Mathematika 64 (2018) 497–501
doi:10.1112/S0025579318000050

CORRIGENDUM

STABLY FREE MODULES OVER $\mathbf{Z}[(C_P \rtimes C_Q) \times C_\infty^M]$ ARE FREE

(Mathematika 63 (2017), 451–461)

J. D. P. EVANS

In the paper [1] I claimed to show that there are no non-trivial stably free modules over integral group rings of the groups $(C_p \rtimes C_q) \times C_\infty^m$. Unfortunately there are a number of erroneous statements in [1] which vitiate the attempted proof. To explain where these occur, recall that in [1] two Milnor fibre squares $(\clubsuit)$ and $(\heartsuit)$ were introduced as follows:

$$(\clubsuit) = \left\{ \begin{array}{ccc} \mathbf{Z}[C_p \rtimes C_q] & \longrightarrow & \mathcal{T}_q \\ \downarrow & & \downarrow \\ \mathbf{Z}[C_q] & \longrightarrow & \mathbf{F}_p[C_q], \end{array} \right. \qquad (\heartsuit) = \left\{ \begin{array}{ccc} \mathbf{Z}[(C_p \rtimes C_q) \times \Gamma] & \longrightarrow & \mathcal{T}_q[\Gamma] \\ \downarrow & & \downarrow \\ \mathbf{Z}[C_q \times \Gamma] & \longrightarrow & \mathbf{F}_p[C_q \times \Gamma]. \end{array} \right.$$

Here $\Gamma = C_\infty^m$, and $\mathcal{T}_q = \mathcal{T}_q(A, \pi)$ is the ring of quasi-triangular $q \times q$ matrices where $A = \mathbf{Z}[\zeta_p]^{C_q}$ is the subring of the cyclotomic integers $\mathbf{Z}[\zeta_p]$ fixed under the Galois action of C_q and $\pi \in \mathrm{Spec}(A)$ is the unique prime over p.

The most obvious errors [1, Corollary 3.4] include a misdescription of the unit group $U(\mathbf{F}_p[C_q \times \Gamma])$, and the possibility of non-trivial rank-one stably free modules over $\mathcal{T}_2[\Gamma]$. A slightly less obvious but more significant error concerns the possibility of lifting units from $\mathbf{F}_p[C_q \times \Gamma]$ to $\mathcal{T}_q[\Gamma]$. In consequence we must amend the original statement of [1] as follows.

THEOREM A. *Let S be a stably free module of rank n over $\mathbf{Z}[(C_p \rtimes C_q) \times C_\infty^m]$ where $m \geqslant 2$. Then:*
- *if q is an odd prime, S is free provided $n \neq 2$; and*
- *if $q = 2$, S is free provided $n \geqslant 3$.*

Nevertheless, when $m = 1$ the original statement continues to hold:

THEOREM B. *Any stably free module over $\mathbf{Z}[(C_p \rtimes C_q) \times C_\infty]$ is free.*

Rather than try to patch up the proof in [1] piecemeal we give a more straightforward approach which isolates the real difficulty and avoids it where possible. We first establish four propositions.

Received 25 May 2017, published online 17 April 2018.

PROPOSITION 1. *$U(\mathbf{F}_p[C_q \times \Gamma])/U(\mathcal{T}_q[\Gamma])$ is finite; in fact*

$$|U(\mathbf{F}_p[C_q \times \Gamma])/U(\mathcal{T}_q[\Gamma])| \leqslant |U(\mathbf{F}_p[C_q])/U(\mathcal{T}_q)|.$$

Proof. As q is a divisor of $p-1$, we have

$$\mathbf{F}_p[C_q] \cong \underbrace{\mathbf{F}_p \times \cdots \times \mathbf{F}_p}_{q}.$$

Consequently

$$\mathbf{F}_p[C_q \times \Gamma] \cong \underbrace{\mathbf{F}_p[\Gamma] \times \cdots \times \mathbf{F}_p[\Gamma]}_{q}. \tag{$*$}$$

Observe that $U(\mathcal{T}_q[\Gamma])$ contains a copy of $\Gamma^{(q)} = \underbrace{\Gamma \times \cdots \times \Gamma}_{q}$, namely the diagonal matrices

$$\Delta(\gamma_1, \ldots, \gamma_q) = \begin{pmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_q \end{pmatrix},$$

where $\gamma_i \in \Gamma$. Combining this with the obvious inclusion $U(\mathcal{T}_q) \subset U(\mathcal{T}_q[\Gamma])$ gives an injection $U(\mathcal{T}_q) \times \Gamma^{(q)} \hookrightarrow U(\mathcal{T}_q[\Gamma])$. Hence we now have a surjection

$$U(\mathbf{F}_p[C_q \times \Gamma])/U(\mathcal{T}_q) \times \Gamma^{(q)} \twoheadrightarrow U(\mathbf{F}_p[C_q \times \Gamma])/U(\mathcal{T}_q[\Gamma]). \tag{$**$}$$

The ring isomorphism ($*$) now gives an isomorphism of unit groups

$$U(\mathbf{F}_p[C_q \times \Gamma]) \cong \underbrace{U(\mathbf{F}_p[\Gamma]) \times \cdots \times U(\mathbf{F}_p[\Gamma])}_{q}.$$

Now $\Gamma = C_\infty^m$ is a t.u.p. group so $\mathbf{F}_p[\Gamma]$ has only trivial units (cf. [**2**, Appendix C]). Hence

$$\begin{aligned} U(\mathbf{F}_p[C_q \times \Gamma]) &\cong \underbrace{(U(\mathbf{F}_p) \times \Gamma) \times \cdots \times (U(\mathbf{F}_p) \times \Gamma)}_{q} \\ &\cong \underbrace{U(\mathbf{F}_p) \times \cdots \times U(\mathbf{F}_p)}_{q} \times \Gamma^{(q)} \end{aligned}$$

so that, by ($*$), there are bijections

$$\begin{aligned} U(\mathbf{F}_p[C_q \times \Gamma])/U(\mathcal{T}_q) \times \Gamma^{(q)} &\leftrightarrow U(\mathbf{F}_p[C_q]) \times \Gamma^{(q)}/U(\mathcal{T}_q) \times \Gamma^{(q)} \\ &\leftrightarrow U(\mathbf{F}_p[C_q])/U(\mathcal{T}_q). \end{aligned}$$

From ($**$) we obtain a surjection $U(\mathbf{F}_p[C_q])/U(\mathcal{T}_q) \twoheadrightarrow U(\mathbf{F}_p[C_q \times \Gamma])/U(\mathcal{T}_q[\Gamma])$. The stated result now follows as $U(\mathbf{F}_p[C_q])/U(\mathcal{T}_q)$ is finite. □

PROPOSITION 2. *Let p be an odd prime and q be a divisor of $p-1$. Then, for all $n \geqslant 3$,*

$$\mathrm{GL}_n(\mathbf{F}_p[C_q \times \Gamma]) = U(\mathbf{F}_p[C_q \times \Gamma]) \cdot E_n(\mathbf{F}_p[C_q \times \Gamma]).$$

Proof. Given rings A, B such that $\mathrm{GL}_n(A) = U(A)E_n(A)$ and $\mathrm{GL}_n(B) = U(B)E_n(B)$, we have $\mathrm{GL}_n(A \times B) = U(A \times B)E_n(A \times B)$. The result thus follows from ($*$) by induction on q, the case $q = 1$ being Suslin's theorem [3], namely that

$$\mathrm{GL}_k(\mathbf{F}[\Gamma]) = U(\mathbf{F}[\Gamma]) \cdot E_k(\mathbf{F}[\Gamma])$$

for any field $\mathbf{F}$ and any integer $k \geqslant 3$. □

PROPOSITION 3. $\mathrm{GL}_n(\mathbf{F}_p[C_q \times \Gamma])/\mathrm{GL}_n(\mathcal{T}_q[\Gamma])$ *is finite for $n \geqslant 3$; in fact*

$$|\mathrm{GL}_n(\mathbf{F}_p[C_q \times \Gamma])/\mathrm{GL}_n(\mathcal{T}_q[\Gamma])| \leqslant |U(\mathbf{F}_p[C_q])/U(\mathcal{T}_q)|.$$

Proof. Evidently $U(\mathcal{T}_q[\Gamma])E_n(\mathcal{T}_q[\Gamma]) \subset \mathrm{GL}_n(\mathcal{T}_q[\Gamma])$ and so there is a natural surjection $\mathrm{GL}_n(\mathbf{F}_p[C_q \times \Gamma])/U(\mathcal{T}_q[\Gamma])E_n(\mathcal{T}_q[\Gamma]) \twoheadrightarrow \mathrm{GL}_n(\mathbf{F}_p[C_q \times \Gamma])/\mathrm{GL}_n(\mathcal{T}_q[\Gamma])$. Also, the ring homomorphism $\natural : \mathcal{T}_q(A,\pi)[\Gamma] \to \mathbf{F}_p[C_q \times \Gamma]$ is surjective and so induces surjections $\natural_* : E_k(\mathcal{T}_q(A,\pi)[\Gamma]) \to E_k(\mathbf{F}_p[C_q \times \Gamma])$ for all $k \geqslant 2$. By Proposition 2 we may write

$$\mathrm{GL}_n(\mathbf{F}_p[C_q \times \Gamma]) = U(\mathbf{F}_p[C_q \times \Gamma])E_n(\mathbf{F}_p[C_q \times \Gamma]).$$

We obtain a surjection $U(\mathbf{F}_p[C_q \times \Gamma])/U(\mathcal{T}_q[\Gamma]) \twoheadrightarrow \mathrm{GL}_n(\mathbf{F}_p[C_q \times \Gamma])/\mathrm{GL}_n(\mathcal{T}_q[\Gamma])$ and so the stated result now follows from Proposition 1. □

PROPOSITION 4. *Let p be an odd prime and q be a divisor of $p-1$. Then, for all $n \geqslant 1$,*

$$\mathrm{GL}_n(\mathbf{F}_p[C_q \times C_\infty]) = U(\mathbf{F}_p[C_q \times C_\infty]) \cdot E_n(\mathbf{F}_p[C_q \times C_\infty]).$$

Proof. We follow the same line of argument as Proposition 2 with the exception that, in establishing the induction base, we do not use Suslin's theorem. Instead we note that, as $\mathbf{F}_p[C_\infty]$ is a Euclidean domain, we may use the Smith normal form to show that $\mathrm{GL}_k(\mathbf{F}_p[C_\infty]) = U(\mathbf{F}_p[C_\infty]) \cdot E_k(\mathbf{F}_p[C_\infty])$. □

As in [1], we denote the set of isomorphism classes of locally free $\mathbf{Z}[C_p \rtimes C_q]$-modules of rank k by $\mathcal{LF}_k(\clubsuit)$. By Milnor's classification, this corresponds to the two-sided quotient

$$\mathcal{LF}_k(\clubsuit) = \mathrm{GL}_k(\mathbf{Z}[C_q]) \backslash \mathrm{GL}_k(\mathbf{F}_p[C_q])/\mathrm{GL}_k(\mathcal{T}_q).$$

Likewise, the locally free $\mathbf{Z}[(C_p \rtimes C_q) \times \Gamma]$-modules of rank k correspond to the quotient

$$\mathcal{LF}_k(\heartsuit) = \mathrm{GL}_k(\mathbf{Z}[C_q \times \Gamma]) \backslash \mathrm{GL}_k(\mathbf{F}_p[C_q \times \Gamma])/\mathrm{GL}_k(\mathcal{T}_q[\Gamma]).$$

In particular, if neither $\mathbf{Z}[C_q]$ nor $\mathcal{T}_q$ admits non-trivial stably free modules of rank k, then any stably free module of rank k over $\mathbf{Z}[C_p \rtimes C_q]$ is locally free. Consequently, the set $\mathcal{SF}_k(\mathbf{Z}[C_p \rtimes C_q])$ of stably free modules of rank k over $\mathbf{Z}[C_p \rtimes C_q]$ is a subset of $\mathcal{LF}_k(\clubsuit)$. Similarly, $\mathcal{SF}_k(\mathbf{Z}[(C_p \rtimes C_q) \times \Gamma])$ is a subset of $\mathcal{LF}_k(\heartsuit)$ if neither $\mathbf{Z}[C_q \times \Gamma]$ nor $\mathcal{T}_q[\Gamma]$ admits non-trivial stably free modules of rank k.

There are obvious mappings of fibre squares $i : (\clubsuit) \hookrightarrow (\heartsuit)$ and $r : (\heartsuit) \to (\clubsuit)$ such that $r \circ i = \mathrm{Id}$. Consequently, there is a commutative ladder of mappings

$$\begin{array}{ccccccccc}
\mathcal{LF}_1(\clubsuit) & \xrightarrow{s_{1,1}} & \mathcal{LF}_2(\clubsuit) & \xrightarrow{s_{2,1}} & \mathcal{LF}_3(\clubsuit) & \xrightarrow{s_{3,1}} & \mathcal{LF}_4(\clubsuit) & \xrightarrow{s_{4,1}} & \\
\downarrow i_1 & & \downarrow i_2 & & \downarrow i_3 & & \downarrow i_4 & & \\
\mathcal{LF}_1(\heartsuit) & \xrightarrow{\sigma_{1,1}} & \mathcal{LF}_2(\heartsuit) & \xrightarrow{\sigma_{2,1}} & \mathcal{LF}_3(\heartsuit) & \xrightarrow{\sigma_{3,1}} & \mathcal{LF}_4(\heartsuit) & \xrightarrow{\sigma_{4,1}} &
\end{array}$$

where $s_{k,1}$ and $\sigma_{k,1}$ are the obvious stabilization mappings. We note that the mappings i_k are injective in view of the fact that $r \circ i = \mathrm{Id}$.

The argument is now divided into two cases: q is odd, and $q = 2$. First, suppose q is an odd prime dividing $p - 1$. As noted in [1], in $(\clubsuit)$, the rings $\mathbf{Z}[C_q]$ and $\mathcal{T}_q$ both have property SFC. Consequently, $\mathcal{SF}_k(\mathbf{Z}[C_p \rtimes C_q])$ is a subset of $\mathcal{LF}_k(\clubsuit)$ for all $k \geqslant 1$. Similarly, in the fibre square $(\heartsuit)$, the rings $\mathbf{Z}[C_q \times \Gamma]$ and $\mathcal{T}_q[\Gamma]$ also have SFC and once again $\mathcal{SF}_k(\mathbf{Z}[(C_p \rtimes C_q) \times \Gamma])$ is a subset of $\mathcal{LF}_k(\heartsuit)$ for all $k \geqslant 1$. The essence of the argument now consists of the following five statements.

(I) For all n, $\mathcal{LF}_n(\clubsuit)$ is finite and $s_{n,1} : \mathcal{LF}_n(\clubsuit) \to \mathcal{LF}_{n+1}(\clubsuit)$ is bijective.
(II) $i_1 : \mathcal{LF}_1(\clubsuit) \to \mathcal{LF}_1(\heartsuit)$ is bijective.
(III) $i_n : \mathcal{LF}_n(\clubsuit) \to \mathcal{LF}_n(\heartsuit)$ is bijective for all $n \geqslant 3$.
(IV) $\sigma_{n,1} : \mathcal{LF}_n(\heartsuit) \to \mathcal{LF}_{n+1}(\heartsuit)$ is injective provided $n \neq 2$.
(V) If $m = 1$ (that is, $\Gamma = C_\infty$) then $i_2 : \mathcal{LF}_2(\clubsuit) \to \mathcal{LF}_2(\heartsuit)$ is bijective.

To prove (I) we note that, as $C_p \rtimes C_q$ is finite, the finiteness of $\mathcal{LF}_n(\clubsuit)$ follows from the Jordan–Zassenhaus theorem, together with Milnor's classification of projectives. Moreover, as $\mathbf{Z}[C_p \rtimes C_q]$ satisfies the Eichler condition, the Swan–Jacobinski theorem shows that each $s_{k,1} : \mathcal{LF}_k(\clubsuit) \to \mathcal{LF}_{k+1}(\clubsuit)$ is bijective. It follows from Proposition 1 that $|\mathcal{LF}_1(\heartsuit)| \leqslant |\mathcal{LF}_1(\clubsuit)|$. Thus (II) is true as i_1 is injective and $\mathcal{LF}_1(\clubsuit)$ is finite. Likewise it follows from Proposition 3 that $|\mathcal{LF}_n(\heartsuit)| \leqslant |\mathcal{LF}_n(\clubsuit)|$ for $n \geqslant 3$. Thus (III) is true as i_n is injective and $\mathcal{LF}_n(\clubsuit)$ is finite; (IV) now follows from (I), (II) and (III) by diagram chasing using the fact that i_2 is injective. Finally, (V) follows by the same argument as (III) on substituting Proposition 4 for Proposition 2.

To proceed with the proof of Theorem A, put $\sigma_{n,k} = \sigma_{n+k-1,1} \circ \cdots \circ \sigma_{n,1}$ whenever $k \geqslant 1$. It follows from (IV) that $\sigma_{n,k}$ is injective provided $n \geqslant 3$. A straightforward diagram chase using (I), (II) and (III) also shows that each $\sigma_{1,k}$ is injective. Now suppose that S is a stably free module of rank $n \neq 2$ over $\Lambda = \mathbf{Z}[(C_p \rtimes C_q) \times \Gamma])$ and denote its class in $\mathcal{LF}_n(\heartsuit)$ by $[S]$. Then

$S \oplus \Lambda^k \cong \Lambda^{n+k}$ for some $k \geqslant 1$ so that $\sigma_{n,k}[S] = \sigma_{n,k}[\Lambda^n]$. As $\sigma_{n,k}$ is injective, $S \cong \Lambda^n$. Consequently, when $n \neq 2$ there are no non-trivial stably free modules of rank n over $\mathbf{Z}[(C_p \rtimes C_q) \times C_\infty^m]$, and this proves the first part of Theorem A.

In the case $q = 2$ (i.e. dihedral groups) we cannot claim $\mathcal{T}_2[\Gamma]$ has property SFC. To see why, consider the square

$$\begin{array}{ccc} \mathcal{T}_2(A,\, \pi)[\Gamma] & \longrightarrow & M_2(A[\Gamma]) \\ \downarrow & & \natural\downarrow \\ \mathcal{T}_2((A/\pi)[\Gamma]) & \xrightarrow{\;i\;} & M_2((A/\pi)[\Gamma]). \end{array}$$

As $(A/\pi)[\Gamma]$ is commutative, we have $\mathrm{GL}_2((A/\pi)[\Gamma]) = U((A/\pi)[\Gamma]) \cdot \mathrm{SL}_2((A/\pi)[\Gamma])$. The unit group $U((A/\pi)[\Gamma])$ lifts back to $\mathcal{T}_2((A/\pi)[\Gamma])$. However, it is not clear whether we can lift the elements of $\mathrm{SL}_2((A/\pi)[\Gamma])$. Thus, it is conceivable that $\mathcal{T}_2(A,\, \pi)[\Gamma]$ has non-trivial stably free modules of rank 1. Nevertheless, using Suslin's theorem as before, it is clear that $\mathcal{T}_2(A,\, \pi)[\Gamma]$ admits no non-trivial stably free module of rank $\geqslant 2$. Consequently, we observe that $\mathcal{SF}_k(\mathbf{Z}[(C_p \rtimes C_q) \times \Gamma])$ is a subset of $\mathcal{LF}_k(\heartsuit)$ for all $k \geqslant 2$. We now proceed as above.

Finally, the proof of Theorem B follows exactly the same lines except that now, in the case where $m = 1$ and $\Gamma = C_\infty$, we see from (V) that $\sigma_{2,1}$ is also injective. Consequently, each $\sigma_{2,k}$ is injective. Thus, there are no non-trivial stably free modules of any rank over $\mathbf{Z}[(C_p \rtimes C_q) \times C_\infty]$.

References

1. J. D. P. Evans, Stably free modules over $\mathbf{Z}[(C_p \rtimes C_q) \times C_\infty^m]$ are free. *Mathematika* **63** (2017), 451–461.
2. F. E. A. Johnson, *Syzygies and Homotopy Theory*, Springer (London, 2011).
3. A. A. Suslin, On the structure of the special linear group over polynomial rings. *Math. USSR, Izvestiya* **11**(2) (1977), 221–238.

J. D. P. Evans,
Department of Mathematics,
University College London,
Gower Street,
London WC1E 6BT,
U.K.
E-mail: john.evans.09@ucl.ac.uk