Canad. Math. Bull. Vol. **29** (3), 1986

# A PRACTICAL TWO-DIMENSIONAL ERGODIC THEOREM

BY

MANNY SCAROWSKY and ABRAHAM BOYARSKY*

ABSTRACT. Let $\tau:[0,1]\to[0,1]$ be defined by $\tau(x) = 2x$ on $[0,\frac{1}{2}]$ and $\tau(x) = 2(1-x)$ on $[\frac{1}{2},1]$, and let $T:[0,1]\times[0,1]\to[0,1]\times[0,1]$ be defined by $T(x,y) = (\tau(x),\tau(y))$. Let

$$\theta_M = \left\{\frac{2\alpha}{p^M},\ 0 < 2a < p^M,\ (a,p) = 1\right\},$$

where $p$ is a prime $> 2$, and $a$ and $M$ are integers. Consider $T$ restricted to $\theta_M \times \theta_N$, $1 < M < N$. Let $X = ((2a)/(p^M),(2b)/(p^N)) \in \theta_M \times \theta_N$ and let per$(X)$ denote the length of the period of $X$.

Then,

$$\left|\frac{1}{\mathrm{per}(X)}\sum_{i=1}^{\mathrm{per}(X)} g(T^i(X)) - \int_0^1\int_0^1 g\,dm\times dm\right| \le C\left(\frac{1}{p^{N-M}} + \frac{1}{p^{M-1}}\right).$$

where $m$ is Lebesque measure on $[0,1]$, and $C$ is independent of $p$, $N$, $M$, $a$ and $b$. Thus, as $p\to\infty$ or as $N-M$ and $M\to\infty$,

$$\frac{1}{\mathrm{per}(X)}\sum_{i=1}^{\mathrm{per}(X)} g(T^i(X)) \to \int_0^1\int_0^1 g\,dm\times dm.$$

1. **Introduction**. Let $(X,\mathcal{B},\mu,\tau)$ be a dynamical system, i.e., $(X,\mathcal{B},\mu)$ is a finite measure space and $\tau:X\to X$ is a measure preserving transformation: for any $A\in\mathcal{B}$, $\mu(\tau^{-1}A) = \mu(A)$. Let $f\in\mathcal{L}_1(X,\mathcal{B},\mu)$ and suppose that $\tau$ is ergodic. Then it follows from the Birkhoff Ergodic Theorem that

$$(1)\qquad \lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1} f(\tau^k x) = \int_X f\,d\mu \text{ a.e. } \mu$$

Let $S$ be the support of $\mu$. Then (1) says that for a.e. $x\in S$ with respect to $\mu$, the orbit of $x$ *exhibits* $\mu$ in the sense that for any $A\in\mathcal{B}$,

$$(2)\qquad \lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1}\chi_A(\tau^k x) = \mu(A) \text{ a.e. } \mu,$$

Received by the editors January 15, 1985, and, in final revised form, May 6, 1985.

*The research of this author was supported by an NSERC grant No. A9072 and an FCAC grant from the Quebec Department of Education.

AMS Subject Classification (1980): Primary 28D10. Secondary 26A18.

© Canadian Mathematical Society 1985.



https://doi.org/10.4153/CMB-1986-054-x Published online by Cambridge University Press

where $\chi_A$ is the characteristic function of the set $A$. This is a general statement about $(X, \mathcal{B}, \mu, \tau)$ but it has a crucial drawback if $\mu$ is continuous, for then every point $x$ has $\mu$-measure 0 and no matter what starting point is used, we cannot be sure that it will exhibit $\mu$ in the sense of (2). Thus, (2) holds for a.e. $x \in S$, yet it is in general impossible to specify a single $x$ where (2) actually holds.

If $\mu$ is absolutely continuous, as in the case for example when $\tau$ is expanding [1], then it has been observed that computer orbits have histograms which approximate the histograms derived from $\mu$. Since the number of points in the computer memory is finite, the computer orbits must actually be periodic. This has led to the conjecture that long periodic orbits exhibit the absolutely continuous invariant measure $\mu$.

In [2], we proved this conjecture in the special case where $\tau$ is the symmetric triangle map from $[0, 1]$ onto $[0, 1]$ defined by

$$\tau(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2(1 - x) & \frac{1}{2} \leq x \leq 1 \end{cases}$$

We restricted $\tau$ to sets of the form $\mathcal{D}_N = \{a/(10^N)\}$, $a$ and $N$ are integers, $N \geq 1$, $0 \leq a \leq 10^N$, and proved that $\tau | \mathcal{D}_N$ has a largest periodic orbit $\theta_N$ with the property that the fraction of points in $\theta_N$ that fall into any interval $[c, d] \subset [0, 1]$ is asymptotic to $|d - c| = m[c, d]$ as $N \to \infty$, where $m$ is Lebesgue measure on $[0, 1]$. This led to the following 1-dimensional practical ergodic theorem: given $\epsilon > 0$ and $0 < p < 1$, there exists an integer $N(\epsilon, p)$ such that for any interval $[c, d] \subset [0, 1]$,

$$\left| \lim_{m \to \infty} \frac{1}{m} \sum_{k=0}^{m-1} \chi_{[c,d]}(\tau^k(x)) - m[c, d] \right| < \epsilon \tag{3}$$

for $p$ of all points in $\mathcal{D}_N$, where $N \geq N(\epsilon, p)$, and the points of $\mathcal{D}_N$ for which (3) holds are known, thereby lending useful meaning to the a.e. statement in the Birkhoff Ergodic Theorem.

This result can be generalised. If we let $\theta_M = \{(2a)/(p^M),\ 0 < 2a < p^M, (a, p) = 1\}$, where $p$ is a prime $> 2$, $a$ and $M$ are integers, then 1) all points of $\theta_M$ are periodic, with period $k(M) = p^{M-1}k$ (where $k$ is the minimal integer such that $p | 2^k \pm 1$; for simplicity we assume $p^2 \nmid 2^k \pm 1$), and 2)

$$\frac{1}{\text{period}(x)} \sum_{i=1}^{\text{period}(x)} g(\tau^i(x)) = \int g\,dm + 0\left(\frac{1}{p^{M-1}}\right),$$

where $g$ is a simple function, $x \in \theta_M$, and the constant in the 0-term is 1. In this note we prove an analogous result for a two-dimensional map. We shall need the following.

https://doi.org/10.4153/CMB-1986-054-x Published online by Cambridge University Press

LEMMA 1. *Let $I = [a, b]$, $l$ an integer $\geq 1$. Then*

$$\left| {}^{\#}\{i \leq [a, b): i \text{ is divisible by } l\} - \frac{m(I)}{l} \right| \leq 1,$$

*where ${}^{\#}\{\ \}$ denotes the number of points in the set $\{\ \}$.*

**Proof.** Let $b > a$. Since the left-hand side of the inequality is periodic as a function of $b$, with period $\mathscr{L}$, we may assume that $b - a \leq \mathscr{L}$. Then

$$\begin{aligned} &\left| {}^{\#}\{i \in [a, b): i \text{ divisible by } l\} - \frac{m(I)}{l} \right| \\ &\quad \leq \max\left( {}^{\#}\{i \in I: i \text{ divisible by } l\} - \frac{m(I)}{l}, \frac{m(I)}{l} - {}^{\#}\{i \in I: i \text{ divisible by } l\} \right) \\ &\quad \leq \max\left( {}^{\#}\{i \in I: i \text{ divisible by } l\}, \frac{m(I)}{l} \right) \\ &\quad \leq 1 \end{aligned}$$

Q.E.D.

With a little bit of care one can show that in fact

$$\left| {}^{\#}\{i \in [a, b]: i \text{ divisible by } l\} - \frac{m(I)}{l} \right| < 1.$$

2. **Main Result**. We shall consider the map $T:[0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ defined by $T(x, y) = (\tau(x), \tau(y))$ restricted to the domain $\theta = \theta_M \times \theta_N$, where we assume that $1 < M < N$. (By symmetry, the same results hold for $\theta_N \times \theta_M$.) As before, $p$ is an odd prime. It is easy to see that if $X \in \theta$, where $X = ((2a)/(p^M), (2b)/(p^N))$, then period $(X)$ = period $((2b)/(p^N)) \equiv k(N)$. We shall assume that $p^2 \nmid 2^k \pm 1$. This implies that $k(N) = p^{N-1} k(1)$ [2].

We now state the main result.

THEOREM. *Let $g$ be a simple function on $[0, 1] \times [0, 1]$. Let $X = ((2a)/(p^M), (2b)/(p^N))$, $M < N$, and let $\text{per}(X)$ denote the length of the period of $X$. Then*

$$\left| \frac{1}{\text{per}(X)} \sum_{i=1}^{\text{per}(x)} g(T^i(X)) - \int_0^1 \int_0^1 g\, dm \times dm \right| \leq C\left( \frac{1}{p^{N-M}} + \frac{1}{p^{M-1}} \right),$$

*where $C$ is independent of $p$, $N$, $M$, $a$ and $b$. Thus, as $p \to \infty$ or as $N - M$ and $M \to \infty$, we have*

$$\frac{1}{\text{per}(X)} \sum_{i=1}^{\text{per}(X)} g(T^i(X)) \to \int_0^1 \int_0^1 g dm \times dm.$$

https://doi.org/10.4153/CMB-1986-054-x Published online by Cambridge University Press

**Proof**. Without loss of generality we can assume b = 1. By the standard reasoning it is enough to prove the result for $g = \chi_{I_1 \times I_2}$. For technical reasons we let $1 \leq i \leq 2\text{per}(X)$. Thus, we have to show

$$(4)\qquad \frac{1}{2\text{per}(X)} \sum_{i=1}^{2\text{per}(X)} \chi_{I_1 \times I_2}\left(T^i\left(\frac{2a}{p^M}, \frac{2}{p^N}\right)\right) = m(I_1)m(I_2) + 0\left(\frac{1}{p^{N-M}} + \frac{1}{p^{M-1}}\right).$$

Note that the numerator on the left hand side of (4) is

$$^{\#}\left\{i\colon \tau^i\left(\frac{2a}{p^M}\right) \in I_1,\ \tau^i\left(\frac{2}{p^N}\right) \in I_2,\ 1 \leq i \leq 2k(N)\right\}.$$

Let

$$\frac{2^i}{p^N} = q_i + \frac{r_i}{p^N},$$

where $q_i, r_i \in N$, $0 < r_i < p^N$, $(r_i, p) = 1$, $1 \leq i \leq 2k(N)$. Then

$$\frac{2^i a}{p^M} = q_i a p^{N-M} + \frac{r_i a}{p^M}.$$

From this it follows that

$$\tau^i\left(\frac{2}{p^N}\right) = \begin{cases} \dfrac{\tau_i}{p^N} & , r_i \text{ even} \\[2ex] \dfrac{p^N - r_i}{p^N}, & r_i \text{ odd} \end{cases}$$

and

$$\tau^i\left(\frac{2a}{p^N}\right) = \begin{cases} \dfrac{(\tau_i a)(\operatorname{mod} p^M)}{p^M} & , (r_i a)(\operatorname{mod} p^M) \text{ even} \\[2ex] \dfrac{p^M - (r_i a)(\operatorname{mod} p^M)}{p^M}, & (r_i a)(\operatorname{mod} p^M) \text{ odd.} \end{cases}$$

Therefore, we want to estimate

$$(5)\qquad K = {}^{\#}\left\{i\colon \begin{Bmatrix} (r_i a)(\operatorname{mod} p^M) \\ p^M - (r_i a)(\operatorname{mod} p^M) \end{Bmatrix} \in p^M I_1,\ \begin{Bmatrix} r_i \\ p^M - r_i \end{Bmatrix} \in p^N I_2\right\}$$

where for each $i$ and $M, N$ respectively we choose whichever line is even, and $1 \leq i \leq 2k(N)$. This breaks up (5) into 4 terms, a typical one being

$$L = {}^{\#}\{i\colon r_i a(\operatorname{mod} p^M) \in p^M I_1,\ r_i \in p^N I_2,\ r_i a(\operatorname{mod} p^M) \text{ even},\ r_i \text{ even},\ 1 \leq i \leq 2k(N)\}.$$

https://doi.org/10.4153/CMB-1986-054-x Published online by Cambridge University Press

To estimate this number we first note that it can be shown that

$$r_i = jp + \bar{r}_{i'},$$

where $0 \leq j < p^{N-1}$, $0 < \bar{r}_{i'} < p$, $1 \leq i' \leq 2k$. (The $\bar{r}_{i'}$ are obtained by letting $N = 1$ in the definition of $\bar{r}_{i'}$; then they are the $r_i$, $p - r_i$ for $i = 1, 2, \ldots, k$, or possibly the $r_i$ repeated twice.) Thus we want to estimate

$$\begin{aligned} L = {}^{\#}\{(j, i')\colon &(jpa + \bar{r}_{i'}a)(\operatorname{mod} p^M) \in p^M I_1,\ jp + \bar{r}_{i'} \in p^N I_1 \text{ and} \\ &\text{both } (jpa + \bar{r}_{i'}a)(\operatorname{mod} p^M) \text{ and } j + \bar{r}_{i'} \text{ are even}, \\ &0 \leq j < p^{N-1},\ 1 \leq i' \leq 2k\}. \end{aligned}$$

Now,

$$\begin{aligned} (6) \qquad L = \sum_{\bar{r}_{i'} \text{ even}} {}^{\#}\{j \text{ even}\colon &(jpa + \bar{r}_{i'}a)(\operatorname{mod} p^M) \in p^M I_1,\ jp + \bar{r}_{i'} \in p^N I_2, \\ &\qquad\qquad\qquad\qquad j + \bar{r}_i \text{ even} \\ &(jpa + \bar{r}_{i'}a)(\operatorname{mod} p^M) \text{ even},\ 0 \leq j < p^{N-1},\ 1 \leq i' \leq 2k\}, \\ &\qquad + \text{ a similar sum with } \bar{r}_{i'} \text{ odd, } j \text{ odd} \end{aligned}$$

There exist integers $j', j''$, $0 \leq j' < p^{M-1}$, $0 \leq j'' < p^{N-M}$ such that

$$j = j' + p^{M-1} j''.$$

Thus ${}^{\#}\{\ \}$ in (6) is equal to

$$\begin{aligned} &{}^{\#}\{j', j'')\colon (j'pa + \bar{r}_{i'})(\operatorname{mod} p^M) \in p^M I_1,\ (j'p + p^M j'') \in p^N I_2 - \bar{r}_{i'}, \\ &\qquad j' + j'' \text{ even},\ 0 \leq j' < p^{M-1},\ 0 \leq j'' < p^{N-M}\} \\ &\qquad\qquad\qquad\qquad + \text{ a similar term with } j' + j'' \text{ odd.} \\ &= \sum_{\substack{j' \text{ even} \\ (j'pa + \bar{r}_{i'})(\operatorname{mod} p^M) \in p^M I_1}} {}^{\#}\left\{j''\colon j'' \in p^{N-M} I_2 - \frac{\bar{r}_{i'}}{p^M} - \frac{j'}{p^{M-1}},\ j'' \text{ odd}\right\} + \text{ a similar term with } j' \text{ odd, } j'' \text{ even.} \end{aligned}$$

Now the first sum above equals

$$(7) \qquad \sum_{\substack{j' \text{ even} \\ (j'pa + \bar{r}_{i'})(\operatorname{mod} p^M) \in p^M I_1}} {}^{\#}\{j''\colon j'' \in p^{N-M} I_2,\ j'' \text{ odd}\}.$$

By Lemma 1, this equals

$$\sum_{\substack{j' \text{ even} \\ (j'pa + \bar{r}_{i'})(\operatorname{mod} p^M) \in p^M I_1}} \left(m\left(\frac{p^{N-M} I_2}{2}\right) + 0(1)\right) + \text{ a similar sum with } j' \text{ odd.}$$

We note that $\{j'pa + \bar{r}_{i'})(\operatorname{mod} p^M)$, $0 \leq j' < p^{M-1}\}$ can be shown to equal $j'''p + d$, where $0 \leq j''' < p^{M-1}$, and $0 < d < p$. Thus by the result mentioned above, the sum over $j'$ contributes

$$m(p^M I_1) + 0(1).$$

https://doi.org/10.4153/CMB-1986-054-x Published online by Cambridge University Press

Thus,

$$K = \sum_{\bar{r}_i, \text{even}} 2(m(p^{M-1}I_1) + 0(1))\left(m\left(\frac{p^{N-M}I_2}{2}\right) + 0(1)\right). \tag{8}$$

Dividing by $2k(N) = 2p^{N-1}k(1)$, we obtain:

$$\frac{1}{2\text{per}(X)} \sum_{i=1}^{2\text{per}(X)} \chi_{I_1 \times I_2}\left(T^i\left(\frac{2a}{p^M}, \frac{2}{p^N}\right)\right) = \left(m(I_1) + 0\left(\frac{1}{p^{M-1}}\right)\right)\left(m(I_2) + 0\left(\frac{1}{p^{N-M}}\right)\right),$$

which yields the desired result.

Let $F_M$ count the number of points in $[a, b)$ with denominator $p^M$, i.e.

$$F_M[a, b) = \frac{[p^M a] - [p^M b]}{p^M}.$$

Then the above proof shows the left hand side of (4) is equal to $2\mu_{M-1}(I_1)\mu_{N-M}(I_2/2)$ where $\mu_M$ is the one-dimensional measure induced by $F_M$ on $[0, 1]$. Thus the 2 dimensional measure induced on $[0, 1] \times [0, 1]$ by

$$\frac{1}{\text{per}(X)} \sum_{i=1}^{\text{per}(X)} g\left(T^i\left(\frac{2a}{p^M}, \frac{2b}{p^N}\right)\right)$$

is actually the product of 2 1-dimensional measures. Finally, we remark that the constants corresponding to the 0-terms in (8) are 2.

COROLLARY. *Let $g$ be a uniform limit of simple functions on $[0, 1] \times [0, 1]$. Then*

$$\frac{1}{\text{per}(X)} \sum_{i=1}^{\text{per}(X)} g(T^i(X)) \rightarrow \int_0^1 \int_0^1 g dm x dm,$$

*as $p \rightarrow \infty$ or as $N - M$ and $M \rightarrow \infty$.*

**Remark**. It can be shown that the limiting statement in the Theorem is not true if we only assume $M \rightarrow \infty$.

## REFERENCES

1. A. Lasota and J. A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc., **183** (1973), pp. 481–485.

2. M. Scarowsky and A. Boyarsky, *Long periodic orbits of the triangle map*, Proc. Amer. Math. Soc. (in press).

DEPARTMENT OF MATHEMATICS
LOYOLA CAMPUS
CONCORDIA UNIVERSITY
7141 SHERBROOKE ST. WEST
MONTREAL, CANADA H4B 1R6

https://doi.org/10.4153/CMB-1986-054-x Published online by Cambridge University Press