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On the Berger-Coburn-Lebow Problem for Hardy Submodules

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Abstract. In this paper we shall give an affirmative solution to a problem, posed by Berger, Coburn and Lebow, for C^*-algebras on Hardy submodules.

1 Introduction

Let $\mathbb{T}^2$ denote the torus, the Cartesian product of two copies of the unit circle in $\mathbb{C}$, that is, $\mathbb{T}^2 = \{(z,w) \in \mathbb{C}^2 : |z| = |w| = 1\}$, and $\mathbb{Z}_+$ denote the set of all positive integers. $L^2(\mathbb{T}^2)$ will denote the usual Lebesgue space with respect to the normalized Lebesgue measure σ of $\mathbb{T}^2$. $H^2 = H^2(\mathbb{T}^2)$ will denote the Hardy space over $\mathbb{T}^2$, the space of all f in $L^2(\mathbb{T}^2)$ whose Fourier coefficients

$$\hat{f}(i,j) = \int_{\mathbb{T}^2} f(z,w)\bar{z}^i \bar{w}^j \, d\sigma$$

are 0 whenever at least one component of (i,j) is negative. It is well known that H^2 is a Hilbert space. $P_{\mathcal{M}}$ denotes the orthogonal projection from H^2 onto a closed subspace $\mathcal{M}$, and $\mathcal{M}^\perp = H^2/\mathcal{M} = H^2 \ominus \mathcal{M}$ the orthogonal complement of $\mathcal{M}$ in H^2. Let $H^2(z)$ and $H^2(w)$ denote the usual one-variable Hardy spaces with the variables z and w, respectively. It is well known that $H^2 = H^2(z) \otimes H^2(w)$. Let $\mathcal{B}(\mathcal{M})$ denote the set of all bounded linear operators on $\mathcal{M}$. A closed subspace $\mathcal{M}$ of H^2 is said to be a Hardy submodule or an invariant subspace of H^2 if $\mathcal{M}$ is invariant under the multiplication operators by the coordinate functions z and w. Let V_z and V_w denote the restriction operators to the Hardy submodule $\mathcal{M}$ of the Toeplitz operators T_z and T_w, respectively. Note that we consider V_z and V_w as operators on $\mathcal{M}$. Let $\mathcal{A}(V_z, V_w; \mathcal{M}) = \mathcal{A}(\mathcal{M}) = \mathcal{A}(V_z, V_w)$ denote the C^*-subalgebra of $\mathcal{B}(\mathcal{M})$ generated by V_z and V_w. The two C^*-algebras $\mathcal{A}(\mathcal{M}_1)$ and $\mathcal{A}(\mathcal{M}_2)$ are said to be unitarily equivalent if there exists a unitary operator U from $\mathcal{M}_1$ onto $\mathcal{M}_2$ such that $U^*\mathcal{A}(\mathcal{M}_2)U = \mathcal{A}(\mathcal{M}_1)$.

In [3], Berger, Coburn and Lebow studied the C^*-algebras generated by commuting isometries. In Section 13 of [3], they posed the following problem:

Berger-Coburn-Lebow problem ([3]) *If $\mathcal{M}$ is any Hardy submodule of finite codimension, then is $\mathcal{A}(V_z, V_w; \mathcal{M})$ unitarily equivalent to $\mathcal{A}(T_z, T_w; H^2)$?*

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We call this problem the BCL problem, for short. In Theorem 9.3 of [3] they solved the BCL problem affirmatively for the following special case:

Theorem 1.1 ([3]) *In the following case, $\mathcal{A}(V_z, V_w; \mathcal{M})$ is unitarily equivalent to $\mathcal{A}(T_z, T_w; H^2)$; $\mathcal{M} = H^2(\mathbb{S})$ is the closed subspace of H^2, which consists of those functions whose Fourier transforms are supported in $\mathbb{S}$. Where $\mathbb{S}$ is a subsemigroup in $\mathbb{Z}_+ \times \mathbb{Z}_+ = \{(m, n) \in \mathbb{Z}^2 : m, n \geq 0\}$ such that if (m, n) is in $\mathbb{S}$ then so are $(m + 1, n)$ and $(m, n + 1)$, and $(\mathbb{Z}_+ \times \mathbb{Z}_+) \setminus \mathbb{S}$ is finite.*

Moreover, if the set of all common zeros of $\mathcal{M}$ consists of one point, one can give an affirmative answer to the BCL problem with a slight modification of their technique. It should be noted that two different Hardy submodules, which are of finite codimension, are not unitarily equivalent as modules ([2]). There are many studies of the equivalence of Hardy submodules (see Agrawal-Clark-Douglas [2], Douglas-Paulsen [4], Douglas-Yang [5], Izuchi [7, 8] and Paulsen [10]).

In this paper we shall solve the BCL problem completely. Section 2 is a preliminary part. In Section 3, we deal with Hardy submodules of finite codimension. In Section 4, we study some operators which will be used in Section 5. In Section 5, we give an affirmative answer to the BCL problem.

2 Preliminaries

The following theorem given by Yang is a breakthrough in the study of operator theory on H^2:

Theorem 2.1 (Yang [13]) *If $\mathcal{M}$ is a Hardy submodule generated by a finite number of polynomials, then $[V_z^*, V_w]$ and $[V_z^*, V_z][V_w^*, V_w]$ are Hilbert-Schmidt class operators.*

This theorem is very strong, because we need no informations of the set of all common zeros.

Next we shall study two C^*-algebras defined by Hardy submodules. The next proposition is well known.

Proposition 2.1 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two Hardy submodules. If $\mathcal{M}_1$ is orthogonal to $\mathcal{M}_2$, then $\mathcal{M}_1 = \{o\}$ or $\mathcal{M}_2 = \{o\}$.*

Corollary 2.1 *V_z and V_w have no non-trivial joint reducing subspace.*

Corollary 2.2 *The C^*-algebra $\mathcal{A}(V_z, V_w)$ is irreducible.*

Let $\mathcal{K}(\mathcal{H})$ denote the set of all compact operators on a Hilbert space $\mathcal{H}$. By Theorem 2.1, we have the following:

Corollary 2.3 *If $\mathcal{M}$ is a Hardy submodule generated by a finite number of polynomials, then $\mathcal{K}(\mathcal{M})$ is contained in $\mathcal{A}(V_z, V_w)$.*

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Definition 2.1 For a Hardy submodule $\mathcal{M}$, let $\mathcal{N} = H^2/\mathcal{M} = \mathcal{M}^{\perp}$. we define two operators on $\mathcal{N}$ as follows:

$$S_z = P_{\mathcal{N}} T_z|_{\mathcal{N}},\ S_w = P_{\mathcal{N}} T_w|_{\mathcal{N}}.$$

Note that $\mathcal{N}$ is a backward shift invariant subspace, that is, $T_z^*\mathcal{N} \subseteq \mathcal{N}$ and $T_w^*\mathcal{N} \subseteq \mathcal{N}$. S_z plays an important role in the study of operators on H^2 and the model theory for contraction operators on a Hilbert space, (see Douglas-Yang [5], Guo-Yang [6], Izuchi-Nakazi-Seto [9] and Yang [13, 14, 15, 16, 17]).

Theorem 2.2 (Yang [13]) *If $\mathcal{M}$ is a Hardy submodule generated by a finite number of polynomials, then $[S_z^*, S_w]$ is a Hilbert-Schmidt class operator.*

The next fact is analogous to Proposition 2.1.

Proposition 2.2 *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two invariant subspaces under T_z^* and T_w^*, that is, there exist Hardy submodules $\mathcal{M}_i$ such that $\mathcal{N}_i = H^2/\mathcal{M}_i$ $(i = 1, 2)$. If $\mathcal{N}_1$ is orthogonal to $\mathcal{N}_2$, then $\mathcal{N}_1 = \{o\}$ or $\mathcal{N}_2 = \{o\}$.*

Proof For any $f_1 \in \mathcal{N}_1$ and $f_2 \in \mathcal{N}_2$, and for any i, j, k and $l \in \mathbb{Z}_+$,

$$\langle T_z^k T_z^{*i} f_1, T_w^j T_w^{*l} f_2\rangle = \langle T_z^{*i} T_w^{*j} f_1, T_z^{*k} T_w^{*l} f_2\rangle = 0.$$

Hence, we have

$$\begin{cases} \langle T_z^k f_1, T_w^j f_2\rangle = 0 & (i = l = 0) \\ \langle T_z^k f_1, T_w^{*l} f_2\rangle = 0 & (i = j = 0) \\ \langle T_z^{*i} f_1, T_w^j f_2\rangle = 0 & (k = l = 0) \\ \langle T_z^{*i} f_1, T_w^{*l} f_2\rangle = 0 & (j = k = 0). \end{cases}$$

Therefore

$$\int f_1 \overline{f_2} z^i w^j \, d\sigma = 0,$$

for any i and $j \in \mathbb{Z}$, that is, $f_1\overline{f_2} = 0$. Since $\log|f| \in L^1$ for any non-zero $f \in H^2$ (*cf.* Rudin [11]), we have $f_1 = 0$ or $f_2 = 0$, that is, $\mathcal{N}_1 = \{o\}$ or $\mathcal{N}_2 = \{o\}$. ■

The following two facts proved by Yang in [16] are immediate consequences of Proposition 2.2:

Corollary 2.4 ([16]) *S_z and S_w have no non-trivial joint reducing subspace.*

Corollary 2.5 ([16]) *The C^*-algebra $C^*(S_z, S_w)$ generated by S_z and S_w is irreducible.*

By Theorem 2.2, we have the following:

Corollary 2.6 ([16]) *Let $\mathcal{M}$ be a Hardy submodule generated by a finite number of polynomials. If $[S_z^*, S_w] \neq 0$ then $C^*(S_z, S_w)$ contains $\mathcal{K}(\mathcal{N})$, where $\mathcal{N} = H^2/\mathcal{M}$.*

In the case where $[S_z^*, S_w] = 0$, the following fact was shown in [9].

Theorem 2.3 ([9]) *Let $\mathcal{M}$ be a Hardy submodule and $\mathcal{N} = H^2/\mathcal{M}$. If $\mathcal{N}$ satisfies the condition $[S_z^*, S_w] = 0$, then one and only one of the following occurs.*

(i) $\mathcal{M} = q_1(z)H^2$,
(ii) $\mathcal{M} = q_2(w)H^2$,
(iii) $\mathcal{M} = q_1(z)H^2 + q_2(w)H^2$,

where $q_1(z)$ and $q_2(w)$ are one variable inner functions.

Definition 2.2 Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$. $H^\infty(\mathbb{D})$ will denote the Banach algebra of all bounded analytic functions on $\mathbb{D}$. A completely non-unitary contraction T is said to be a C_0 class operator if there is a non-zero function f in $H^\infty(\mathbb{D})$ such that $f(T) = 0$. A function m_T in $H^\infty(\mathbb{D})$ is said to be the minimal function of T if $m_T(T) = 0$ and $f/m_T \in H^\infty(\mathbb{D})$, for any function $f \in H^\infty(\mathbb{D})$ such that $f(T) = 0$.

Since $S_z^{*n} \to 0$ strongly as $n \to \infty$, S_z and S_w are completely non-unitary contractions. By Theorem 2.3, we have the following:

Corollary 2.7 ([5]) *If $[S_z^*, S_w] = 0$, then $S_z \in C_0$ or $S_w \in C_0$. Moreover, if $S_z \in C_0$, then $q_1(z)$ in Theorem 2.3 is the minimal function of S_z.*

The next lemma will be used often later.

Lemma 2.1 ([9]) *If $q_1(z)$ and $q_2(w)$ are one variable inner functions, then*

$$\begin{aligned} q_1(z)H^2 + q_2(w)H^2 &= q_1(z)H^2 \oplus q_2(w) \sum_{j \geq 0} \oplus w^j \left(H^2(z) \ominus q_1(z)H^2(z)\right) \\ &= q_2(w)H^2 \oplus q_1(z) \sum_{i \geq 0} \oplus z^i \left(H^2(w) \ominus q_2(w)H^2(w)\right). \end{aligned}$$

Moreover $q_1(z)H^2 + q_2(w)H^2$ is closed.

3 The Case of $\dim(H^2/\mathcal{M}) < +\infty$

In this section we deal with the case where $\dim(H^2/\mathcal{M}) < +\infty$. Ahern and Clark completely described Hardy submodules of finite codimension by the method of commutative algebra in [1]. To begin with, we show the following lemma:

Lemma 3.1 *Let $\mathcal{M}$ be a Hardy submodule. Then $\dim(H^2/\mathcal{M}) < +\infty$ if and only if there exist two finite Blaschke products $q_1(z)$ and $q_2(w)$ such that*

$$q_1(z)H^2 + q_2(w)H^2 \subseteq \mathcal{M}.$$

Proof Suppose that $\dim(H^2/\mathcal{M})$ is finite. Then $S_z \in C_0$. Let $q_1(z)$ be the minimal function of S_z. Then $q_1(z)$ is a finite Blaschke product. Since $0 = q_1(S_z) = S_{q_1(z)} = P_{\mathcal{N}} T_{q_1(z)}|_{\mathcal{N}}$, we have $q_1(z)\mathcal{N} \subseteq \mathcal{M}$. Hence $q_1(z)H^2 \subseteq \mathcal{M}$.
Conversely, it is clear by Lemma 2.1. ■

For any Hardy submodule $\mathcal{M}$ of finite codimension, we define two subspaces as follows:

$$\mathcal{M}_0 = q_1(z)H^2 + q_2(w)H^2, \quad \mathcal{F}_{\mathcal{M}} = \mathcal{M} \ominus \mathcal{M}_0,$$

where $q_1(z)$ and $q_2(w)$ are the minimal functions of S_z and S_w, respectively. Since

$$\begin{aligned}\mathcal{F}_{\mathcal{M}} &\subseteq \left(H^2 \ominus \mathcal{M}_0\right) \\ &= \left(H^2(z) \ominus q_1(z)H^2(z)\right) \otimes \left(H^2(w) \ominus q_2(w)H^2(w)\right),\end{aligned}$$

we have $\dim \mathcal{F}_{\mathcal{M}} < +\infty$. Here, by using Lemma 3.1, we shall give an alternative proof of the following theorem proved by Ahern and Clark.

Theorem 3.1 (Ahern-Clark [1]) *Let $\mathcal{M}$ be a Hardy submodule. If $\dim\left(H^2/\mathcal{M}\right)$ is finite then the polynomial ideal $\mathcal{I} = \mathbb{C}[z,w] \cap \mathcal{M}$ is dense in $\mathcal{M}$ and the set of common zeros $Z(\mathcal{I})$ is a finite subset of $\mathbb{D}^2$. Conversely, if $\mathcal{I}$ is a polynomial ideal such that $Z(\mathcal{I})$ is a finite subset of $\mathbb{D}^2$ then its closure $\mathcal{M}$ in H^2 has a finite codimension and $\mathbb{C}[z,w] \cap \mathcal{M} = \mathcal{I}$.*

Proof The first part of Theorem 3.1 is an immediate consequence of Lemma 3.1. We shall show the second part. Let ϕ be the canonical inclusion map from $\mathbb{C}[z]$ to $\mathbb{C}[z,w]$, and $\tilde{\phi}$ be the following canonical injective map:

$$\tilde{\phi}\colon \mathbb{C}[z]/\phi^{-1}(\mathcal{I}) \hookrightarrow \mathbb{C}[z,w]/\mathcal{I}.$$

By the Nullstellensatz, $\mathbb{C}[z,w]/\mathcal{I}$ is of finite dimension. Hence $\mathbb{C}[z] \cap \mathcal{I} = \phi^{-1}(\mathcal{I}) \neq (0)$. By Lemma 3.1, we have $\dim\left(H^2/\mathcal{M}\right)$ is finite. Next, we shall show $\mathbb{C}[z,w] \cap \mathcal{M} = \mathcal{I}$. Let $\mathcal{J} = \mathbb{C}[z,w] \cap \mathcal{M}$. In Lemma 4 of [1], it has been shown that $\dim\left(H^2/\mathcal{M}\right) = \dim\left(\mathbb{C}[z,w]/\mathcal{J}\right)$. Since $\left(\mathbb{C}[z,w]/\mathcal{J}\right)^* = \mathcal{J}^\perp \subseteq \mathcal{I}^\perp = \left(\mathbb{C}[z,w]/\mathcal{I}\right)^*$, and $\mathcal{I}^\perp$ can be considered as a subspace of $H^2/\mathcal{M}$, we have $\mathbb{C}[z,w]/\mathcal{I} = \mathbb{C}[z,w]/\mathcal{J}$. Hence $\mathcal{I} = \mathcal{J}$. ■

Combining Corollary 2.3 and Theorem 3.1, we have that if $\dim\left(H^2/\mathcal{M}\right)$ is finite then $\mathcal{A}(V_z, V_w)$ contains the set of all compact operators on $\mathcal{M}$. Though we know Theorem 2.1, next, we shall show that the commutator of V_z^* and V_w is compact in the case of finite codimension.

Corollary 3.1 *Let $\mathcal{M}$ be a Hardy submodule of finite codimension. Then $[V_z^*, V_w]$ and $[V_z^*, V_z][V_w^*, V_w]$ are finite rank operators.*

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Proof Let $D = [V_z^*, V_w]$ and $\mathcal{M}_0 = q_1(z)H^2 + q_2(w)H^2$. Since

$$\begin{aligned} D &= D|_{\mathcal{M}_0} + D|_{\mathcal{M}\ominus\mathcal{M}_0} \\ &= D|_{q_1(z)H^2} + D|_{q_2(w)H^2(w)(H^2(z)\ominus q_1(z)H^2(z))} + D|_{\mathcal{M}\ominus\mathcal{M}_0} \\ &= D|_{q_1(z)H^2} + 0 + \text{ finite rank.} \end{aligned}$$

We shall show that $D|_{q_1(z)H^2}$ is a finite rank operator. Let $\{e_i\}_{i=0}^{k-1}$ be a basis of $H^2(z)\ominus q_1(z)H^2(z)$. Since $T_z^*q_1(z) \in H^2(z) \ominus q_1(z)H^2(z)$, we have

$$\begin{aligned} P_{\mathcal{M}}T_z^*q_1(z)g(w) &= \sum_{i,j}\langle T_z^*q_1(z)g(w), q_2(w)w^je_i\rangle q_2(w)w^je_i \\ &= \sum_{i,j}\langle T_z^*q_1(z), e_i\rangle\langle g(w), q_2(w)w^j\rangle q_2(w)w^je_i \\ &= T_z^*q_1(z)\sum_j\langle g(w), q_2(w)w^j\rangle q_2(w)w^j. \end{aligned}$$

Therefore

$$\begin{aligned} Dq_1(z)g(w) &= (V_z^*V_w - V_wV_z^*)q_1(z)g(w) \\ &= P_{\mathcal{M}}T_z^*q_1(z)wg(w) - wP_{\mathcal{M}}T_z^*q_1(z)g(w) \\ &= T_z^*q_1(z)\sum_j\langle wg(w), q_2(w)w^j\rangle q_2(w)w^j \\ &\qquad - T_z^*q_1(z)\sum_j\langle g(w), q_2(w)w^j\rangle q_2(w)w^{j+1} \\ &= \langle wg(w), q_2(w)\rangle q_2(w)T_z^*q_1(z), \end{aligned}$$

and it is easy to check $Dq_1(z)z^ig(w) = 0$ for any $i \geq 1$. Hence D is a finite rank operator. By similar calculations, $[V_z^*, V_z][V_w^*, V_w]$ is a finite rank operator. ■

The next lemma will be used later.

Lemma 3.2 *Let $\mathcal{M}$ be a Hardy submodule of finite codimension, and let $q_1(z)$ be the minimal function of S_z. Then $\mathcal{A}(V_z, V_w)$ contains the projection onto $q_1(z)H^2$.*

Proof Trivially,

$$\mathcal{M}_0 = \big(\mathcal{M}_0 \ominus q_1(z)H^2\big) \oplus \big(q_1(z)H^2 \ominus q_1(z)\mathcal{M}_0\big) \oplus q_1(z)\mathcal{M}_0.$$

Since $P_{\mathcal{M}_0} = P_{\mathcal{M}} - P_{\mathcal{F}_{\mathcal{M}}} \in \mathcal{A}(V_z, V_w)$, we have

$$\begin{aligned} P_{\mathcal{M}_0} - P_{q_1(z)\mathcal{M}_0} &= P_{\mathcal{M}_0} - P_{q_1(z)H^2} + \text{ finite rank} \\ P_{q_1(z)H^2} &= P_{q_1(z)\mathcal{M}_0} + \text{ finite rank} \\ &= \big(V_{q_1(z)}P_{\mathcal{M}_0}\big)\big(V_{q_1(z)}P_{\mathcal{M}_0}\big)^* + \text{ finite rank} \in \mathcal{A}(V_z, V_w). \end{aligned}$$ ■

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4 The Construction of Operators on the Space $q_1(z)H^2 + q_2(w)H^2$

In this section we shall study some operators which will be used later.

Definition 4.1 Let $q_1(z)$ and $q_2(w)$ be two finite Blaschke products, and $\mathcal{M} = q_1(z)H^2 + q_2(w)H^2$. We define a projection Q as follows:

$$Q\colon q_1(z)H^2 + q_2(w)H^2 \to \left(q_1(z)H^2 + q_2(w)H^2\right) \ominus q_1(z)H^2.$$

Then we have $Q \in \mathcal{A}(V_z, V_w; \mathcal{M})$ by Lemma 3.2.

In the following argument, without loss of generality, we may assume that $q_1(0) = 0$.

Lemma 4.1 *If $q_1(z)$ is a finite Blaschke product of degree k and $q_1(0) = 0$, then there exists a basis $\{e_i\}_{i=0}^{k-1}$ of $H^2(z) \ominus q_1(z)H^2(z)$ which satisfies*

$$\begin{cases} ze_{k-1} = q_1(z) \\ ze_i \in H^2(z) \ominus q_1(z)H^2(z) \quad (0 \le i \le k-2). \end{cases}$$

Proof Since,

$$\left\langle z\left(H^2(z) \ominus q_1(z)H^2(z)\right), zq_1(z)H^2(z)\right\rangle = 0,$$

we have

$$z\left(H^2(z) \ominus q_1(z)H^2(z)\right) \subseteq H^2(z) \ominus q_1(z)H^2(z) \oplus \mathbb{C}q_1(z).$$

We can choose a basis $\{e_i\}_{i=0}^{k-1}$ of $H^2(z) \ominus q_1(z)H^2(z)$ which satisfies $e_{k-1} = T_z^* q_1(z)$. Then

$$ze_i = \sum_{j=0}^{k-1} a_{i,j}e_j + b_i q_1(z) \;\; (0 \le i \le k-2)$$

$$ze_{k-1} = q_1(z).$$

By simple calculations, we have $b_0 = b_1 = \cdots = b_{k-2} = 0$. ∎

Here we shall study some properties of the operator QV_zQ. Let $\{e_i\}$ be the basis obtained in Lemma 4.1.

Lemma 4.2 *Suppose that $\mathcal{M}$ is a Hardy submodule of finite codimension. Let p be the projection from $H^2(z)$ onto $H^2(z) \ominus q_1(z)H^2(z)$. Then*

$$QV_zQ = \left(pT_zp\right) \otimes P_{q_2(w)H^2(w)}.$$

Proof Since $Q = p \otimes P_{q_2(w)H^2(w)}$ by Lemma 2.1, we have

$$\begin{aligned} QV_zQ &= QT_zQ \\ &= \left(p \otimes P_{q_2(w)H^2(w)}\right)\left(T_z \otimes I\right)\left(p \otimes P_{q_2(w)H^2(w)}\right) \\ &= \left(pT_zp\right) \otimes P_{q_2(w)H^2(w)}. \end{aligned}$$

∎

Lemma 4.3 *Suppose that $\mathcal{M}$ is a Hardy submodule of finite codimension. Let $C^*(QV_zQ)$ be the C^*-algebra generated by QV_zQ, let p_i be the projection onto $q_2(w)H^2(w)e_i$ and let S be the truncated shift operator defined as follows:*

$$\begin{aligned} S\colon q_2(w)g(w)e_i &\mapsto q_2(w)g(w)e_{i+1} \\ q_2(w)g(w)e_{k-1} &\mapsto 0. \end{aligned}$$

Then p_i and S are contained in $C^*(QV_zQ)$.

Proof Since the C^*-algebra generated by pT_zp is irreducible in $pH^2(z) = H^2(z) \ominus q_1(z)H^2(z)$ by Corollary 2.5, that is the full matrix algebra $M_k(\mathbb{C})$. Then, by Lemma 4.2, we have $C^*(QV_zQ) = M_k(\mathbb{C}) \otimes P_{q_2(w)H^2(w)}$. ■

5 An Affirmative Answer to the Berger-Coburn-Lebow Problem

In this section, we shall solve the BCL problem affirmatively. First, we will consider the case where $[S_z^*, S_w] = 0$. Using this result, next, we will solve the BCL problem completely.

Definition 5.1 Let $q_1(z)$ and $q_2(w)$ be two finite Blaschke products, and $k = \deg q_1(z)$ and $l = \deg q_2(w)$. We define an operator as follows:

$$\begin{aligned} U\colon q_1(z)H^2 + q_2(w)H^2 &\to z^kH^2 + w^lH^2 \\ q_1(z)f(z,w) &\mapsto z^kf(z,w) \\ q_2(w)w^je_i &\mapsto z^iw^{j+l}, \end{aligned}$$

where $\{e_i\}_{i=0}^{k-1}$ is the basis obtained in Lemma 4.1. By Lemma 2.1, U is a unitary operator from $q_1(z)H^2 + q_2(w)H^2$ onto $z^kH^2 + w^lH^2$.

Theorem 5.1 *If $\mathcal{M} = q_1(z)H^2 + q_2(w)H^2$ for two finite Blaschke products $q_1(z)$ and $q_2(w)$ such that $\deg q_1(z) = k$ and $\deg q_2(w) = l$, then $\mathcal{A}(V_z, V_w; \mathcal{M})$ is unitarily equivalent to $\mathcal{A}(T_z|_{U\mathcal{M}}, T_w|_{U\mathcal{M}}; U\mathcal{M})$ with U, that is, $U\mathcal{A}(q_1(z)H^2 + q_2(w)H^2)U^* = \mathcal{A}(z^kH^2 + w^lH^2)$.*

Proof We shall show that $U^*\mathcal{A}(T_z|_{U\mathcal{M}}, T_w|_{U\mathcal{M}}; U\mathcal{M})U = \mathcal{A}(V_z, V_w)$. In this proof, T_z (resp. T_w) denotes $T_z|_{U\mathcal{M}}$ (resp. $T_w|_{U\mathcal{M}}$), and $\mathcal{A}(T_z, T_w)$ denotes

$$\mathcal{A}(T_z|_{U\mathcal{M}}, T_w|_{U\mathcal{M}}; U\mathcal{M}),$$

for short.

First, we shall show $U^*\mathcal{A}(T_z, T_w)U \subseteq \mathcal{A}(V_z, V_w)$. Since

$$\begin{aligned} U^*T_zUq_1(z)f(z,w), &= U^*T_zz^kf(z,w), = U^*z^{k+1}f(z,w), \\ &= q_1(z)zf(z,w), = V_zq_1(z)f(z,w), \end{aligned}$$

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and for any $j \geq 0$ and $0 \leq i \leq k-1$,

$$U^*T_zUq_2(w)w^je_i = U^*T_zz^iw^{j+l} = U^*z^{i+1}w^{j+l}$$
$$= \begin{cases} q_2(w)w^je_{i+1} & (i+1 \leq k-1) \\ q_1(z)w^{j+l} & (i+1 = k). \end{cases}$$

Hence,

$$U^*T_zU = V_z|_{q_1(z)H^2} + S + V^*_{q_2(w)}V_zV^l_w|_{q_2(w)H^2(w)e_{k-1}}.$$

By Lemmas 3.2 and 4.3, we have, $U^*T_zU \in \mathcal{A}(V_z, V_w)$ and trivially $U^*T_wU = V_w$. Therefore

$$U^*\mathcal{A}(T_z, T_w)U \subseteq \mathcal{A}(V_z, V_w).$$

Next, we shall show $U\mathcal{A}(V_z, V_w)U^* \subseteq \mathcal{A}(T_z, T_w)$. Since

$$UV_zU^*z^kf(z,w) = UV_zq_1(z)f(z,w), = Uq_1(z)zf(z,w,), $$
$$= z^{k+1}f(z,w), = T_zz^kf(z,w),$$

and for any $0 \leq i \leq k-1$ and $j \geq 0$,

$$UV_zU^*w^{j+l}z^i = UV_zq_2(w)w^je_i, = Uq_2(w)w^jze_i,$$
$$= \begin{cases} Uq_2(w)w^j\sum_{m=0}^{k-1}a_{i,m}e_m & (i \leq k-2) \\ Uq_1(z)q_2(w)w^j & (i = k-1) \end{cases},$$
$$= \begin{cases} w^{j+l}\sum_{m=0}^{k-1}a_{i,m}z^m & (i \leq k-2) \\ z^kq_2(w)w^j & (i = k-1). \end{cases}$$

Hence, for $0 \leq i \leq k-2$, we have

$$UV_zU^*w^{j+l}z^i = w^{j+l}\sum_{m=0}^{k-1}a_{i,m}z^m = \Big(\sum_{m=0}^{k-1}a_{i,m}T_z^{*i}T_z^m\Big)w^{j+l}z^i,$$
$$UV_zU^*w^{j+l}z^{k-1} = z^kq_2(w)w^j = \big(T_w^{*l}T_{q_2(w)}T_z\big)w^{j+l}z^{k-1},$$

and

$$UV_zU^* = T_z|_{z^kH^2} + \sum_{i=0}^{k-2}\Big(\sum_{j=0}^{k-1}a_{i,j}T_z^{*i}T_z^j|_{w^lH^2(w)z^i}\Big) + T_w^{*l}T_{q_2(w)}T_z|_{w^lH^2(w)z^{k-1}}.$$

Since

$$\operatorname{ran}\big(P_{U\mathcal{M}} - T_zT_z^*\big) = w^lH^2(w) \oplus \text{ finite},$$
$$\operatorname{ran}\big(P_{U\mathcal{M}} - T_z^2T_z^{*2}\big) = w^lH^2(w) \oplus zw^lH^2(w) \oplus \text{ finite},$$
$$\vdots$$
$$\operatorname{ran}\big(P_{U\mathcal{M}} - T_z^kT_z^{*k}\big) = w^lH^2(w) \oplus \cdots \oplus z^{k-1}w^lH^2(w) \oplus \text{ finite},$$

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we have $P_{w^l H^2(w) z^i} \in \mathcal{A}(T_z, T_w)$ for $0 \leq i \leq k-1$ and $UV_zU^* \in \mathcal{A}(T_z, T_w)$. Therefore

$$U\mathcal{A}(V_z, V_w)U^* \subseteq \mathcal{A}(T_z, T_w).$$

Hence

$$U\mathcal{A}(V_z, V_w)U^* = \mathcal{A}(T_z, T_w). \qquad \blacksquare$$

Theorem 5.2 *Suppose that $\mathcal{M}$ is a Hardy submodule of finite codimension. Then $\mathcal{A}(V_z, V_w; \mathcal{M})$ is unitarily equivalent to $\mathcal{A}(T_z, T_w; H^2)$.*

Proof Considering the following decomposition of $\mathcal{M}$,

$$\mathcal{M} = \left(q_1(z)H^2 + q_2(w)H^2\right) \oplus \mathcal{F}_{\mathcal{M}} = \mathcal{M}_0 \oplus \mathcal{F}_{\mathcal{M}},$$

one can check easily that there exists a set $\mathcal{F}$ of monomials such that

$$H^2(\mathbb{S}) = \left(z^k H^2 + w^l H^2\right) \oplus \mathcal{F}$$

is a Hardy submodule defined in Theorem 1.1, and $\dim \mathcal{F} = \dim \mathcal{F}_{\mathcal{M}}$. Let $\tilde{U}$ be the unitary operator from $\mathcal{M}$ onto $H^2(\mathbb{S})$ defined as follows:

$$\tilde{U} = \tilde{U}P_{\mathcal{M}_0} + \tilde{U}P_{\mathcal{F}_{\mathcal{M}}}$$

where $\tilde{U}P_{\mathcal{M}_0} = U$ defined in Definition 5.1, and let $\tilde{U}P_{\mathcal{F}_{\mathcal{M}}}$ be any unitary from $\mathcal{F}_{\mathcal{M}}$ onto $\mathcal{F}$. It suffices to show $\tilde{U}\mathcal{A}(V_z, V_w)\tilde{U}^* = \mathcal{A}\left(T_z|_{H^2(\mathbb{S})}, T_w|_{H^2(\mathbb{S})}; H^2(\mathbb{S})\right)$ by Theorem 1.1. In the following argument, T_z (resp. T_w) denotes $T_z|_{H^2(\mathbb{S})}$ (resp. $T_w|_{H^2(\mathbb{S})}$), and $\mathcal{A}(T_z, T_w)$ denotes $\mathcal{A}\left(T_z|_{H^2(\mathbb{S})}, T_w|_{H^2(\mathbb{S})}; H^2(\mathbb{S})\right)$, for short.

$$\begin{aligned}
\tilde{U}V_z\tilde{U}^* &= \left(\tilde{U}P_{\mathcal{M}_0} + \tilde{U}P_{\mathcal{F}_{\mathcal{M}}}\right) V_z \left(\tilde{U}P_{\mathcal{M}_0} + \tilde{U}P_{\mathcal{F}_{\mathcal{M}}}\right)^* \\
&= \tilde{U}P_{\mathcal{M}_0}V_z\left(\tilde{U}P_{\mathcal{M}_0}\right)^* + \text{ finite rank} \\
&= \tilde{U}P_{\mathcal{M}_0}V_zP_{\mathcal{M}_0}\tilde{U}^* + \text{ finite rank} \\
&= UT_zP_{\mathcal{M}_0}U^* + \text{ finite rank.}
\end{aligned}$$

Since $P_{U\mathcal{M}_0} \in \mathcal{A}(T_z, T_w)$, and by Theorem 5.1, we have

$$UT_zP_{\mathcal{M}_0}U^* \in \mathcal{A}(T_zP_{U\mathcal{M}_0}, T_wP_{U\mathcal{M}_0}) \subseteq \mathcal{A}(T_z, T_w).$$

Hence

$$\tilde{U}\mathcal{A}(V_z, V_w)\tilde{U}^* \subseteq \mathcal{A}(T_z, T_w).$$

$$\begin{aligned}
\tilde{U}^*T_z\tilde{U} &= \left(\tilde{U}P_{\mathcal{M}_0} + \tilde{U}P_{\mathcal{F}_{\mathcal{M}}}\right)^* T_z \left(\tilde{U}P_{\mathcal{M}_0} + \tilde{U}P_{\mathcal{F}_{\mathcal{M}}}\right) \\
&= \left(\tilde{U}P_{\mathcal{M}_0}\right)^* T_z\tilde{U}P_{\mathcal{M}_0} + \text{ finite rank} \\
&= P_{\mathcal{M}_0}\tilde{U}^*T_z\tilde{U}P_{\mathcal{M}_0} + \text{ finite rank} \\
&= U^*T_zP_{U\mathcal{M}_0}U + \text{ finite rank.}
\end{aligned}$$

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Since $P_{\mathcal{M}_0} \in \mathcal{A}(V_z, V_w)$, and by Theorem 5.1, we have

$$U^* T_z P_{U\mathcal{M}_0} U \in \mathcal{A}(V_z P_{\mathcal{M}_0}, V_w P_{\mathcal{M}_0}) \subseteq \mathcal{A}(V_z, V_w).$$

Hence

$$\tilde{U}^* \mathcal{A}(T_z, T_w) \tilde{U} \subseteq \mathcal{A}(V_z, V_w).$$

Therefore

$$\tilde{U}^* \mathcal{A}(T_z, T_w) \tilde{U} = \mathcal{A}(V_z, V_w).$$

■

Corollary 5.1 *If $\mathcal{M}$ is a Hardy submodule of finite codimension in H^2, then $\mathcal{A}(V_z, V_w)/\mathcal{K}(\mathcal{M})$ is isomorphic to $\mathcal{A}(T_z, T_w)/\mathcal{K}(H^2)$.*

Proof By Theorem 5.2, there exists a unitary operator U from $\mathcal{M}$ onto H^2 such that $U\mathcal{A}(V_z, V_w)U^* = \mathcal{A}(T_z, T_w)$. Since $U\mathcal{K}(\mathcal{M})U^* = \mathcal{K}(H^2)$, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}(\mathcal{M}) & \longrightarrow & \mathcal{A}(V_z, V_w) & \longrightarrow & \mathcal{A}(V_z, V_w)/\mathcal{K}(\mathcal{M}) & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \mathrm{Ad}\, U|_{\mathcal{K}(\mathcal{M})}} & & \Big\downarrow {\scriptstyle \mathrm{Ad}\, U} & & \Big\downarrow & & \\
0 & \longrightarrow & \mathcal{K}(H^2) & \longrightarrow & \mathcal{A}(T_z, T_w) & \longrightarrow & \mathcal{A}(T_z, T_w)/\mathcal{K}(H^2) & \longrightarrow & 0.
\end{array}$$

■

Concluding remarks In [3], Berger, Coburn and Lebow defined an index (called the BCL index) for two essentially double commuting isometries, and they showed that the absolute value of this index is a unitary invariant for the C^*-algebras generated by these isometries, and conjectured that this index is a unitary invariant.

Yang made a study of the BCL index on H^2 in [12]. By Yang's Berger-Shaw type theorem (Theorem 2.1), the BCL index can be considered in the case that the Hardy submodules are finitely generated by polynomials. In fact, Yang has studied the BCL index under a certain very mild condition in [15], [16] and [17]. He showed that the BCL index $\mathrm{ind}(V_z, V_w)$ has the following explicit formula:

$$\mathrm{ind}(V_z, V_w) = \dim\left(\ker(S_z) \cap \ker(S_w)\right) - \dim\left(\ker(V_z^*) \cap \ker(V_w^*)\right).$$

From this it follows that $\mathrm{ind}(V_z, V_w) = -1$.

Combining their study, we pose the following question:

Question If $\mathcal{M}$ is any Hardy submodule generated by a finite number of polynomials, then is $\mathcal{A}(V_z, V_w; \mathcal{M})$ unitarily equivalent to $\mathcal{A}(T_z, T_w; H^2)$?

We will study this conjecture at a later time.

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