5

Multivariate Fourier–Laplace integrals

5.1 Overview

In this chapter we generalize the univariate saddle point techniques of Chapter 4 to ***multivariate Fourier–Laplace integrals*** of the form

$$\mathcal{I}(\lambda) = \int_C A(\boldsymbol{z}) \exp(-\lambda\phi(\boldsymbol{z}))\, d\boldsymbol{z}\,, \tag{5.1}$$

where the ***amplitude*** A and ***phase*** ϕ are now analytic functions of a vector argument $\boldsymbol{z}$ in d variables and C is a d-chain in $\mathbb{C}^d$ (see Section A.3 of Appendix A for definitions involving integration of chains on manifolds). In one variable, the comprehensive Theorem 4.1 covers all degrees of vanishing of the phase and amplitude functions. The range of possibilities for the phase function ϕ in higher dimensions is much greater, however, and we restrict ourselves here to the case of ***nondegenerate phase*** where the $d \times d$ ***Hessian*** matrix $\mathcal{H} = \left(\dfrac{\partial^2\phi}{\partial z_j\, \partial z_k}\right)$ of ϕ is nonsingular at the points in the domain of integration determining asymptotics. The Taylor series for ϕ at a point $\boldsymbol{p} \in \mathbb{C}^d$ is

$$\phi(\boldsymbol{z}) = \phi(\boldsymbol{p}) + (\boldsymbol{z}-\boldsymbol{p})^T(\nabla\phi)(\boldsymbol{p}) + \frac{1}{2}(\boldsymbol{z}-\boldsymbol{p})^T\,\mathcal{H}(\boldsymbol{p})(\boldsymbol{z}-\boldsymbol{p}) + O\left(|\boldsymbol{z}-\boldsymbol{p}|^3\right),$$

hence the Hessian matrix $\mathcal{H}(\boldsymbol{p})$ represents (twice) the quadratic term in the phase, and nondegeneracy is a generalization of nonvanishing of the quadratic term for a univariate phase function.

Exercise 5.1. Determine whether the phase function $\phi(x, y, z) = z^2+(x+y)z+xy$ is degenerate at the origin.

We begin, analogously to the univariate case, by considering integrals whose phase is restricted to the ***standard quadratic*** $S(\boldsymbol{z}) := z_1^2 + \cdots + z_d^2$. Asymptotic behavior when A is monomial and ϕ is the standard quadratic (Corollary 5.7

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below) is coupled with a big-O bound (Lemma 5.8 below), allowing us to integrate term by term and obtain the following result. Recall that if $\boldsymbol{r} \in \mathbb{N}^d$ we write $|\boldsymbol{r}| = r_1 + \cdots + r_d$.

Theorem 5.1 (standard phase)**.** *Let $A(\boldsymbol{x}) = \sum_{\boldsymbol{r}\in\mathbb{N}^d} a_{\boldsymbol{r}} \boldsymbol{x}^{\boldsymbol{r}}$ be a real analytic function defined on a neighborhood $\mathcal{N}$ of the origin in $\mathbb{R}^d$. If*

$$\mathcal{I}(\lambda) = \int_{\mathcal{N}} A(\boldsymbol{x}) e^{-\lambda S(\boldsymbol{x})}\, d\boldsymbol{x} \tag{5.2}$$

then there is an asymptotic series expansion

$$\mathcal{I}(\lambda) \approx \sum_{n\geq 0} \sum_{|\boldsymbol{r}|=n} a_{\boldsymbol{r}} \beta_{\boldsymbol{r}} \lambda^{-(|\boldsymbol{r}|+d)/2},$$

where

$$\beta_{\boldsymbol{r}} = \begin{cases} 0 & \text{if any } r_j \text{ is odd} \\ \pi^{d/2} \displaystyle\prod_{j=1}^{d} \frac{(2m_j)!}{m_j! 4^{m_j}} & \text{if } \boldsymbol{r} = 2\boldsymbol{m} \end{cases}.$$

After establishing Theorem 5.1 in Section 5.2, we use a change of variables and contour deformation to study the case of nondegenerate phase whose real part has a strict minimum at the origin. Our next result is proven in Section 5.3; note that we change our variables from $\boldsymbol{x}$ to $\boldsymbol{z}$ to reflect the fact that our proof works over the complex numbers.

Theorem 5.2 (Re$\{\phi\}$ has a strict minimum)**.** *Let A and ϕ be complex-valued analytic functions on a compact neighborhood $\mathcal{N}$ of the origin in $\mathbb{R}^d$. Suppose that the real part of ϕ is nonnegative on $\mathcal{N}$ and vanishes only at the origin, and that the Hessian matrix $\mathcal{H}$ of ϕ at the origin is nonsingular. Then $\mathcal{I}(\lambda) = \int_{\mathcal{N}} A(\boldsymbol{z}) e^{-\lambda\phi(\boldsymbol{z})}$ has an asymptotic expansion*

$$\mathcal{I}(\lambda) \approx \sum_{\ell\geq 0} c_\ell \lambda^{-d/2-\ell} \tag{5.3}$$

with leading coefficient

$$c_0 = A(\mathbf{0}) \frac{(2\pi)^{d/2}}{\sqrt{\det \mathcal{H}}}, \tag{5.4}$$

where $\sqrt{\det \mathcal{H}}$ is the product of the principal square roots of the eigenvalues of $\mathcal{H}$.

Exercise 5.2. Show that the expansion in (5.3) can be written as

$$\mathcal{I}(\lambda) \approx \frac{A(\mathbf{0})}{\sqrt{\det 2\pi\lambda\mathcal{H}}} \sum_{\ell\geq 0} c'_\ell \lambda^{-\ell}, \tag{5.5}$$

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where $c_0' = 1$.

Exercise 5.3. Let $c > 0$, where c is independent of λ. What happens to the right side of (5.5) if we change ϕ to ϕ/c and λ to $c\lambda$?

When $\text{Re}\{\phi\}$ is strictly positive except at a finite number of points in the domain of integration of $\mathcal{I}(\lambda)$ then, up to an asymptotically negligible error, we can express $\mathcal{I}(\lambda)$ as a sum of integrals localized to neighborhoods of these vanishing points. A chain of integration in a manifold which can be localized to arbitrarily small neighborhoods can be pulled back to correspondingly localized integrals over $\mathbb{R}^d$, which is why the above results are stated for integrals over neighborhoods of the origin in $\mathbb{R}^d$. In dimension greater than one, however, it is possible for an analytic function to have a real part that vanishes along a set of positive dimension without vanishing everywhere, meaning it may not be possible to localize.

Exercise 5.4. Over which of the following chains in $\mathbb{C}^2$ does $\phi(x, y) = x^2 + y^2$ have a nonnegative real part and, among those where this holds, which have the real part of ϕ vanishing only at the origin?

(a) a small neighborhood of $\mathbf{0}$ in the real × real subspace of $\mathbb{C}^2$
(b) a small neighborhood of $\mathbf{0}$ in the imaginary × imaginary subspace of $\mathbb{C}^2$
(c) a small neighborhood of $\mathbf{0}$ in the diagonal subspace $\{(x, y) \in \mathbb{C}^2 : x = y\}$ of $\mathbb{C}^2$
(d) a small neighborhood of $\mathbf{0}$ in the linear subspace of $\mathbb{C}^2$ spanned over $\mathbb{R}$ by $(1 + i, 0)$ and $(0, 1 + i)$

Such difficulties lead us to state our most general results in the language of stratified spaces and vector flows. A ***vector flow*** on a space X is the solution $\Psi : X \times [0, T] \to X$ to a differential equation $(d/dt)\Psi(x, t) = \boldsymbol{v}(\Psi(\boldsymbol{x}, t))$, where $\boldsymbol{v}$ is a vector field on X (see, e.g., Lemma 5.14); an ***upward gradient flow*** is defined by the vector field $\boldsymbol{v} = \nabla\phi$, while a ***downward gradient flow*** is defined by $\boldsymbol{v} = -\nabla\phi$. These constructions and results, summarized in Appendix D, have been around for over 50 years, though they are not very well known outside of differential topology and singularity theory. To ease exposition we now state our main result using some terminology to be defined in Section 5.4, where the result is proved. If X is an oriented stratified space then $\boldsymbol{q} \in X$ is a ***critical point (in the stratified sense)*** of an analytic map $\phi : X \to \mathbb{C}$ if $\boldsymbol{q}$ lies in a stratum $\mathcal{S}$ and the differential $d\phi|_{\mathcal{S}}$ at $\boldsymbol{q}$ is zero. Appendix D contains a more complete explanation of the vector flows we use.

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Theorem 5.3 (minimum of Re$\{\phi\}$ is not strict but there are finitely many critical points)**.** *Let $\mathcal{V}$ be a smooth complex $(d-k)$-dimensional algebraic variety and suppose that*

(i) $X = \Delta^p \times M^{d-k}$ *is a stratified space of dimension* $p+d-k$ *in* $\mathbb{C}^{p+d-k}$*, where* $\Delta^p \subseteq \mathbb{R}^p$ *is the standard p-simplex and* $M^{d-k} \subseteq \mathcal{V} \subseteq \mathbb{C}^d$ *is a smooth* $(d-k)$*-dimensional analytic submanifold of* $\mathcal{V}$*,*
(ii) the closure of X is represented by an analytic $(p+d-k)$*-chain C,*
(iii) $\phi : X \to \mathbb{C}$ *is an analytic map with* $\min_{x\in X} \operatorname{Re}\phi(x) = 0$*, and*
(iv) $\eta = A(\boldsymbol{z})\, d\boldsymbol{z}$ *is a holomorphic* $(p+d-k)$*-form on X.*

Assume that the set G of critical points of ϕ on C is finite and that the subset $G' \subseteq G$ where $\operatorname{Re}\phi$ vanishes are all in strata of dimension $p+d-k$. Suppose also that

(v) $\det \mathcal{H}(\boldsymbol{q}) \neq 0$ *for all* $\boldsymbol{q} \in G'$*, where* $\mathcal{H}$ *is the Hessian matrix for* ϕ *in some local coordinates near* $\boldsymbol{q}$*,*
(vi) the imaginary part (and thus all) of $\phi(\boldsymbol{q})$ *is zero for all* $\boldsymbol{q} \in G'$*,*
(vii) the boundary ∂C *is supported on strata of dimension at most* $p+d-k-1$*, with its simplices* σ_j *analytic orientation preserving maps having disjoint interiors, and*
(viii) the elements of G' *lie in the interiors of distinct simplices* σ_j*.*

Then the integral

$$\mathcal{I}(\lambda) = \int_C e^{-\lambda\phi(\boldsymbol{z})}\,\eta$$

has an asymptotic expansion

$$\mathcal{I}(\lambda) \approx \sum_{\ell=0}^{\infty} c_\ell \lambda^{-(d-k)/2-\ell} \tag{5.6}$$

as $\lambda \to \infty$*, with leading term*

$$c_0 = (2\pi)^{d/2} \sum_{\boldsymbol{q}\in G'} \frac{A(\boldsymbol{q})}{\sqrt{\det \mathcal{H}(\boldsymbol{q})}}\,. \tag{5.7}$$

The sign for the square root of the determinant is computed by choosing a parametrization Υ *for X near* $\boldsymbol{q}$ *by a neighborhood of the origin in* $\mathbb{R}^{p+d-k}$ *and then multiplying the principal square roots of the eigenvalues of* ϕ *in these coordinates with the Jacobian determinant* $\det d\Upsilon(\boldsymbol{q})$*.*

Without the assumption that the imaginary part of $\phi(\boldsymbol{q})$ *vanishes for* $\boldsymbol{q} \in G'$ *the formula* (5.7) *holds when each summand is multiplied by the term* $e^{-\lambda\phi(\boldsymbol{q})}$ *with modulus* 1*, making* $c_0 = c_0(\lambda)$ *dependent on* λ *but having bounded modulus.*

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Remark 5.4. We apply Theorem 5.3 at two points in this book: first with $p = 0$ and $k = 1$ to derive Theorem 9.12, and second in its more general form to prove asymptotics for multiple points in Chapter 10. In the statement of the theorem, the first four conditions are geometric conditions ensuring the existence of the necessary deformations and analytic extensions, while the last four ensure we know how to do computations.

Example 5.5. Let $X = I \times S^1$, where I is the interval $[-1, 1]$ and S^1 is the unit circle parametrized by $\theta \in [-\pi, \pi]$ with the endpoints identified, so that X can be nicely embedded in $\mathbb{C}^2$. We apply Theorem 5.3 in the case $p = k = 1$ and $d = 2$, with $A(x, y) = 1$ and $\phi : X \to \mathbb{C}$ defined by

$$\phi(t, \theta) = K\theta^2 + iL\theta t \tag{5.8}$$

for real numbers $K > 0$ and L. The phase ϕ is analytic on X, and the 2-chain C representing X can be any cell complex with a subcomplex $I \times \mathcal{N}$ for a compact neighborhood $\mathcal{N}$ of $\theta = 0$ in S^1. There is a single critical point $\boldsymbol{p} = (0, 0)$, at which $\operatorname{Re}\phi$ vanishes, so $G' = G = \{\boldsymbol{p}\}$.

Note that the strip $I \times \{0\}$ on which the phase function vanishes extends out to the bounding circles of the cylinder X, so we are not in a case where the magnitude of the integrand is small away from $\boldsymbol{p}$, and Theorem 5.2 does not apply. The Hessian matrix of ϕ at $\boldsymbol{p}$ is $\begin{pmatrix} 2K & iL \\ iL & 0 \end{pmatrix}$, so Theorem 5.3 implies

$$\mathcal{I}(\lambda) = \int_{\mathcal{N}\times I} e^{-\lambda\phi(\boldsymbol{x})}\, d\boldsymbol{x} \sim \frac{2\pi}{\lambda\, |L|}.$$

The choice of sign on the term $\sqrt{L^2} = |L|$ is arbitrary and depends on properly orienting $\mathcal{N} \times I$ for the application at hand. ◁

Exercise 5.5. Let X be the real sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \subset \mathbb{C}^3$ and let $\phi(x, y, z) = z^2 + ix^2$.

(a) Identify the sets G and G'.
(b) Determine $\mathcal{H}(\boldsymbol{q})$ for $\boldsymbol{q} \in G'$.
(c) Determine $c_0(\lambda)$ when $A(x, y, z) = 1 + x + y$.

5.2 Standard phase

As in the one-dimensional case, we begin with the simplest phase function and a monomial amplitude. We first state a formula for the one-dimensional integral with amplitude $A(x) = x^{2n}$ and standard phase in terms of the explicit constants

$$\beta_{2n} = \sqrt{\pi}\,\frac{(2n)!}{n!\,4^n}.$$

Proposition 5.6. *For all $n \in \mathbb{N}$,*

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2}\,dx = \beta_{2n}.$$

Proof For $n = 0$ this is just the standard Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi},$$

and the general result follows by induction. Indeed, rewriting

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2}\,dx = \int_{-\infty}^{\infty} \frac{-x^{2n-1}}{2}\left(-2x\,e^{-x^2}\,dx\right)$$

and applying integration by parts gives

$$\begin{aligned}\int_{-\infty}^{\infty} x^{2n} e^{-x^2}\,dx &= \frac{2n-1}{2}\int_{-\infty}^{\infty} x^{2n-2}\,e^{-x^2}\,dx \\ &= \frac{2n-1}{2}\cdot\sqrt{\pi}\cdot\frac{(2n-2)!}{(n-1)!\,4^{n-1}} \\ &= \sqrt{\pi}\,\frac{(2n)!}{n!\,4^n}\end{aligned}$$

by induction, as claimed. □

Corollary 5.7 (monomial integral)**.** *Let $S(\boldsymbol{z}) = \sum_{j=1}^{d} z_j^2$ and $\boldsymbol{r} \in \mathbb{N}^d$. Then*

$$\int_{\mathbb{R}^d} \boldsymbol{z}^{\boldsymbol{r}} e^{-\lambda S(\boldsymbol{z})}\,d\boldsymbol{z} = \beta_{\boldsymbol{r}}\lambda^{-(d+|\boldsymbol{r}|)/2}$$

for any $\lambda > 0$, where $\beta_{\boldsymbol{r}} = \prod_{j=1}^{d}\beta_{r_j}$ if all the components r_j are even and $\beta_{\boldsymbol{r}} = 0$ otherwise.

Proof If $n \in \mathbb{N}$ then making the change of variables $x = y\lambda^{-1/2}$ and applying Proposition 5.6 proves

$$\int_{-\infty}^{\infty} x^{2n}\,e^{-\lambda x^2}\,dx = \lambda^{-1/2-n}\int_{-\infty}^{\infty} y^{2n} e^{-y^2}\,dy = \lambda^{-1/2-n}\beta_{2n},$$

while $\int_{-\infty}^{\infty} x^{2n+1}\,e^{-\lambda x^2} = 0$ as its integrand is odd. The integral under consideration factors as

$$\int_{\mathbb{R}^d} \boldsymbol{z}^{\boldsymbol{r}} e^{-\lambda S(\boldsymbol{z})}\,d\boldsymbol{z} = \prod_{j=1}^{d}\left[\int_{-\infty}^{\infty} z_j^{r_j}\,e^{-\lambda z_j^2}\,dz_j\right],$$

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and the result follows from simplifying each factor. □

Before establishing Theorem 5.1 we also need to bound the error terms that appear.

Lemma 5.8 (Big-O Lemma)**.** *Let A be a measurable function satisfying $A(z) = O(|z|^r)$ at the origin. Then the integral of $A(z)e^{-\lambda S(z)}$ over any compact set K may be bounded from above by*

$$\int_K A(z)e^{-\lambda S(z)}\, dz = O\left(\lambda^{-(d+r)/2}\right).$$

The implied constant on the right goes to zero as the implied constant in the hypothesis $A(z) = O(|z|^r)$ goes to zero.

Proof Because K is compact and $A(z) = O(|z|^r)$ at the origin, there exists a constant $C > 0$ such that $|A(z)| \leq C|z|^r$ on all of K. Let

$$K_0 = \left\{z \in K : |z| \leq \lambda^{-1/2}\right\}$$

denote the intersection of K with the ball of radius $\lambda^{-1/2}$, and for $n \geq 1$ let

$$K_n = \left\{z \in K : 2^{n-1}\lambda^{-1/2} \leq |z| \leq 2^n\lambda^{-1/2}\right\}$$

denote the intersection of K with a shell. On K_0 we have $|A(z)| \leq C\lambda^{-r/2}$ and $|e^{-\lambda S(z)}| \leq 1$, so

$$\int_{K_0} A(z)e^{-\lambda S(z)}\, dz \leq \mathrm{Vol}(K_0)\, C\lambda^{-r/2} = \frac{(\pi/\lambda)^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}\, C\,\lambda^{-r/2}.$$

For $n \geq 1$, when $z \in K_n$ we have the upper bounds

$$|A(z)| \leq 2^{rn}C\lambda^{-r/2} \qquad \text{by upper bound on } |z|$$

$$e^{-\lambda S(z)} \leq e^{-2^{2n-2}} \qquad \text{by lower bound on } |z|$$

$$\mathrm{Vol}(K_n) \leq \frac{2^{dn}\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}\lambda^{-d/2} \qquad \text{by upper bound on } |z|.$$

Thus, if $C' = 1 + \sum_{n\geq 0} 2^{(d+r)n}e^{-2^{2n-2}} < \infty$ then

$$\left|\int_K A(z)e^{-\lambda S(z)}\, dz\right| \leq \sum_{n=0}^{\infty}\left|\int_{K_n} A(z)e^{-\lambda S(z)}\, dz\right| \leq \frac{\pi^{m/2}}{\Gamma\left(\frac{d}{2}+1\right)}\, C\, C'\, \lambda^{-(d+r)/2}$$

with the right-hand side going to zero with the implied constant C, as claimed. □

We are now ready to prove Theorem 5.1.

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Proof of Theorem 5.1. Writing $A(\boldsymbol{z})$ as a power series up to degree N plus a remainder term,

$$A(\boldsymbol{z}) = \left(\sum_{n=0}^{N} \sum_{|\boldsymbol{r}|=n} a_{\boldsymbol{r}} \boldsymbol{z}^{\boldsymbol{r}} \right) + R(\boldsymbol{z}),$$

where $R(\boldsymbol{z}) = O\left(|\boldsymbol{z}|^{N+1}\right)$. Using Corollary 5.7 to integrate all the monomial terms and Proposition 5.8 to bound the integral of $R(\boldsymbol{z})e^{-\lambda S(\boldsymbol{z})}$ shows that

$$\mathcal{I}(\lambda) = \sum_{n=0}^{N} \sum_{|\boldsymbol{r}|=n} a_{\boldsymbol{r}} \beta_{\boldsymbol{r}} \lambda^{-(d+n)/2} + O\left(\lambda^{-(d+N+1)/2}\right),$$

proving the stated asymptotic expansion. □

Exercise 5.6. Let $M(t) = \sup\{|A(\boldsymbol{z})| : |\boldsymbol{z}| = t\}$ and let K_R be a ball of fixed radius R. Prove a version of Theorem 5.1 using the fact that if $M(t) = O(t^{\alpha})$ then

$$\int_{K_R} A(\boldsymbol{z}) e^{-\lambda S(\boldsymbol{z})}\, d\boldsymbol{z} \le \int_0^R e^{-\lambda t} M(t)\, dV(K_t)\,,$$

where $dV(K_t) = c_t t^{d-1} dt$ is the volume of a spherical shell of thickness dt and radius t in d dimensions.

5.3 Real part of phase has a strict minimum

We extend our results beyond integrals with standard quadratic phases using complex analytic techniques. If $\mathcal{N}$ is a neighborhood of the origin in $\mathbb{R}^d$ and $\phi : \mathcal{N} \to \mathbb{C}$ is analytic on $\mathcal{N}$ then ϕ can be viewed as a complex analytic function on a neighborhood $\mathcal{N}_{\mathbb{C}}$ of the origin in $\mathbb{C}^d$ using its power series expansion. Suppose that $\phi(\mathbf{0}) = 0$ and the real part of ϕ is nonnegative on $\mathcal{N}$, so that the gradient of ϕ must vanish at the origin. Our first key lemma is that, under an assumption of nondegeneracy, we can change variables so that ϕ becomes the standard quadratic form.

Lemma 5.9 (Complex Morse Lemma)**.** *If $\phi(\boldsymbol{x})$ has vanishing gradient and nonsingular Hessian $\mathcal{H}$ at the origin then there is a bi-holomorphic change of variables $\boldsymbol{x} = \psi(\boldsymbol{y})$ around $\boldsymbol{x} = \boldsymbol{y} = \mathbf{0}$ such that $\phi(\psi(\boldsymbol{y})) = S(\boldsymbol{y}) = \sum_{j=1}^{d} y_j^2$. The Jacobian matrix $J_\psi = d\psi(\mathbf{0})$ satisfies* $(\det J_\psi)^2 = \frac{2^d}{\det \mathcal{H}}$.

Our proof of Lemma 5.9 is adapted from the proof of the real version given in [Ste93, VIII:2.3.2].

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Proof To prove the claim about the Jacobian determinant, we apply the chain rule to the equation $\phi(\psi(\boldsymbol{y})) = S(\boldsymbol{y})$ and conclude that the Hessian matrix of S at the origin equals $J_\psi^T \mathcal{H} J_\psi$. The Hessian of S is twice the identity matrix, so $2I = J_\psi^T \mathcal{H} J_\psi$ and taking determinants gives the stated result.

To prove the change of variables, we begin by writing

$$\phi(\boldsymbol{x}) = \sum_{j,k=1}^{d} x_j x_k \phi_{j,k}(\boldsymbol{x})$$

for analytic functions $\phi_{j,k} = \phi_{k,j}$ with constant terms $\phi_{j,k}(\mathbf{0}) = \mathcal{H}_{j,k}/2$. There is plenty of freedom, but a convenient choice is to take

$$x_j x_k \phi_{j,k}(\boldsymbol{x}) = \sum_{|\boldsymbol{r}|\geq 2} \frac{r_j(r_k - \delta_{j,k})}{|\boldsymbol{r}|(|\boldsymbol{r}|-1)}\, a_{\boldsymbol{r}} \boldsymbol{x}^{\boldsymbol{r}}, \tag{5.9}$$

where $a_{\boldsymbol{r}}$ are the Taylor coefficients of ϕ at $\boldsymbol{x}$ and $\delta_{j,k} = 1$ if $j = k$ and 0 otherwise. For fixed $\boldsymbol{r}$ it is easy to check that

$$\sum_{1\leq j,k\leq d} \frac{r_j(r_k - \delta_{j,k})}{|\boldsymbol{r}|(|\boldsymbol{r}|-1)} = 1,$$

so that $\phi(\boldsymbol{x}) = \sum_{j,k} x_j x_k \phi_{j,k}(\boldsymbol{x})$, and matching coefficients on the terms of order precisely two verifies $\phi_{j,k}(\mathbf{0}) = \mathcal{H}_{j,k}/2$. We may assume without loss of generality that $\phi_{j,j}(\mathbf{0}) = \mathcal{H}_{j,j} \neq 0$ for all j, because there is always a unitary map U such that the Hessian of $\phi \circ U$ has nonvanishing diagonal entries, and if $(\phi \circ U) \circ \psi_0 = S$ for some ψ_0 then $\phi \circ \psi = S$, where $\psi = U \circ \psi_0$.

We conclude with an induction. Since we are assuming that $\phi_{1,1}(\mathbf{0}) \neq 0$, the reciprocal $1/\phi_{1,1}(\boldsymbol{x})$ and a branch of $\sqrt{\phi_{1,1}(\boldsymbol{x})}$ are both analytic in a neighborhood of the origin. If

$$y_1 = \sqrt{\phi_{1,1}(\boldsymbol{x})}\left[x_1 + \sum_{k>1} \frac{x_k \phi_{1,k}(\boldsymbol{x})}{\phi_{1,1}(\boldsymbol{x})}\right]$$

then the terms of y_1^2 of total degree at most one in $x_2, \ldots, x_d$ match those of ϕ, since

$$\phi(\boldsymbol{x}) - y_1^2 = \sum_{j,k=2}^{d} x_j x_k \left(\phi_{j,k}(\boldsymbol{x}) - \frac{\phi_{1,j}(\boldsymbol{x})\phi_{1,k}(\boldsymbol{x})}{\phi_{1,1}(\boldsymbol{x})}\right). \tag{5.10}$$

In the new coordinates $y_1, x_2, x_3, \ldots, x_d$, the Hessian matrix of ϕ is a $(1, d-1)$ block matrix, where the submatrix $\mathcal{H}^{(1)}$ that corresponds to the variables $x_2, \ldots, x_d$ has determinant $\det \mathcal{H}^{(1)} = \det \mathcal{H}/\phi_{1,1} \neq 0$. In fact, if $\mathcal{H}$ is real positive definite then so is $\mathcal{H}^{(1)}$, provided the correct branch of the square root is chosen. Equation (5.10) thus writes ϕ in the form

$$\phi(\boldsymbol{x}) = y_1^2 + \sum_{j,k\geq 2} x_j x_k \phi_{j,k}^{(1)}(\boldsymbol{x}) \tag{5.11}$$

for some analytic functions $\phi_{j,k}^{(1)}$ satisfying $\phi_{j,k}^{(1)}(\mathbf{0}) = \mathcal{H}_{j,k}^{(1)}/2$. By induction, if we assume that

$$\phi(\boldsymbol{x}) = \sum_{j=1}^{r-1} y_j^2 + \sum_{j,k\geq r} x_j x_k \phi_{j,k}^{(r-1)}(\boldsymbol{x})$$

for some $1 \leq r \leq d$ then setting

$$y_r = \sqrt{\phi_{r,r}(\boldsymbol{x})}\left[x_r + \sum_{k>r} \frac{x_k \phi_{r,k}^{(r-1)}(\boldsymbol{x})}{\phi_{r,r}^{(r-1)}(\boldsymbol{x})} \right]$$

gives

$$\phi(\boldsymbol{x}) = \sum_{j=1}^{r} y_j^2 + \sum_{j,k\geq r+1} x_j x_k \phi_{j,k}^{(r)}(\boldsymbol{x})$$

for some analytic functions $\phi_{j,k}^{(r)}$ satisfying $\phi_{j,k}^{(r)}(\mathbf{0}) = \mathcal{H}_{j,k}^{(r)}/2$ with $\mathcal{H}^{(r)}$ nonsingular, leading in the end to a sequence of bi-holomorphic changes of variables writing $\phi(\boldsymbol{x}) = \sum_{j=1}^{d} y_j^2$ as claimed. □

Exercise 5.7. Use the Complex Morse Lemma to find a bi-holomorphic change of variables turning $\phi(x, y, z) = xy + yz + zx + xyz$ into the standard quadratic form $S(u, v, w)$.

We are now ready to prove Theorem 5.2.

Proof of Theorem 5.2. The convergent power series expansion of ϕ allows us to extend it to a neighborhood of the origin in $\mathbb{C}^d$. Under the change of variables ψ from Lemma 5.9,

$$\begin{aligned} \mathcal{I}(\lambda) &= \int_{\psi^{-1}\mathcal{N}} A(\psi(\boldsymbol{y}))e^{-\lambda S(\boldsymbol{y})}(\det d\psi(\boldsymbol{y}))\, d\boldsymbol{y} \\ &= \int_{\psi^{-1}\mathcal{N}} \tilde{A}(\boldsymbol{y})e^{-\lambda S(\boldsymbol{y})}\, d\boldsymbol{y}\,. \end{aligned}$$

We need to check that we can move the chain of integration $C = \psi^{-1}\mathcal{N}$ back to the real plane. If we can, then applying the expansion from Theorem 5.1 and noting that the terms with odd values of $|\boldsymbol{r}|$ all vanish yields the desired expansion in powers $\lambda^{-d/2-\ell}$.

Let $h(\boldsymbol{z}) = \mathrm{Re}\{S(\boldsymbol{z})\}$. Our assumption that the real part of ϕ is nonnegative on $\mathcal{N}$ and vanishes only at the origin implies that the chain C lies in the region

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$\{z \in \mathbb{C}^d : h(z) > 0\}$ except when $z = 0$, meaning there exists $\varepsilon > 0$ such that $h(z) \geq \varepsilon > 0$ for all $z \in \partial C$. Let

$$H(z, t) = \text{Re}\{z\} + (1 - t)\, i\ \text{Im}\{z\}$$

be a homotopy from the identity map to the projection map $\pi(z) = \text{Re}\{z\}$. For any chain σ, the homotopy H induces a chain homotopy $H(\sigma)$ satisfying

$$\partial H(\sigma) = \sigma - \pi(\sigma) + H(\partial\sigma)\,.$$

Taking $\sigma = C$, and using the fact that $h(H(z, t)) \geq h(z)$, we see there is a d-chain C' supported on $\{z \in \mathbb{C}^d : h(z) > \varepsilon\}$ and a $(d + 1)$-chain $\mathcal{D}$ such that

$$\partial\mathcal{D} = C - \pi(C) + C'.$$

Stokes's Theorem (Theorem A.24 in Appendix A) implies that

$$\int_{\partial\mathcal{D}} \omega = \int_{\mathcal{D}} d\omega = 0$$

for any holomorphic d-form ω, which means

$$\int_C \omega = \int_{\pi(C)} \omega - \int_{C'} \omega\,.$$

When $\omega = \tilde{A}(y)e^{-\lambda S(y)}\, dy$ the integral over C' is $O\left(e^{-\lambda\varepsilon}\right)$, giving

$$\mathcal{I}(\lambda) = \int_{\pi(C)} \tilde{A}(y)e^{-\lambda S(y)}\, dy + O\left(e^{-\varepsilon\lambda}\right).$$

The projection π maps any real d-manifold in $\mathbb{C}^d$ locally diffeomorphically into $\mathbb{R}^d$ wherever its tangent space is not parallel to the imaginary subspace of $\mathbb{C}^d$. Because $h(z) \geq 0$ on C, the tangent space to the support of C at the origin is not parallel to the imaginary subspace. The tangent space varies continuously, so in a neighborhood of the origin π is a diffeomorphism. In particular, the chain $\pi(C)$ is a disk Δ in $\mathbb{R}^d$ plus a collection of points whose image under h is bounded above zero (which will contribute an exponentially negligible term to dominant asymptotic behavior). Observing that

$$\tilde{A}(\mathbf{0}) = A(\mathbf{0})\det(d\psi(\mathbf{0})) = \frac{2^d A(\mathbf{0})}{\sqrt{\det \mathcal{H}}}$$

finishes the proof, up to the choice of sign of the square root corresponding to the orientation of Δ.

The stated sign choice of $\tilde{A}(\mathbf{0})$ in this theorem can be verified by proving that the linear map $d\pi \circ d\psi^{-1}$ at the origin sends the standard basis of $\mathbb{R}^d$ to another positively oriented basis if and only if $\det(d\psi(\mathbf{0}))$ is the product of the principal square roots of the eigenvalues of $\mathcal{H}$. We state and prove the

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necessary technical result in Lemma 5.10 below, which completes our proof of this theorem. □

Lemma 5.10. *Let* $W = \{z \in \mathbb{C}^d : \text{Re}\{S(z)\} > 0\}$ *and suppose that* $\alpha \in GL_d(\mathbb{C})$ *maps* $\mathbb{R}^d$ *into* $\overline{W}$. *If* $M = \alpha^T \alpha$ *is the matrix representing the quadratic form* $S \circ \alpha$ *and* π *is the projection map from* $\mathbb{C}^d$ *onto* $\mathbb{R}^d$ *then* $\pi \circ \alpha$ *is orientation preserving on* $\mathbb{R}^d$ *if and only if* $\det \alpha$ *is the product of the principal square roots of the eigenvalues of* M *(rather than the negative of this).*

Proof First suppose $\alpha \in GL_d(\mathbb{R})$. Then M has positive eigenvalues, and the product of their principal square roots is positive. The map π is the identity on $\mathbb{R}^d$, so the claimed statement boils down to saying that α preserves orientation if and only if it has positive determinant, which is true by definition. In the general case, let $\alpha_t = \pi_t \circ \alpha$, where $\pi_t(z) = \text{Re}\{z\} + (1 - t)\,\text{Im}\{z\}$. Since $\pi_t(\mathbb{R}^d) \subseteq \overline{W}$ for all $0 \le t \le 1$, the matrix $M_t = \alpha_t^T \alpha_t$ always has eigenvalues with nonnegative real parts. The product of the principal square roots of the eigenvalues is a continuous function on the set of nonsingular matrices with no negative real eigenvalues. The determinant of α_t is a continuous function of t, and we have seen it agrees with the product of principal square roots of eigenvalues of M_t when $t = 1$ (the real case), so by continuity this is the correct sign choice for all $0 \le t \le 1$. Taking $t = 0$ proves the lemma. □

Exercise 5.8. Suppose ϕ is the logarithm of an analytic function, defined only up to the addition of $(2\pi i)n$ for $n \in \mathbb{Z}$. How does this affect the conclusion of Theorem 5.2?

5.4 General nondegenerate phase with finite critical set

In this section we prove Theorem 5.3 by moving the chain of integration of $\mathcal{I}$ so that $\text{Re}\,\phi$ is minimized only at the finite set of critical points, then applying Theorem 5.2. We first remark on some differences from our previous arguments. While the chain of integration may be defined on a subspace $X \subseteq \mathbb{R}^d$, the form in the integrand will be extended to a neighborhood of X in $\mathbb{C}^d$ and the deformation in general will not be confined to X. To accomplish this, we define a complexification of X with analytic structure. Our deformation is defined by a smooth vector field $\boldsymbol{v}$ which, although not analytic, lies in the complex tangent bundle to the complexification of each stratum. The proof is broken into the following steps.

(1) Define the complexification $X \otimes \mathbb{C}$.
(2) Extend ϕ and η to a neighborhood U of X in $\mathbb{C}^d$.

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(3) Construct a vector flow $\boldsymbol{v}$ on U, tangent to the strata of $X\otimes\mathbb{C}$ and vanishing precisely on G, such that $\langle \mathrm{Re}\, d\phi, \boldsymbol{v}\rangle > 0$ on $U \setminus G$.
(4) Show that $\int_C e^{-\lambda\phi}\eta = \int_{C'} e^{-\lambda\phi}\eta$, where C' is obtained from C by flowing along $\boldsymbol{v}$ for a short time.
(5) Use Theorem 5.2 on the deformed chain C'.

Step (2) follows in a straightforward manner from Step (1). Step (3) is where the special assumptions on X are used. This allows us to follow the methodology of locally constant vector fields and partitions of unity in [ABG70], rather than the more difficult methodology of controlled vector fields in [Mat70]. Step (4) is an application of Stokes's Theorem together with the crucial observation that part of the boundary of the chain representing the deformation lies in a complex manifold of dimension less than d. Step (5), once we reach it, is immediate.

Step 1: Complexification

Lemma 5.11. *Under the hypotheses of Theorem 5.3, there is a complex stratified space $X \otimes \mathbb{C}$ with strata $\mathcal{S} \otimes \mathbb{C}$ as $\mathcal{S}$ ranges over the strata of X, such that $X\otimes\mathbb{C}$ is a neighborhood of X in $\mathbb{C}^d$ and such that the chart maps $\psi\otimes\mathbb{C} : \mathbb{C}^k \to \mathcal{S}\otimes\mathbb{C}$ restricted to $\mathbb{R}^k$ are chart maps for $\mathcal{S}$.*

Proof We complexify Δ^p in $\mathbb{C}^p$ and M^{d-k} in $\mathbb{C}^d$ separately and take the product. First, we note that the set Δ^p is defined by $p + 1$ linear inequalities. Relaxing these by $\varepsilon > 0$ produces a neighborhood $\mathcal{D}$ of Δ^p in $\mathbb{R}^d$. Taking the product with a sufficiently small imaginary interval $[-\varepsilon i, \varepsilon i]$ in each of the p coordinates produces a complex stratified space that is a neighborhood of Δ^p in $\mathbb{C}^d$. Because of our assumptions, we can complexify M^{d-k} to $\mathcal{V}$. Taking the product of $F \otimes \mathbb{C}$ and $M^{d-k} \otimes \mathbb{C}$ as stratified spaces, where F is a face of Δ^p, produces $(F \times M) \otimes \mathbb{C}$; these are the strata $\mathcal{S} \otimes \mathbb{C}$ and fit together to form the stratified space $X \otimes \mathbb{C}$ satisfying the conclusion of the theorem. □

Remark 5.12. In Chapter 9, we always have $k = 1$ and M^{d-k} is a smooth open $(d-1)$-patch in $\mathcal{V}_Q$. In Chapter 10, we have $k = 0$ and M^d is a patch of a middle-dimensional torus in $\mathbb{C}^d \setminus \mathcal{V}$. In general, under the condition that the real tangent space spans a complex space of dimension $d - k$, one may define $M^{d-k} \otimes \mathbb{C}$ to be the *intrinsic complexification* of M, namely the smallest complex manifold containing M. This is known to exist [BER99], following a construction of [BW59], and will be a complex $(d - k)$-manifold.

Exercise 5.9. Suppose X is the circle in $\mathbb{R}^3$ defined by the real solutions to $x^2 + y^2 - 1 = z = 0$. Find the complexification $X \otimes \mathbb{C}$.

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Step 2: Extending analytic maps

Proposition 5.13. *Let $\phi : X \to \mathbb{C}$ be an analytic map on an analytic stratified space X, and let ψ_1 and ψ_2 be chart maps whose ranges are overlapping domains $\mathcal{N}_1$ and $\mathcal{N}_2$ in X. As $\boldsymbol{y}$ takes values in the intersection of the ranges of $\psi_1 \otimes \mathbb{C}$ and $\psi_2 \otimes \mathbb{C}$, the function $\tilde{\phi} = \tilde{\phi}_j$ defined by*

$$\tilde{\phi}_j(\boldsymbol{y}) = \tau_j \circ (\psi_j \otimes \mathbb{C})^{-1}(\boldsymbol{y}) \tag{5.12}$$

is independent of j, where τ_j represents analytic continuation of the map $\phi \circ \psi_j$ from the real parameter space to a complex neighborhood of it. This common value defines an analytic extension of ϕ on a neighborhood of X in $\mathbb{C}^d$.

Proof The maps $\tilde{\phi}_1$ and $\tilde{\phi}_2$ agree when $\boldsymbol{y} \in \mathcal{N}_1 \cap \mathcal{N}_2 \cap X$. Being analytic, they must agree in a neighborhood of X in $X \otimes \mathbb{C}$. □

Step 3: Constructing the vector flow

Lemma 5.14. *Under the assumptions of Theorem 5.3, there is a vector field $\boldsymbol{v}$ on a neighborhood U of X in $\mathbb{C}^d$, tangent to each complexified stratum $\mathcal{S} \otimes \mathbb{C}$ of $X \otimes \mathbb{C}$ and vanishing only on G, with the property that $\langle \operatorname{Re} d\phi, \boldsymbol{v} \rangle > 0$ at every point of $U \setminus G$. For sufficiently small s, there is a well-defined differential flow $\Psi : [0, 1] \times |C| \to \mathbb{C}^d$ satisfying $(d/dt)\Psi(t, x) = s\,\boldsymbol{v}(\Psi(t, x))$, and the map $\boldsymbol{x} \mapsto \Psi(\varepsilon, \boldsymbol{x})$ is a local diffeomorphism for sufficiently small $\varepsilon > 0$.*

Proof As described in Proposition D.14 of Appendix D, there is a diffeomorphic local product structure under the assumptions of Theorem 5.3. The argument discussed in Step 2 of the proof of Proposition D.13 in Appendix D implies that there is a Lipschitz vector flow $\boldsymbol{v}$ on a neighborhood U of X in $\mathbb{C}^d$ defined by (D.2.1). This vector field $\boldsymbol{v}$ is tangent to each complexified stratum, vanishes precisely on G, and satisfies $\langle \operatorname{Re} d\phi, \boldsymbol{v} \rangle > 0$ on $U \setminus G$.

The map $\boldsymbol{x} \mapsto \Psi(\varepsilon, \boldsymbol{x})$ is a local diffeomorphism for sufficiently small $\varepsilon > 0$ because $\boldsymbol{v}$ is smooth and bounded (see, for example, [Lee03, Proposition 9.12]). □

Step 4: Deforming the contour

Composing the flow from Lemma 5.14 with the simplex σ_j gives a homotopy from each σ_j to a new analytic simplex σ'_j, and summing the σ'_j defines a new chain C'. Because $\boldsymbol{v}$ is tangent to $\mathcal{S} \otimes \mathbb{C}$ for each stratum $\mathcal{S}$, the flow preserves each complexified stratum $\mathcal{S} \otimes \mathbb{C}$, and because ∂C is contained in the union of strata of dimensions at most $p + d - k - 1$, it follows that $\Psi(\partial C)$ is contained in the union of complexified strata of dimensions at most $p + d - k - 1$.

Lemma 5.15. *Using the above notation,*

$$\int_C e^{-\lambda\phi}\eta = \int_{C'} e^{-\lambda\phi}\eta\,. \tag{5.13}$$

Proof The homotopy Ψ defined by Lemma 5.14 is a chain with boundary $C - C' - [0,1]\times\partial C$. Since $e^{-\lambda\phi}\eta$ is a holomorphic $p+d-k$-form, it is annihilated by the differential, and Stokes's Theorem implies

$$\begin{aligned} 0 &= \int_\Psi d(e^{-\lambda\phi}\eta) \\ &= \int_{\partial\Psi} e^{-\lambda\phi}\eta \\ &= \int_{C'} e^{-\lambda\phi}\eta - \int_C e^{-\lambda\phi}\eta - \int_{\mathcal{D}} e^{-\lambda\phi}\eta\,, \end{aligned}$$

where $\mathcal{D}$ is a chain representing $[0,1]\times\partial C$. Because $e^{-\lambda\phi}\eta$ is a holomorphic $(p+d-k)$-form, its integral vanishes over any $p+d-k$-chain supported on a $(p+d-k-1)$-dimensional complex manifold (see Exercise A.15 of Appendix A), finishing the proof. □

Exercise 5.10. A simpler version of the deformation argument can be illustrated in two real variables. Suppose that $V(x,y)$ is a smooth function on $\mathbb{R}^2$ and $\eta = V_x(x,y)dx + V_y(x,y)dy$. Assume that a homotopy $H : [0,T]\times[0,1]$ carries the path $\alpha : [0,T] \to \mathbb{R}^2$ to the path $\beta : [0,T] \to \mathbb{R}^2$, meaning $H(t,0) = \alpha(t)$ and $H(t,1) = \beta(t)$ for all $t \in [0,T]$. If $f(u) = H(0,u)$ and $g(u) = H(T,u)$ then what conditions on df and dg guarantee that $\int_\alpha \eta = \int_\beta \eta$ by implying that the integrals over the paths traced out by the endpoints of the path as it moves from α to β are everywhere zero?

Step 5: Evaluating the integral on C'

From the construction of $\boldsymbol{v}$ we see that $C' = \sum_{j=1}^m \sigma'_j$, where each simplex σ'_j contains the same critical point $\boldsymbol{q}$ in its interior as σ_j. By Lemma 5.14 we know that $\operatorname{Re}\phi(\sigma'_j(\boldsymbol{x})) \geq \operatorname{Re}\phi(\sigma_j(\boldsymbol{x})) \geq 0$ for all $j \leq m$ and $\boldsymbol{x} \in \Delta^p$, where the first inequality is strict unless $\sigma_j(\boldsymbol{x})$ is a critical point. Thus, the image $|C'|$ is a stratified space represented analytically by $(p+d-k)$-simplices σ'_j for $1 \leq j \leq m$ and the function $\operatorname{Re}\phi$ is nonnegative and vanishes precisely on G'.

If σ'_j contains no point of G' then the modulus of $\int_{\sigma'_j} e^{-\lambda\phi(\boldsymbol{z})}\eta$ is bounded above by $Me^{-\lambda K}$, where $M = \max\{|A(\boldsymbol{x})| : \boldsymbol{x} \in \Delta^d\}$ and $K = \min\{\operatorname{Re}\phi(\boldsymbol{x}) : \boldsymbol{x} \in \Delta^d\}$. If σ'_j contains $\boldsymbol{q} \in G'$ then translating the preimage of $\boldsymbol{q}$ to the origin turns Δ^{p+d-k} into a neighborhood $\mathcal{N}$ of the origin on which A and $\tilde{\phi} = \phi \circ \sigma'_j$ are analytic and $\operatorname{Re}\tilde{\phi}(\boldsymbol{z}) \geq 0$, with equality only at the origin. Theorem 5.2

then gives an asymptotic expansion for the integral over each simplex. Invariance of $\det J(\Upsilon)/\sqrt{\mathcal{H}}$ under coordinate transformations equates the summand in (5.7) with the result (5.4) in Theorem 5.2, provided the correct sign in (5.7) is chosen to match (5.4). Summing these expansions then finishes the proof of Theorem 5.3. □

Exercise 5.11. The simplest case of Theorem 5.3 occurs when $p = 0, d = 1$, and $k = 0$. Take $\phi(z) = -z^2$, so $e^{-\lambda\phi(z)} = e^{\lambda z^2}$, and let M be the 1-chain consisting of the imaginary axis. Which sign in the equality

$$\int_M e^{\lambda z^2} dz = \pm i\sqrt{\frac{2\pi}{\lambda}}$$

is correct when parametrizing M by $z = it$ for $t \in \mathbb{R}$, and which sign is correct when parametrizing M by $z = -it$?

5.5 Higher order terms in the expansions

All of our explicit asymptotic computations ultimately reduce to computing terms in the asymptotic expansions of Fourier–Laplace integrals. It is therefore useful to have a closed formula for the higher-order terms that can appear. The following result is derived in [Hör83, Thm. 7.7.5] using smooth methods, and we refer the reader to that source for a proof.

Lemma 5.16. *Let $X \subseteq \mathbb{R}^d$ be an open neighborhood of the origin and let ϕ and A be smooth functions on X such that* $\mathrm{Re}\{\phi\} \geq 0$ *on X. Further suppose that ϕ has a unique critical point on the support of A at the origin, that $\phi(\mathbf{0}) = 0$, and that the Hessian $\mathcal{H}$ of ϕ at $\mathbf{0}$ is nonsingular. Then for any positive integer M and $\lambda > 0$ there exist constants $L_k(A, \phi)$ such that*

$$\left|\int_X A(\boldsymbol{x}) e^{-\lambda\phi(\boldsymbol{x})} d\boldsymbol{x} - \lambda^{-d/2}\frac{(2\pi)^{d/2}}{\sqrt{\det\mathcal{H}}}\sum_{0\leq k<M}\lambda^{-k}L_k(A,\phi)\right|$$
$$\leq C(\phi)\lambda^{-d/2-M}\sum_{|\beta|\leq 2M}\sup|\mathcal{D}^\beta A|,$$

where the constant $C(\phi)$ has a uniform bound when ϕ stays in a bounded set of $(3N+1)$-differentiable functions on X for which $\|\boldsymbol{x}\|/\|\nabla\phi(\boldsymbol{x})\|$ has a uniform bound. Setting

$$\underline{\phi}(\boldsymbol{x}) = \phi(\boldsymbol{x}) - (1/2)\boldsymbol{x}\cdot\mathcal{H}\cdot\boldsymbol{x}^T,$$

which vanishes to order three at $\mathbf{0}$*, we have*

$$L_k(A,\phi) = (-1)^k \sum_{0\le\ell\le 2k} \frac{\mathcal{D}^{\ell+k}\left(A(\boldsymbol{x})\cdot \underline{\phi}(\boldsymbol{x})^{\ell}\right)}{2^{\ell+k}\ell!(\ell+k)!}\Bigg|_{\boldsymbol{x}=\mathbf{0}}, \tag{5.14}$$

where $\mathcal{D}$ *is the differential operator*

$$\mathcal{D} = -\sum_{1\le i,j\le d} \left(\mathcal{H}^{-1}\right)_{ij} \partial_i \partial_j.$$

The total number of derivatives of A *in the term* $L_k(A,\phi)$ *is at most* $2k$ *and the total number of derivatives of* ϕ *is at most* $2k+2$. □

Interpreting this in our context, we obtain the following.

Corollary 5.17 (full expansion of Fourier–Laplace integral)**.** *Assume the hypotheses of Theorem 5.2 and suppose further that* ϕ *has a single critical point on* $\mathcal{N}$ *which lies at the origin. Then the constants in* (5.3) *satisfy*

$$c_k = \frac{(2\pi)^{d/2}}{\sqrt{\det \mathcal{H}}} L_k(A,\phi)$$

for each $k \ge 0$*, where* L_k *is as defined by* (5.14).

The fact that Lemma 5.16 requires only smoothness also makes it easy to localize.

Lemma 5.18. *If* ϕ *has a finite number of critical points on* $\mathcal{N}$ *where the hypotheses of Corollary 5.17 hold, then an asymptotic expansion for* $\mathcal{I}(\lambda)$ *is obtained by summing the expansions corresponding to each of these critical points.*

Proof The proof of Theorem 5.2 shows that the contribution from the boundary of the domain of integration may be ignored – we may localize to a neighborhood $\mathcal{N}'$ of the critical point that is diffeomorphic to an open ball in $\mathbb{R}^d$. Replacing A by the product $A\alpha$ of A with a compactly supported smooth function α that is equal to one on $\mathcal{N}'$, the result follows from Lemma 5.16. □

Exercise 5.12. Verify that when $A(\mathbf{0}) \neq 0$ the leading constant of (5.3) matches the expression in Corollary 5.17.

Notes

A number of the results in this section originally appeared in [PW10]. In the case of purely real or imaginary phase, the results of this chapter are fairly

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standard; see [BH86; Won01] for real phase or [Ste93] for imaginary phase. We have not seen the complex phase result Theorem 5.2 stated before, though such a result was certainly understood to be true. The remaining statements and proofs via complex deformation methods are new, though not surprising.

Theorem 5.4.8 from the first edition has been replaced by Theorem 5.3. The proof has been split into two parts. The first part, involving stratified Morse deformations, is a standard stratified construction and is summarized in Appendix D. The second part, deforming a chain by complexifying and manipulating the chain through complex space, is new as far as we know and spelled out in Section 5.4.

According to B. Lamel *(personal communication)*, intrinsic complexifications of strata of any stratified space (see Remark 5.12) should fit together to form a complex stratified space, provided the (real) tangent spaces E at every point satisfy $E \cap iE = \{0\}$. This would imply that Lemma 5.11 and hence Theorem 5.3 holds for any stratified space of dimension m analytically embedded in $\mathbb{C}^m$. However, this appears to require some condition about extending the (real) chart maps of each stratum beyond its boundary, which can be done with Δ^p and M^{d-1} but not for arbitrary stratified spaces (where the boundary might be a singularity of the stratum). We could not find such an argument in the existing literature, which is why Theorem 5.3 is restricted to products of stratified spaces for which we have explicit complexifications.

The general approach, namely to stay in the analytic category and use deformations suggested by stratified Morse theory for complex spaces, is an extension of our treatment of univariate Fourier–Laplace integrals in Chapter 4. The main motivation for doing things the way we have is that the analysis of multiple points in Chapter 10 requires us to integrate over the product of a chain in $\mathcal{V}$ with an abstract simplex; when integrating terms with imaginary phase over manifolds with boundary, one needs a way to eliminate boundary terms. A result similar to Theorem 5.3 was proved in [PV19, Theorem 4.2], via an approach which avoids Morse theoretic contour deformation arguments, replacing these by iterated integrals and single parameter steepest descent curves.

The first edition of the book assumed a strong torality hypothesis when deriving asymptotics (see, e.g., Corollaries 9.2.4 and 9.2.9 there), which allowed use of known techniques in the case where the phase is purely imaginary and the contour of integration has no boundary. In the present edition this overly strong hypothesis has been replaced by a weaker notion for which Theorem 5.3 has been specifically designed (see Theorem 9.12).

Additional exercises

Exercise 5.13. Consider an integrand of the form $A(x,y,z)e^{-\lambda S(x,y,z)}\,d\boldsymbol{z}$, where $A(x,y,z) = (x^2+y^2+z^2)^\alpha$ is not smooth at the origin when $\alpha \notin \mathbb{Z}_{>0}$. What kind of asymptotic estimate or expansion can be obtained in this case?

Exercise 5.14. Prove that for critical points on the boundary of the chain of integration, where the chain is locally diffeomorphic to a halfspace and ϕ has vanishing one-sided normal derivative, the conclusion of Theorem 5.2 holds with the leading coefficient multiplied by $1/2$. (See Example 10.66 in Chapter 10 for an application of this result.)

Exercise 5.15. (non-isolated critical points) Consider the integral

$$\int_{-\varepsilon}^{\varepsilon}\int_0^1 e^{-\lambda\phi(\theta,t)}dtd\theta,$$

where $\phi(\theta,t) = (1-t)g_1(\theta)+tg_1(\theta)$ and each g_i is analytic and vanishes to order 2 at $\theta = 0$, with positive second derivative. Calculate the first-order asymptotic as $\lambda \to \infty$ in terms of derivatives of g_1 and g_2 at 0. (This foreshadows the computations in Section 10.5, in particular Proposition 10.62.)

Exercise 5.16. Use the vanishing to order 3 of $\underline{\phi}$ at the origin to write a simpler expression for $L_1(A,\phi)$. Further simplify it in the cases where A vanishes to orders one, two, and three, respectively, at the origin.

Exercise 5.17. (alternative method to compute higher order terms) This exercise outlines an alternative way of computing higher order terms in the asymptotic expansion of a Fourier–Laplace integral. Assume for simplicity that the phase ϕ has a single critical point, occurring at $\mathbf{0}$, and that the amplitude A vanishes outside the closure of a neighborhood of $\mathbf{0}$. Let S be the standard quadratic $S(\boldsymbol{z}) = \sum_{i=1}^d z_i^2$.

(i) Prove that when $\phi = S$, the differential operator

$$\sum_{|\boldsymbol{r}|=k} \frac{\partial_1^{2r_1}\cdots\partial_d^{2r_d}}{4^k r_1!\cdots r_d!},$$

when applied to A and evaluated at $\mathbf{0}$, gives the coefficient c_k from Theorem 5.3.

(ii) The Complex Morse Lemma gives an analytic change of variables $\boldsymbol{z} = \psi(\boldsymbol{y})$ such that $S(\boldsymbol{y}) = (\phi\circ\psi)(\boldsymbol{y})$. Apply the result of (i) and solve a triangular system to compute the derivatives of ψ at $\mathbf{0}$. Note that this changes the amplitude function A to $(A\circ\psi)\det\psi'$.

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(iii) Use Corollary 5.7 and steps (i) and (ii) to derive an explicit formula for asymptotics in the case $d = k = 1$. Check your result against the formula given in Corollary 5.17.

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