorem 1].

 $(K^{p^{a-1}})_*^p$  is of dimension  $(p-1)p^{a-1}-1$ . Also it may be shown that  $|(K^{q^{b-1}})_*^q|$  is a deformation retract of  $I_*^{\beta}$ . Hence  $|(K^{p^{a-1}})_*^p| \times |(K^{q^{b-1}})_*^q|$  is a deformation retract of  $I_*^{\alpha+\beta} = I_*^{\alpha} \times I_*^{\beta}$ . Therefore  $H^i(I_*^{\alpha+\beta}) = 0$  for  $i \ge (p-1)p^{a-1} + (q-1)q^{b-1} - 1$ . By [3, p. 44] we have the following result.

THEOREM 2.2.  $\nu(R_*^{\alpha+\beta}) \leq (p-1)p^{a-1} + (q-1)q^{b-1} - 1.$ 

3. The index of SO( $\alpha + \beta$ ). The following results are known [1, proposition 10.2 and théorème 19.1], where  $B_x^* = H^*(B_x, J_p)$ ,  $B_x$  is the classifying space of

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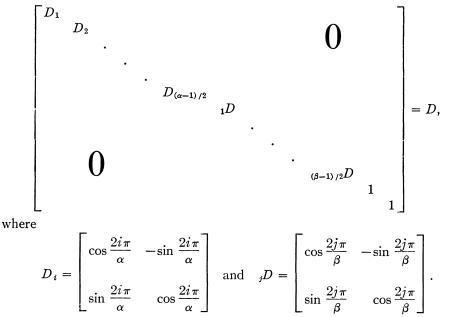
a compact topological group X, and  $J_p(\ )$  and  $\bigwedge(\ )$  refer to the polynomial and to the exterior algebra over  $J_p$ , the integers modulo p, respectively:

$$\begin{aligned} H^*(\mathrm{SO}(\alpha+\beta), J_p) &\cong \bigwedge (u_{\alpha+\beta-1}, u_3, u_7, \ldots, u_{2(\alpha+\beta)-5}), & \dim u_i = i, \\ B_{\mathrm{SO}(\alpha+\beta)} &\cong J_p(v_{\alpha+\beta}, v_4, v_8, \ldots, v_{2(\alpha+\beta)-4}), & \dim v_i = i. \end{aligned}$$

Let T be the maximal torus in SO( $\alpha + \beta$ ), G the subgroup of T of elements of order  $\alpha\beta$ .  $S = \{1, s, \ldots, s^{\alpha\beta-1}\}$  may be embedded in G, since

$$s = \begin{bmatrix} \alpha & \beta \\ 010...0 & \dots \\ 0010..0 & \dots \\ 00...1 & 0 \\ 10...0 & \\ 0 & 010..0 \\ 0 & 0010..0 \\ \dots \\ 0....1 \\ 10...0 \end{bmatrix} \beta^{\alpha}$$

is orthogonal and similar to



Let  $n = \frac{1}{2}(\alpha + \beta)$ . We have:

$$B_T^* \cong J_p(t_1, t_2, \dots, t_n), \qquad \text{dim } t_i = 2,$$
  

$$B_G^* \cong \bigwedge (a_1, \dots, a_n) \otimes J_p(b_1, b_2, \dots, b_n), \qquad \text{dim } a_i = 1, \text{ dim } b_i = 2,$$
  

$$B_S^* \cong \bigwedge (a) \otimes J_p(b), \qquad \text{dim } a = 1, \text{ dim } b = 2.$$

Let M be a compact topological group, L a subgroup of M, and  $E_M$  the N-universal space of M for a sufficiently large N. The projection  $\rho(L, M)$  of  $B_L = E_M/L$  onto  $B_M = E_M/M$  induces  $\rho^*(L, M): B_M^* \to B_L^*$ .

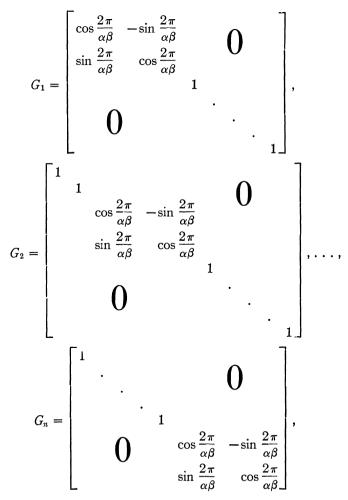
By [1, p. 200] we have:

$$\rho^*(T, SO(\alpha + \beta))v_{4i} = \sigma_i(t_1^2, \ldots, t_{n-1}^2)$$
 for  $i = 1, 2, \ldots, n-1$ ,

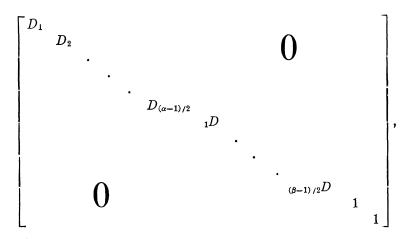
where  $\sigma_i(t_1^2, \ldots, t_{n-1}^2)$  is the *i*th symmetric function in the arguments  $t_1^2, \ldots, t_{n-1}^2$ . We have

$$\rho^*(T, \operatorname{SO}(\alpha + \beta))v_{\alpha+\beta} = t_1t_2 \dots t_n \quad \text{and} \quad \rho^*(G, T)t_i = b_i.$$

Since G is generated by  $G_1, G_2, \ldots, G_n$ , where



and since s is orthogonal and similar to



we see that

$$\rho^*(S, G)b_i = i\beta b$$
, for  $i = 1, ..., \frac{1}{2}(\alpha - 1)$ ,  $\rho^*(S, G)b_i = (i - \frac{1}{2}(\alpha - 1))\alpha b$ ,  
for  $i = \frac{1}{2}(\alpha - 1) + 1, ..., n - 1$ , and  $\rho^*(S, G)b_n = \alpha\beta b$ . Hence  
 $\rho^*(S, G)b_i \equiv 0 \pmod{p}$ 

for  $i > \frac{1}{2}(\alpha - 1)$ . This generalizes [5, (a)].

We have the following lemma [4; 5].

LEMMA  $\prod_{i=1}^{m} (1 + (ib)^2) \equiv 1 + Ab^{2m} \pmod{p}$  where  $m = \frac{1}{2}(p-1)$  and A is a non-vanishing constant.

$$\rho^{*}(S, SO(\alpha + \beta))(B^{*}_{SO(\alpha+\beta)}) = \rho^{*}(S, G)\rho^{*}(G, T)\rho^{*}(T, SO(\alpha + \beta))B^{*}_{SO(\alpha+\beta)}$$
$$= \rho^{*}(S, G) \bigg[ J_{p} \bigg( \prod_{i=1}^{n-1} (1+b_{i}^{2}) \bigg) \otimes J_{p}(b_{1}, b_{2}, \dots, b_{n}) \bigg]$$
$$= \rho^{*}(S, G) \bigg[ J_{p} \bigg( \prod_{i=1}^{\frac{1}{2}(\alpha-1)} (1+b_{i}^{2}) \bigg) \bigg].$$

This equals, since  $\alpha = p^{a}$ ,  $1 + A'b^{(p-1)\alpha/p}$  [8, § IV, Lemma 2.5], where A' is a non-vanishing constant. Therefore, by [2, Proposition 10.3],

$$H^*(\mathrm{SO}(\alpha+\beta)/S) = \bigwedge(a) \otimes J_p(b)/\mathscr{I}(b^{(p-1)\alpha/p}) \otimes P',$$

where  $\mathscr{I}(b^{(p-1)\alpha/p})$  is the ideal generated by  $b^{(p-1)\alpha/p}$  and

$$P' = \wedge (u_{\alpha+\beta-1}, u_3, \ldots, \hat{u}_{(2(p-1)\alpha)/p-1}, \ldots, u_{4n-5}).$$

By an argument similar to that in [9], we may show the following.

THEOREM 1.  $\nu(SO(\alpha + \beta)) = 2(p - 1)\alpha/p = 2(p - 1)p^{\alpha - 1}$ .

4. A mapping problem. When  $g: S^{\alpha+\beta-1} \to \mathbf{R}$  is given, the map

$$G: SO(\alpha + \beta) \rightarrow \mathbf{R}^{\alpha + \beta}$$

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is defined by  $G(w_1, \ldots, w_{\alpha+\beta}) = (g(w_1), \ldots, g(w_{\alpha+\beta}))$ . Let  $\mathbf{E} = G^{-1}(F(\mathbf{R}^{\alpha+\beta}))$ and  $\mathbf{A} = \mathrm{SO}(\alpha + \beta) - \mathbf{E} = G^{-1}(\mathbf{R}_*^{\alpha+\beta})$ . It is clear that  $F|\mathbf{A}: \mathbf{A} \to \mathbf{R}_*^{\alpha+\beta}$  is free equivariant. Therefore,  $\nu(\mathbf{A}) \leq \nu(\mathbf{R}_*^{\alpha+\beta}) \leq (p-1)p^{\alpha-1} + (q-1)q^{b-1} - 1$ .

If X is a locally Euclidean manifold, with Čech or Alexander (co)homology groups over  $J_p$  with closed supports, [7, (6.2)] becomes

with exact rows and commutativity in the squares, where U is an open set in X, X' = X/S, U' = U/S, S is a properly discontinuous transformation group on X and U and  $N = \dim X$ .

(b) dim (X' - U') = M implies  $H_j(X' - U') = 0$  for j > M.

Let  $\mathscr{A}^{i}(X, s)$  be the *i*th  $J_{p}$ -Smith class of X in  $H^{i}(X/S, J_{p})$  [9, Definition 3.5]. Let  $\phi: X \to Y$  be a free equivariant mapping. Then

$$\phi^* \mathscr{A}^i(Y, s) = \mathscr{A}^i(X, s)$$

## **[9**, (3.6)].

With the above notation we have the following result.

THEOREM 1. Let  $\mathbf{E}' = \mathbf{E}/S$ . If

$$2(p-1)p^{a-1} > (p-1)p^{a-1} + (q-1)q^{b-1} - 1,$$

that is, if  $\alpha > \beta$ , then  $H_{N-j}(\mathbf{E}') \neq 0$  for

$$(p-1)p^{a-1} + (q-1)q^{b-1} - 1 \leq j \leq 2(p-1)p^{a-1} - 1,$$

where  $N = \dim SO(\alpha + \beta) = \frac{1}{2}(\alpha + \beta)(\alpha + \beta - 1).$ 

*Proof.* Substitute  $X' = SO(\alpha + \beta)/S$ , U' = A' in the diagram (a). Since  $\nu(\mathbf{R}_*^{\alpha+\beta}) \leq (p-1)p^{\alpha-1} + (q-1)q^{b-1} - 1$ ,  $\mathscr{A}^i(\mathbf{A}, s) = 0$  for

$$i \ge (p-1)p^{a-1} + (q-1)q^{b-1} - 1$$

On the other hand,  $\mathscr{A}^{i}(\mathrm{SO}(\alpha + \beta), s) \neq 0$  for  $i \leq 2(p-1)p^{\alpha-1} - 1$ . The exactness and commutativity of the diagram (a) will yield the desired result.

*Remarks.* (1) If  $\beta > \alpha$ , that is, if

$$2(q-1)q^{b-1} > (p-1)p^{a-1} + (q-1)q^{b-1} - 1,$$

then we may use the  $J_q$ -indices to obtain a similar result.

(2) dim  $\mathbf{E} \geq \frac{1}{2}(\alpha + \beta - 1)(\alpha + \beta) - (p - 1)p^{\alpha - 1} - (q - 1)q^{b - 1} + 1.$ 

(3) By [6, p. 41] and the technique in [4, Corollary 4 or 8, § IV, Corollary 2.10], we may show that there exists a point  $\bar{x} \in F(\mathbf{R}^{\alpha+\beta})$  such that if  $E_0 = G^{-1}(\bar{x})$ , then dim  $E_0 \ge \dim \mathbf{E} - \dim F(\mathbf{R}^{\alpha+\beta})$ .

(4) A (p + q)-dimensional rectangular parallelotope has (p + q) possible edge lengths. If q or more edge lengths are the same, the (p + q)-dimensional rectangular parallelotope is called a q-semicube.

If a = b = 1 and p > q, then by the method used in [4, Corollary 5] we may show that: If K is a compact convex body in  $\mathbb{R}^{p+q}$ , then the dimension of the set of circumscribing q-semicubes is at least  $\frac{1}{2}(p+q)(p+q-3)+3$ .

There is a set of dimension at least

$$\frac{1}{2}(p+q)(p+q-3) + 1 = \frac{1}{2}(p+q-1)(p+q-2)$$

of circumscribing q-semicubes with the same edge length.

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