BULL. AUSTRAL. MATH. SOC. 34A60, 65L05
VOL. 42 (1990) [169–170]

High Accuracy Numerical Methods for Ordinary Differential Equations with Discontinuous Right-hand Side

DAVID E. STEWART

Ordinary Differential Equations (ODEs) with discontinuous right-hand side, where the discontinuities occur in the *state* variables arise in a number of applications.

The appropriate formulation of such a problem (Filippov [4]) is to replace the ODE "$x' = f(x)$" by a *differential inclusion* $x' \in F(x)$ where F is a set-valued function with nonempty compact convex values and is upper semi-continuous. Solutions exist, although they are not necessarily unique.

Numerical methods to compute solutions of differential inclusions have already been developed and convergence results proven by Taubert [**6**, **7**], Elliott [**3**] and Niepage and Wendt [**5**]. However, these methods are no better than 1st order accurate, and 1st order accuracy holds only with additional assumptions. Further, unless solutions are unique, these methods only guarantee that a *sub-sequence* of the numerical solutions converge.

It is assumed that the structure of the discontinuous ODE is as follows: there is a collection of disjoint open sets R_i with dense union in $\mathbf{R}^n$, such that $x \in R_i$ implies that $f(x) = f_i(x)$ with f_i smooth, and the R_i can be described by functions: $R_i = \{x \mid h_i(x) < h_j(x), j \neq i\}$. The *active set* at x is $I(x) = \{i \mid x \in \overline{R_i}\}$. The Filippov formulation becomes $x' \in \operatorname{co}\{f_i(x) \mid i \in I(x)\}$. We suppose that $I(x(t))$ is a piecewise constant function. Where $I(x(t))$ is constant, we can select elements of the above differential inclusion that keep the active set constant. At a *switching point* the set of possible new values for $I(x(t))$ are determined by means of a Linear Complementarity Problem. Given a new active set from this set, we can continue by solving a new smooth ODE to maintain the new active set, until another switching point occurs.

A complete algorithm using this approach has been developed and convergence results proven. In particular, if a particular solution has only finitely many switching points and some non-degeneracy conditions are satisfied, then by suitable choices of new active sets, a sequence of numerical solutions can be generated that converge to the given solution with the same order of accuracy the smooth ODE solver used.

Received 6th April, 1990. Thesis submitted to the University of Queensland, July 1989. Degree approved March 1990. Supervisor Professor L. Bass.

---

https://doi.org/10.1017/S000497270002829X Published online by Cambridge University Press

This algorithm has been implemented and numerical results obtained.

A "decomposition" extension to the main algorithm is described and convergence results are given for this extension. This extension has been implemented and numerical results obtained for a friction problem with multiple friction surfaces.

REFERENCES

[1] Frank H. Clarke, 'Optimal control and the true Hamiltonian', *SIAM Rev.* **21** (1979), 157–166.

[2] Frank H. Clarke, *Optimization and Nonsmooth Analysis: Canadian Mat. Soc. Ser. #1* (Wiley–Interscience, New York, 1983).

[3] Charles M. Elliott, 'On the convergence of a one–step method for the numerical solution of ordinary differential inclusions', *IMA J. Numer. Anal.* **5** (1985), 3–21.

[4] A.F. Filippov, 'Differential equations with discontinuous right–hand side', *Amer. Math. Soc. Transl. (Orig. in Russian in Math. Sbornik* 5 *pp.99–127 (1960)* **42** (1964), 199–231.

[5] H-D. Niepage and W. Wendt, 'On the discrete convergence of multistep methods for differential inclusions', *Numer. Funct. Anal. Optim.* **9** (1987), 591–617.

[6] Klaus Taubert, 'Differenzverfahren für Schwingungen mit trockener und zäher Reibung und für Regelungssysteme', *Numer. Math.* **26** (1976), 379–395.

[7] Klaus Taubert, 'Converging multistep methods for initial value problems involving multivalued maps', *Computing* **27** (1981), 123–136.

Department of Mathematics
The University of Queensland
Queensland 4072 Australia

https://doi.org/10.1017/S000497270002829X Published online by Cambridge University Press