ON THE SYMMETRIES OF SPHERICAL HARMONICS

BURNETT MEYER

Introduction

Let $\mathfrak{G}$ be a finite group of transformations of three-dimensional Euclidean space, such that the distance between any two points is preserved by all transformations of the group. Such a group is a group of orthogonal linear transformations of three variables, or, geometrically speaking, a group of rotations and rotatory inversions. Thirty-two groups of this type are important in crystallography and are known as the crystallographic classes.

A function is said to have the symmetry of a given group if it remains invariant under all transformations of the group. Our problem is to determine all spherical harmonics of a given degree m and a given symmetry. It is sufficient to find a basis of these harmonics for all m and for all groups $\mathfrak{G}$.

Section I of this paper enumerates and classifies all groups of the desired type. In §II we find the number of elements in a basis of all homogeneous polynomials of a given degree which have a given symmetry, applying a theorem of Molien.

In §III we find the number of elements in a basis of all spherical harmonics of a given degree which have a given symmetry. This is accomplished by associating with each group a generating function.

In §IV we solve the problem proposed, using the results of §III. The required basis is found in terms of partial derivatives of $1/r$, r denoting the distance from the origin. For certain simpler symmetries the basis is also expressed in terms of the associated Legendre functions.

A particular case of this problem arose and was solved in another research problem, the aim of which was to compute approximately the electrostatic capacity of the cube (**12**, pp. 76–78). In generalizing this particular case, we were led to our results which were announced, without proof, in two notes (**10**; **11**).

Work in this problem has been done previously by Poole (**13**), Laporte (**7**), Bethe (**1**), Ehlert (**4**), and Hodgkinson (**6**), and recently by Stiefel (**15**). For the geometrical and algebraic background see Molien (**9**) and the bibliographies in Coxeter (**3a**) and Speiser (**14**).

The present paper differs in two respects from preceding work on the subject. First, all groups are treated in a uniform manner, whereas previous papers are restricted to certain groups. Second, the generating function of §III enables

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us to discuss fully the question of linear independence and to link the subject to general theorems of group theory and analysis.

The major portion of this paper consists of material contained in the doctoral dissertation of the author. He wishes to thank Professor George Pólya for suggesting the problem and for his helpful guidance. In addition, some results are presented which were obtained recently by Professor Pólya and by the author.

I. Groups

1. Rotations and rotatory inversions. The purpose of this section is to enumerate and classify all finite groups of distance-preserving transformations of three-dimensional Euclidean space into itself which leave one point fixed. The elements of such groups are either *rotations* or *rotatory inversions*. A rotatory inversion is a rotation followed by central symmetry with respect to a point on the axis of rotation.

Without loss of generality the origin may be chosen as the fixed point. Our problem may then be restated in algebraic form: We seek all finite groups of orthogonal linear transformations in x, y, and z.

The matrix of a rotatory inversion may be written JR, in which

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and R is a rotation. We sometimes use the term "rotatory inversion with angle θ." This means a rotation through angle θ, followed by central symmetry with respect to the origin.

2. Three types of finite groups. For proofs of statements in this section, see Weyl (**17**, pp. 77–80, 149–156).

(a). Type 1: Groups consisting of rotations only. We first consider groups, all the elements of which are rotations. It has been proved by Felix Klein that there are only five classes of groups of this type. They are $\mathfrak{C}_n$, $\mathfrak{D}_n$, $\mathfrak{T}$, $\mathfrak{O}$, and $\mathfrak{J}$, the cyclic, dihedral, tetrahedral, octahedral, and icosahedral groups, respectively. In the following paragraphs we will indicate how the rotational axes of each group are to be placed with respect to the x, y, and z axes.

For the group $\mathfrak{C}_n$, the n-fold axis will be taken as the z-axis.

For $\mathfrak{D}_n$, the n-fold axis will also be the z-axis and one of the 2-fold axes will be the x-axis.

All rotations of the group $\mathfrak{O}$ transform a cube (or a regular octahedron) into itself. In this paper the cube will be placed with its centre at the origin and with its faces parallel to the coordinate planes.

The tetrahedral group, $\mathfrak{T}$, consists of the rotations which transform a regular tetrahedron into itself. The tetrahedron will be placed so that its 3-fold axes coincide with the 3-fold axes of $\mathfrak{O}$, and the 2-fold axes will be taken as the coordinate axes.

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The icosahedral group, $\mathfrak{J}$, consists of the rotations which transform a regular icosahedron (or a regular dodecahedron) into itself. We shall place the icosahedron, as explained in Coxeter **(3**, pp. 52–53**)**, with its centre at the origin, and the coordinate axes passing through the midpoints of opposite edges in such a way that the edges through which the x-axis passes are parallel to the y-axis.

(b). Groups containing rotatory inversions. There are two types of groups containing rotatory inversions.

The groups of Type 2 are those with a centre of symmetry and are obtained by adjoining J to a group of Type 1. If g is the order of a rotational group $\mathfrak{G}$, then $2g$ is the order of $\mathfrak{G}_i$, the group of Type 2 derived from $\mathfrak{G}$. The groups of Type 2 are $\mathfrak{C}_{ni}$, $\mathfrak{D}_{ni}$, $\mathfrak{T}_i$, $\mathfrak{O}_i$, and $\mathfrak{J}_i$.

The groups of Type 3 are derived from a rotational group $\mathfrak{G}_2$, which has a subgroup, $\mathfrak{G}_1$, of index 2. We denote by $\mathfrak{G}_1[\mathfrak{G}_2$ the group consisting of the rotations of $\mathfrak{G}_1$ and all elements of the form JR, R being a rotation in $\mathfrak{G}_2$ but not in $\mathfrak{G}_1$.

3. Crystallographic classes. All finite groups of distance-preserving transformations have been enumerated in the preceding section. Certain of these groups are important in crystallography and are known as the crystallographic classes. The transformations in such groups must not only be distance-preserving but they must also transform a point lattice into itself. It can be shown **(17**, pp. 98–104**)** that of the groups previously discussed only those having all their axes of rotation or rotatory inversion of orders 2, 3, 4, and 6 are crystallographic classes. There are thirty-two such groups.

II. Invariant Polynomials

1. The generating function of Molien. Let $\mathfrak{G}$ be one of the finite groups of orthogonal linear transformations in x, y, and z discussed in §I. Given a non-negative integer m, we consider those homogeneous polynomials of degree m which have the symmetry of $\mathfrak{G}$; that is, they are invariant with respect to all transformations of $\mathfrak{G}$. We define an *invariant basis* of degree m for $\mathfrak{G}$ as a finite subset of these polynomials, the elements of which are linearly independent but on which all other invariant polynomials of degree m are linearly dependent.

The number of polynomials constituting such a basis depends on m and $\mathfrak{G}$, but not on the particular subset chosen; we call this number g_m. The purpose of this section is to determine g_m for arbitrary m and for all the groups discussed in §I.

Molien **(9)** solved this problem by finding a generating function

$$g(t) = \sum_{m=0}^{\infty} g_m t^m.$$

Since his derivation can be simplified somewhat, a proof will be given in the following section.

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2. Molien's theorem. In the following proof due to Burnside (**2**, p. 300) we use the terminology and notation of Macduffee (**8**, pp. 17–19).

Let $\mathfrak{G}$ be an abstract group of order n, and let Γ be a representation of it by linear transformations of k variables; such a representation is said to be of degree k. Let the matrices of Γ be $A_1, A_2, \ldots, A_n$.

We denote by $f_i(t)$ the polynomial det $(E - tA_i)$, which we shall call the *characteristic polynomial* of A_i. Then

$$f_i(t) = \det (E - tA_i) = \prod_{j=1}^{k} (1 - \lambda_j t) = 1 - \chi(A_i)t + \ldots,$$

in which $\chi(A_i)$ is the trace and the λ_j are the characteristic roots of A_i.

Now consider $P_m(A_i)$, the mth power-matrix (**8**, pp. 84–87) of A_i. The characteristic roots of $P_m(A_i)$ are all the possible products of the mth degree of powers of $\lambda_1, \lambda_2, \ldots, \lambda_k$. The trace of $P_m(A_i)$ is, then, the sum of all possible products of the mth degree of powers of the λ's; that is, it is the coefficient of t^m in the expansion of

$$\frac{1}{(1 - \lambda_1 t)(1 - \lambda_2 t) \ldots (1 - \lambda_k t)} = \frac{1}{f_i(t)}$$

in powers of t.

The matrices $P_m(A_1), P_m(A_2), \ldots, P_m(A_n)$ form a group Γ^m, which is isomorphic with Γ; that is, it also represents $\mathfrak{G}$. Our problem is now to find the number of independent invariant linear forms of Γ^m, since these are an invariant basis of Γ of degree m. But the number of independent invariant linear forms of a group of matrices is equal to the sum of the traces of the matrices, divided by the order of the group (**14**, pp. 158–161). Therefore, g_m is the coefficient of t^m in the expansion of

$$\frac{1}{n}\sum_{i=1}^{n} \frac{1}{f_i(t)}$$

in powers of t. But since this proof is valid for all m we have proved

MOLIEN'S THEOREM: *Let g_m be the number of elements of an invariant basis of degree m for a group $\mathfrak{G}$ of order n of orthogonal matrices. Let $f_i(t)$ $(i = 1, 2, \ldots, n)$ be the characteristic polynomials of the matrices of $\mathfrak{G}$. Then*

$$g(t) = \sum_{m=0}^{\infty} g_m t^m = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{f_i(t)}.$$

3. Characteristic polynomials. We have seen that the generating function $g(t)$ corresponding to the group $\mathfrak{G}$ is the arithmetic mean of reciprocals of the characteristic polynomials of the matrices which are the elements of $\mathfrak{G}$.

Let A be the matrix of a rotation or a rotatory inversion of angle θ of $\mathfrak{G}$ with a given axis. We may perform the transformation A in the following manner: First, a rotation S may be performed that brings the axis of rotation or of rotatory inversion of A to coincidence with the z-axis; second, a transformation A' may be performed, A' being a rotation or rotatory inversion with

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angle θ about the z-axis; then the rotation S^{-1} is performed. Hence, $A = S^{-1}A'S$. Thus, A and A' are equivalent matrices and have the same characteristic polynomial (**14**, pp. 147–148). We may, then, in computing the contribution of any orthogonal matrix to the generating function, always take the axis of rotation or rotatory inversion as the z-axis.

The characteristic polynomial for rotation through an angle θ is

$$\det\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-t\begin{pmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{pmatrix}\right]=(1-t)(1-2t\cos\theta+t^2).$$

For rotatory inversion with angle θ, the characteristic polynomial is

$$\det\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-t\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{pmatrix}\right]=(1+t)(1+2t\cos\theta+t^2).$$

If it is necessary to distinguish between the generating functions of several groups $\mathfrak{G}_1, \mathfrak{G}_2, \ldots$, the notations $g(t; \mathfrak{G}_1), g(t; \mathfrak{G}_2), \ldots$ will be used.

4. Groups of Type 1. By computing the generating functions for a few small values of n, it is conjectured that the generating function for $\mathfrak{C}_n$ is

$$g(t;\ \mathfrak{C}_n) = \frac{1+t^n}{(1-t)(1-t^2)(1-t^n)}.$$

This may be verified by factoring the denominator of the above fraction into linear factors involving the nth roots of unity, expanding the fraction into a sum of partial fractions, and recombining these in pairs to obtain

$$\frac{1}{n}\sum_{\nu=0}^{n-1}\frac{1}{(1-t)\{1-2t\cos(2\pi\nu/n)+t^2\}}.$$

The above expansion is clearly $g(t;\ \mathfrak{C}_n)$. The algebraic calculations in the foregoing partial fraction expansion are elementary but tedious, as the cases of n even and n odd must be considered separately in the intermediate stages.

The generating functions of all groups of Type 1 can be expressed as sums and differences of those of $\mathfrak{C}_n$ for various values of n. Suppose that $\mathfrak{G}$ is of order n and has p_1 different q_1-fold axes, p_2 different q_2-fold axes, etc. Then,

$$g(t;\ \mathfrak{G}) = (1/n)[p_1q_1g(t;\ \mathfrak{C}_{q_1}) + p_2q_2g(t;\ \mathfrak{C}_{q_2}) + \ldots - (p_1+p_2+\ldots-1)\,g(t;\ \mathfrak{C}_1)];$$

observe that the group $\mathfrak{C}_1$ contains only the identity element.

5. Groups of Type 2. It will be recalled that the group $\mathfrak{G}_i$ has a subgroup $\mathfrak{G}$, which consists entirely of rotations. For every rotation of angle θ of $\mathfrak{G}$, the group $\mathfrak{G}_i$ has a rotatory inversion of angle θ. But the contribution of a rotatory inversion of angle θ to the generating function is obtained by changing

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t to $-t$ in the function corresponding to a rotation of angle θ. Hence, if n is the order of the group $\mathfrak{G}$,

$$g(t;\ \mathfrak{G}_i) = (1/2n)[ng(t;\ \mathfrak{G}) + ng(-t;\ \mathfrak{G})] = \tfrac{1}{2}[g(t;\ \mathfrak{G}) + g(-t;\ \mathfrak{G})],$$

that is, the even part of $g(t; \mathfrak{G})$.

6. Groups of Type 3. It will be recalled that the group $\mathfrak{G}_1[\mathfrak{G}_2$ has the following structure: $\mathfrak{G}_2$ is a rotational group and $\mathfrak{G}_1$ a subgroup of index 2. $\mathfrak{G}_1[\mathfrak{G}_2$ consists of the rotations of $\mathfrak{G}_1$ plus a rotatory inversion with angle θ corresponding to each rotation of angle θ belonging to $\mathfrak{G}_2$ but not to $\mathfrak{G}_1$. Therefore, if $\mathfrak{G}_1$ is of order n,

$$\begin{aligned} g(t;\ \mathfrak{G}_1[\mathfrak{G}_2) &= (1/2n)[ng(t;\ \mathfrak{G}_1) + 2ng(-t;\ \mathfrak{G}_2) - ng(-t;\ \mathfrak{G}_1)] \\ &= \tfrac{1}{2}[g(t;\ \mathfrak{G}_1) - g(-t;\ \mathfrak{G}_1)] + g(-t;\ \mathfrak{G}_2). \end{aligned}$$

The first term is the odd part of $g(t; \mathfrak{G}_1)$.

III. The Generating Function

1. Invariant harmonic basis. Let $\mathfrak{G}$ be one of the finite groups discussed in §I. We define an *invariant harmonic basis* of degree m for $\mathfrak{G}$ as a set of linearly independent spherical harmonics of degree m which are invariants of $\mathfrak{G}$ and on which all other invariant spherical harmonics of degree m are linearly dependent. Let h_m be the number of elements in an invariant harmonic basis of degree m for $\mathfrak{G}$. We wish to determine h_m for arbitrary m.

2. The main theorem. The main theorem of this section obtains a generating function for the h_m; this generating function is surprisingly similar in form to the generating function of Molien, as is seen in the

Theorem. *Let $\mathfrak{G}$ be a finite group of orthogonal linear transformations in x, y, and z. Let*

$$h(t) = \sum_{m=0}^{\infty} h_m t^m,$$

in which h_m is the number of elements in an invariant harmonic basis for $\mathfrak{G}$ of degree m. Then

$$h(t) = (1 - t^2)g(t),$$

in which $g(t)$ is the generating function of Molien.

3. Operations preserving invariance. Before proceeding to the proof of the main theorem, we shall need some lemmas.

Lemma 1. *Let $P(x, y, z)$ and $Q(x, y, z)$ be homogeneous polynomials which are invariants of $\mathfrak{G}$. Then $P + Q$ and PQ are also homogeneous invariant polynomials.*

Lemma 2. *Let $P(x, y, z)$ and $Q(x, y, z)$ be homogeneous polynomial invariants of a group $\mathfrak{G}$ of orthogonal linear transformations. Then the polynomial*

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$$R(x, y, z) = P\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) Q(x, y, z) \tag{1}$$

is also a homogeneous polynomial invariant of $\mathfrak{G}$.

Proof. Let A represent the matrix of any orthogonal transformation of $\mathfrak{G}$. By the chain rule of differentiation, it is easily seen that the operators

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z},$$

are contragredient with x, y, z (**16**, pp. 149–154). But since A is an orthogonal matrix, the operators are cogredient with the variables. Hence, the right member of (1) is an invariant of A, and, therefore, of every transformation in $\mathfrak{G}$.

Of course, the polynomial $R(x, y, z)$ in the above lemma may turn out to be identically zero.

Since $x^2 + y^2 + z^2$ is an invariant of all orthogonal matrices, the above lemmas have the following special cases, which will be of importance in the following sections:

LEMMA 3. *If $P(x, y, z)$ is a homogeneous polynomial invariant of $\mathfrak{G}$, of degree m, then $(x^2 + y^2 + z^2)^j P(x, y, z)$ is also a homogeneous polynomial invariant of $\mathfrak{G}$ of degree $m + 2j$; we let j denote a non-negative integer.*

LEMMA 4. *If $P(x, y, z)$ is a homogeneous polynomial invariant of $\mathfrak{G}$ of degree m, then $\Delta P(x, y, z)$ is also a homogeneous polynomial invariant of $\mathfrak{G}$ of degree $m - 2$; we let Δ denote the Laplace operator,*

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

4. Proof of the first part of the main theorem. We shall prove the main theorem in the form

$$g_m = h_m + h_{m-2} + h_{m-4} + \dots .$$

In this first part we shall prove that

$$g_m \leqslant h_m + h_{m-2} + h_{m-4} + \dots .$$

For any degree m, an invariant basis has g_m elements and an invariant harmonic basis has h_m elements. For brevity, let $h_m = \alpha$ and $g_m = \alpha + \beta$. We shall choose first an invariant harmonic basis of degree m, say $H_1, H_2, \dots, H_\alpha$. Then β homogeneous polynomials of degree m, say $I_1, I_2, \dots, I_\beta$, are chosen so that the complete set

$$H_1, H_2, \dots, H_\alpha, I_1, I_2, \dots, I_\beta$$

form an invariant basis.

Let $P(x, y, z)$ be an invariant homogeneous polynomial of $\mathfrak{G}$ of degree m. Then

$$P = a_1 H_1 + a_2 H_2 + \dots + a_\alpha H_\alpha + b_1 I_1 + b_2 I_2 + \dots + b_\beta I_\beta.$$

Furthermore, P is harmonic if and only if $b_1 = b_2 = \dots = b_\beta = 0$. That is,

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$$\Delta P = b_1\Delta I_1 + b_2\Delta I_2 + \ldots + b_\beta \Delta I_\beta = 0$$

only if $b_1 = b_2 = \ldots = b_\beta = 0$. But this means that $\Delta I_1, \Delta I_2, \ldots, \Delta I_\beta$ are linearly independent homogeneous polynomials of degree $m - 2$ and are invariants of $\mathfrak{G}$ by Lemma 4 of the last section. Hence, $\beta = g_m - h_m \leqslant g_{m-2}$, or $g_m \leqslant h_m + g_{m-2}$. By repeated application of this inequality to $g_{m-2}, g_{m-4}, \ldots$ we obtain

$$g_m \leqslant h_m + h_{m-2} + h_{m-4} \ldots,$$

and the first part of the main theorem is proved.

5. Proof of the second part of the main theorem. In this section we shall show that

$$g_m \geqslant h_m + h_{m-2} + h_{m-4} + \ldots. \tag{2}$$

This result, combined with that of the last section, will prove the main theorem.

The inequality (2) will be proved if we can construct

$$h_m + h_{m-2} + h_{m-4} + \ldots$$

invariant homogeneous polynomials of degree m which are linearly independent. For this construction, we take first h_m independent invariant harmonics of degree m, then h_{m-2} invariant harmonics of degree $m - 2$ each multiplied by $x^2 + y^2 + z^2$, then h_{m-4} invariant harmonics of degree $m - 4$ each multiplied by $(x^2 + y^2 + z^2)^2$, and so on. Thus we obtain $h_m + h_{m-2} + h_{m-4} + \ldots$ homogeneous polynomials of degree m, all invariant by Lemma 3. It remains to show that they are linearly independent.

Let $\mathscr{H}_{m-\mu}$ be a linear combination of the $h_{m-\mu}$ harmonics selected above, containing $h_{m-\mu}$ arbitrary constants, $\mu = 0, 2, 4, \ldots$. We wish to show that if

$$\mathscr{H}_m + (x^2 + y^2 + z^2)\, \mathscr{H}_{m-2} + (x^2 + y^2 + z^2)^2\, \mathscr{H}_{m-4} + \ldots = 0 \tag{3}$$

identically, all the $h_m + h_{m-2} + h_{m-4} + \ldots$ constants are necessarily zero.

To this end we multiply equation (3) by $\mathscr{H}_{m-\mu}$, in which μ takes one of the values $0, 2, 4, \ldots$, and integrate the resulting equation over the surface of the unit sphere. Because of the orthogonality of surface harmonics of different degrees on the unit sphere,

$$\iint (\mathscr{H}_{m-\mu}\, \mathscr{H}_m + \mathscr{H}_{m-\mu}\, \mathscr{H}_{m-2} + \ldots + \mathscr{H}_{m-\mu}^2 + \ldots)\, d\sigma = \iint \mathscr{H}_{m-\mu}^2\, d\sigma = 0.$$

Therefore, $\mathscr{H}_{m-\mu}$ vanishes on the surface of the unit sphere. It follows, from the uniqueness theorem for harmonic functions, that $\mathscr{H}_{m-\mu}$ vanishes identically. But if $\mathscr{H}_{m-\mu}$ vanishes identically, then the $h_{m-\mu}$ constants it contains must all be zero. Now let μ assume, in turn, the values $0, 2, 4, \ldots$. Thus, all the $h_m + h_{m-2} + h_{m-4} \ldots$ constants involved vanish, and the second part of the main theorem is proved.

6. Tables of generating functions. The factor $1 - t^2$ appears in the denominators of all the generating functions $g(t)$ found in §II. Since $h(t) = (1 - t^2)g(t)$,

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TABLE I

GENERATING FUNCTIONS OF THE FINITE GROUPS

(The absence of planes of symmetry is indicated by — or 0, according as the group is purely rotational or skew.)

n even	n odd	$h(t)$	p
$\mathfrak{J}_i$		$\dfrac{1}{1-t^{10}}\ \dfrac{1}{1-t^6}$	15
$\mathfrak{J}$		$\dfrac{1}{1-t^{10}}\ \dfrac{1+t^{15}}{1-t^6}$	—
$\mathfrak{O}_i$		$\dfrac{1}{1-t^4}\ \dfrac{1}{1-t^6}$	9
$\mathfrak{O}$		$\dfrac{1}{1-t^4}\ \dfrac{1+t^9}{1-t^6}$	—
$\mathfrak{T}_i$		$\dfrac{1}{1-t^4}\ \dfrac{1+t^6}{1-t^6}$	3
$\mathfrak{T}[\mathfrak{O}$		$\dfrac{1}{1-t^4}\ \dfrac{1}{1-t^3}$	6
$\mathfrak{T}$		$\dfrac{1}{1-t^4}\ \dfrac{1+t^6}{1-t^3}$	—
$\mathfrak{C}_n[\mathfrak{D}_n$		$\dfrac{1}{1-t}\ \dfrac{1}{1-t^n}$	n
$\mathfrak{C}_n$		$\dfrac{1}{1-t}\ \dfrac{1+t^n}{1-t^n}$	—
$\mathfrak{D}_{n\,i}$	$\mathfrak{D}_n[\mathfrak{D}_{2n}$	$\dfrac{1}{1-t^2}\ \dfrac{1}{1-t^n}$	$n+1$
$\mathfrak{D}_n$		$\dfrac{1}{1-t^2}\ \dfrac{1+t^{n+1}}{1-t^n}$	—
$\mathfrak{D}_n[\mathfrak{D}_{2n}$	$\mathfrak{D}_{n\,i}$	$\dfrac{1}{1-t^2}\ \dfrac{1+t^{n+1}}{1-t^{2n}}$	n
$\mathfrak{C}_{n\,i}$	$\mathfrak{C}_n[\mathfrak{C}_{2n}$	$\dfrac{1}{1-t^2}\ \dfrac{1+t^n}{1-t^n}$	1
$\mathfrak{C}_n[\mathfrak{C}_{2n}$	$\mathfrak{C}_{n\,i}$	$\dfrac{1}{1-t^2}\ \dfrac{1+2t^{n+1}+t^{2n}}{1-t^{2n}}$	0

we find $h(t)$ at once. Table I gives the generating functions $h(t)$ for all finite groups.

7. Further properties of $h(t)$. Besides the basic property of $h(t)$ just discussed, this function shows, in various ways, the structure of the abstract group and the geometrical properties of the symmetry. The proofs of the following theorems are omitted for brevity; the theorems may be verified by consulting Table I.

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THEOREM 1.

$$\lim_{t \to 1} (1 - t)^2 h(t) = \frac{2}{n},$$

n being the order of the group.

THEOREM 2.

$$\lim_{t \to \infty} t\, h(t) = 1 \text{ or } 0$$

according as $\mathfrak{G}$ *is or is not of Type* 1.

THEOREM 3. *The function* $h(t)$ *is an even function if and only if* $\mathfrak{G}$ *is of Type* 2.

THEOREM 4. *Let* $\mathfrak{G}_1 \subset \mathfrak{G}$, *and let* $h_1(t)$ *and* $h(t)$ *be their respective generating functions. Then*

$$h_1(t) \gg h(t).$$

Besides the classification of the finite groups given in §I, the groups of Type 2 and 3 may also be classified according to the number of planes of symmetry. The groups *generated* by planes of symmetry are

$$\mathfrak{C}_1[\mathfrak{C}_2,\ \mathfrak{C}_n[\mathfrak{D}_n,\ \mathfrak{D}_{ni}(n \text{ even}),\ \mathfrak{D}_n[\mathfrak{D}_{2n}(n \text{ odd}),\ \mathfrak{T}[\mathfrak{O},\ \mathfrak{O}_i,\ \mathfrak{J}_i.$$

They have, respectively, 1, n, $n + 1$, $n + 1$, 6, 9, and 15 planes of symmetry which divide the space into 2, $2n$, $4n$, $4n$, 24, 48, and 120 compartments, so that the number of compartments equals the order of the group.

Groups of Types 2 and 3 which have no plane of symmetry are called *skew* groups. They are $\mathfrak{C}_n[\mathfrak{C}_{2n}$ (n even) and $\mathfrak{C}_{ni}$ (n odd). Let p denote the number of planes of symmetry in the group.

THEOREM 5.

$$\lim_{t \to \infty} t^{p+1} h(t) = 1$$

if, and only if, $\mathfrak{G}$ *is not a skew group.*

THEOREM 6. *The function* $h(t)$ *vanishes for a finite value of* t, *not a root of unity, if and only if* $\mathfrak{G}$ *is skew.*

THEOREM 7. *The function* $h(t)$ *vanishes at no finite point if and only if* $\mathfrak{G}$ *is generated by reflections.*

THEOREM 8. *Let* $\mathfrak{G}$ *be a group generated by reflections,* $\mathfrak{G}_1$ *the purely rotational subgroup of* $\mathfrak{G}$ *of index* 2, $h(t)$ *and* $h_1(t)$ *their respective generating functions. Then*

$$h_1(t) = (1 + t^p)\, h(t).$$

IV. INVARIANT HARMONIC BASES

1. Invariant basis and invariant harmonic basis. Let $\mathfrak{G}$ denote one of the groups discussed in I, and let $\mathscr{H}_m$ denote the set of spherical harmonics of degree m which form an invariant harmonic basis of degree m for $\mathfrak{G}$. Then, by the proof of the second part of the main theorem of §III, the set

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$$\{\mathscr{H}_m, \quad (x^2+y^2+z^2)\,\mathscr{H}_{m-2}, \quad (x^2+y^2+z^2)^2\,\mathscr{H}_{m-4}, \quad \ldots\}$$

is an invariant basis of degree m for $\mathfrak{G}$. In this manner, the problem of finding an invariant basis of degree m is reduced to that of finding an invariant harmonic basis of each of the degrees $m, m-2, m-4, \ldots$.

The remainder of this paper will be devoted to finding invariant harmonic bases of arbitrary degree for each of the finite groups of §I. We shall construct the bases using the Maxwell representation of spherical harmonics in terms of partial derivatives of $1/r$. For the cyclic and dihedral groups and groups of Types 2 and 3 derived from them we shall develop an equivalent representation of the bases, which is simpler in certain respects, in terms of the associated Legendre functions.

2. Invariant harmonic bases in the Maxwell representation.

Operating basis. Let $Q(x,y,z)$ be a homogeneous polynomial of degree m in x, y, z. Then

$$Q\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

is a differential operator formed by replacing the variables of the polynomial by the appropriate partial derivators. Let $r = (x^2+y^2+z^2)^{\frac{1}{2}}$. The expression

$$r^{2m+1}\,Q\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r} \tag{1}$$

is a spherical harmonic of degree m or is identically zero. Furthermore, the above expression is identically zero if and only if $Q(x,y,z)$ is divisible by $x^2 + y^2 + z^2$ (**5**, pp. 127–129.) The polynomial $Q(x,y,z)$ will be called the *operating polynomial* corresponding to the spherical harmonic (1). If Q is an invariant of a group $\mathfrak{G}$ of orthogonal linear transformations, then the spherical harmonic (1) will be also an invariant of $\mathfrak{G}$ by Lemma 2, §III.

These considerations lead to the following basic

THEOREM. *Let*

$$Q_1(x,y,z), \quad Q_2(x,y,z), \quad \ldots, \quad Q_{h_m}(x,y,z)$$

be h_m homogeneous polynomials of degree m which are invariants of $\mathfrak{G}$ and which are linearly independent mod $(x^2+y^2+z^2)$. *Then the h_m spherical harmonics*

$$r^{2m+1}\,Q_j\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r} \qquad (j = 1, 2, \ldots, h_m) \tag{2}$$

form an invariant harmonic basis of degree m for $\mathfrak{G}$.

The statement that the Q_i are linearly independent mod $(x^2+y^2+z^2)$ means that a linear combination of them,

$$\sum_i A_i Q_i,$$

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is divisible by $x^2 + y^2 + z^2$ only if $A_i = 0$ for all i. We call $Q_1, Q_2, \ldots, Q_{h_m}$ an *operating basis* of degree m for $\mathfrak{G}$.

3. Invariant harmonic bases for $\mathfrak{C}_n$, $\mathfrak{D}_n$, and derived groups. A result of §III,

$$h(t;\ \mathfrak{C}_n) = \frac{1+t^n}{(1-t)(1-t^n)} = (1-t)^{-1}(1+t^n)(1+t^n+t^{2n}+\ldots)$$
$$= (1-t)^{-1}(1+2t^n+2t^{2n}+\ldots),$$

suggests that we seek an invariant operating polynomial for $\mathfrak{C}_n$ of degree one, and two invariant operating polynomials of each of the degrees $n, 2n, 3n, \ldots$. Let $\rho = (x^2+y^2)^{\frac{1}{2}}$. Since the rotation generating $\mathfrak{C}_n$, expressed in spherical coordinates, is $\phi \to \phi + 2\pi/n$, the polynomials

$$z,\ \rho^n \cos n\phi,\ \rho^n \sin n\phi,\ \rho^{2n}\cos 2n\phi,\ \rho^{2n}\sin 2n\phi, \ldots$$

are clearly invariants of $\mathfrak{C}_n$. For simplicity of notation, let

$$C_n = \rho^n \cos n\phi = x^n - \binom{n}{2}x^{n-2}y^2 + \binom{n}{4}x^{n-4}y^4 - \ldots,$$
$$C'_n = \rho^n \sin n\phi = \binom{n}{1}x^{n-1}y - \binom{n}{3}x^{n-3}y^3 + \binom{n}{5}x^{n-5}y^5 - \ldots.$$

We shall call z, C_n, and C'_n the fundamental operating polynomials for $\mathfrak{C}_n$.

We consider the expression

$$(3)\qquad (1-z)^{-1}(1+C_n+C'_n+C_{2n}+C'_{2n}+\ldots)$$
$$= (1+z+z^2+\ldots)(1+C_n+C'_n+C_{2n}+C'_{2n}+\ldots),$$

or rather the double series resulting from term-by-term multiplication of the two infinite series in the last line, without any reference to convergence, as the *set of its terms*. Each term is an invariant operating polynomial for $\mathfrak{C}_n$. That is, our assertion is that an operating basis consists of the following $2[m/n]+1$ polynomials:

$$(4)\qquad z^m,\ z^{m-n}C_n,\ z^{m-n}C'_n,\ z^{m-2n}C_{2n},\ z^{m-2n}C'_{2n}, \ldots.$$

By the theorem of the last section, all that remains to be shown is that the polynomials (4) are linearly independent mod $(x^2+y^2+z^2)$. In this section and the next two, we shall find series analogous to (3) for all finite groups. Such a series constitutes a complete solution of our problem.

It remains to show that the polynomials of the set (4) are linearly independent mod $(x^2+y^2+z^2)$; that is, we wish to show

$$(5)\qquad A_0 z^m + \sum_{\nu=1}^{[m/n]} z^{m-\nu n}(A_\nu C_{\nu n} + B_\nu C'_{\nu n}) \equiv 0 \mod (x^2+y^2+z^2)$$

only if $A_0 = A_1 = A_2 = \ldots = B_1 = B_2 = \ldots = 0$.

Without loss of generality, we may assume that m is even, since if m is odd, we may multiply the congruence by z, obtaining an equivalent congruence.

We first consider the case of n even. Then the left member of (5) is even in z. The congruence (5) is then equivalent to the equation

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$$A_0(-x^2-y^2)^{\frac{1}{2}m} + \sum_{\nu=1}^{[m/n]} (-x^2-y^2)^{\frac{1}{2}(m-\nu n)}(A_\nu C_{\nu n} + B_\nu C'_{\nu n}) = 0 \tag{6}$$

identically. Now if we let $x = \cos\phi$, $y = \sin\phi$, (6) becomes

$$(-1)^{\frac{1}{2}m} A_0 + \sum_{\nu=1}^{[m/n]} (-1)^{\frac{1}{2}(m-\nu n)}(A_\nu \cos \nu n\phi + B_\nu \sin \nu n\phi) = 0. \tag{7}$$

If equation (7) is multiplied, in turn, by

$$1, \quad \cos\phi, \quad \sin\phi, \quad \cos 2\phi, \quad \sin 2\phi, \quad \ldots \tag{8}$$

and the resulting equation is integrated from $-\pi$ to π, we see, by the orthogonality of the set (8), that all the coefficients A_i and B_i must be zero, which was the aim of our proof.

We now consider the case of n odd. Then, equation (5) has the form

$$F_1(x, y, z^2) + zF_2(x, y, z^2) \equiv 0 \qquad \mathrm{mod}\,(x^2+y^2+z^2). \tag{9}$$

The above congruence remains true if we substitute $-z$ for z,

$$F_1(x, y, z^2) - zF_2(x, y, z^2) \equiv 0 \qquad \mathrm{mod}\,(x^2+y^2+z^2). \tag{10}$$

If we now add and subtract (9) and (10), we obtain

$$\begin{matrix} F_1(x, y, z^2) \equiv 0 \\ F_2(x, y, z^2) \equiv 0 \end{matrix} \qquad \mathrm{mod}\,(x^2+y^2+z^2). \tag{11}$$

The proof may now be completed in a similar manner to the case of n even, since both congruences of (11) are of the form (5) (with changed values for the parameters m and n).

We pass to the group $\mathfrak{D}_n$. We assume that one of the 2-fold axes, perpendicular to the n-fold axis, is the x-axis, as explained in §I. Then we obtain the group $\mathfrak{D}_n$ by adjoining to $\mathfrak{C}_n$ the matrix

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

or $\phi \to -\phi$ and $\theta \to \pi - \theta$; that is, the group consists of the matrices of $\mathfrak{C}_n$ and the coset $K\mathfrak{C}_n$. Hence, an invariant of $\mathfrak{C}_n$ which is also an invariant of K will be an invariant of $\mathfrak{D}_n$. Upon applying the transformation K to the fundamental operating polynomials of $\mathfrak{C}_n$, we note that

$$z \to -z, \quad C_{wn} \to C_{wn}, \quad C'_{wn} \to -C'_{wn} \quad (w = 1, 2, 3, \ldots).$$

Hence, z^2, C_{wn}, and zC'_{wn} are invariants of $\mathfrak{D}_n$.

In §III the generating function for $\mathfrak{D}_n$ was found to be

$$h(t;\, \mathfrak{D}_n) = \frac{1+t^{n+1}}{(1-t^2)(1-t^n)} = (1-t^2)^{-1}(1+t^{n+1})(1+t^n+t^{2n}+\ldots)$$
$$= (1-t^2)^{-1}(1+t^n+t^{n+1}+t^{2n}+t^{2n+1}+\ldots).$$

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We consider now the expression

$$(12) \qquad (1 - z^2)^{-1} (1 + C_n + zC'_n + C_{2n} + zC'_{2n} + \ldots)$$

in the same way as we did (3), namely as the set of all the terms of the double series resulting from the term-by-term multiplication of the two factors. It is clear, by comparison of the above expression with the generating function, that we have the correct number of operating polynomials for each degree m. We have just seen that each term in the series is an invariant of $\mathfrak{D}_n$, and, since the above operating polynomials are a subset of those found for $\mathfrak{C}_n$, their linear independence mod $(x^2 + y^2 + z^2)$ has already been proved.

Similarly, the invariant operating polynomials for the other groups derived from $\mathfrak{C}_n$ will be a subset of those for $\mathfrak{C}_n$ and may be found in a completely analogous manner. In all cases the symmetry elements of these groups will be placed, with relation to the coordinate axes, as explained in §I, 2. In Table II, the matrix which generates each group from an appropriate subgroup (either $\mathfrak{C}_n$ or $\mathfrak{D}_n$) is given. The transformation in spherical coordinates corresponding to this matrix is also given. Table III shows which of the invariant operating polynomials for $\mathfrak{C}_n$ are also invariants for each of the groups. In this table, $+$ means that the expression is an invariant; $-$ means that the expression is not an invariant. In most cases (all except $\mathfrak{D}_{ni}$ and $\mathfrak{D}_n[\mathfrak{D}_{2n}]$) these expressions which are not invariants go over into their negatives upon application of the transformation adjoined to the subgroup. Finally, the series analogous to (3) and (12) are given in Table IV for each of the groups. For convenience, Tables II and IV also include the groups of the Cubic System which will be discussed in the next section.

Thus we see that these three tables, together with Table I, present in tabular form, for each of the groups, a complete derivation of the invariant operating polynomials similar to that carried out in detail for $\mathfrak{D}_n$.

4. Invariant harmonic basis for groups of the Cubic System. In §III we found that

$$h(t;\, \mathfrak{T}) = \frac{1 + t^6}{(1 - t^3)(1 - t^4)}\,.$$

With this in mind we seek an invariant of $\mathfrak{T}$ for each of the degrees three, four, and six. Geometrical considerations enable us to find these invariants.

It is clear that in any purely rotational group $\mathfrak{G}$, the set of all axes of rotation must be permuted by any transformation of the group; that is, the permutation group of these axes form a representation of $\mathfrak{G}$. Moreover, it may be that the set of all rotational axes of the group may be decomposed into several disjoint subsets in such a manner that all rotations of the group permute the axes of each of these subsets among themselves. Suppose there are q axes in one of these subsets, with direction numbers $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2, \ldots, q)$. Then the expression

$$\prod_{i=1}^{q} (\alpha_i x + \beta_i y + \gamma_i z)$$

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TABLE II

MATRICES ADJOINED TO SUBGROUPS TO GENERATE GROUPS

n even	Subgroup	Generating matrix	Transformation in spherical coordinates	n odd
$\mathfrak{D}_n$	$\mathfrak{C}_n$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\phi \to -\phi$ $\theta \to \pi - \theta$	$\mathfrak{D}_n$
$\mathfrak{C}_n[\mathfrak{D}_n$	$\mathfrak{C}_n$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\phi \to -\phi$	$\mathfrak{C}_n[\mathfrak{D}_n$
$\mathfrak{C}_{ni}$	$\mathfrak{C}_n$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\theta \to \pi - \theta$	$\mathfrak{C}_n[\mathfrak{C}_{2n}$
$\mathfrak{D}_{ni}$	$\mathfrak{D}_n$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\theta \to \pi - \theta$	$\mathfrak{D}_n[\mathfrak{D}_{2n}$
$\mathfrak{C}_n[\mathfrak{C}_{2n}$	$\mathfrak{C}_n$	$\begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} & 0 \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\phi \to \phi + \frac{\pi}{n}$ $\theta \to \pi - \theta$	$\mathfrak{C}_{ni}$
$\mathfrak{D}_n[\mathfrak{D}_{2n}$	$\mathfrak{D}_n$	$\begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} & 0 \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\phi \to \phi + \frac{\pi}{n}$ $\theta \to \pi - \theta$	$\mathfrak{D}_{ni}$
$\mathfrak{T}$	$\mathfrak{D}_2$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$		$\mathfrak{T}$
$\mathfrak{O}$	$\mathfrak{T}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$		$\mathfrak{O}$
$\mathfrak{T}[\mathfrak{O}$	$\mathfrak{T}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$		$\mathfrak{T}[\mathfrak{O}$
$\mathfrak{T}_i$	$\mathfrak{T}$	J		$\mathfrak{T}_i$
$\mathfrak{O}_i$	$\mathfrak{O}$	J		$\mathfrak{O}_i$

is transformed into a constant multiple of itself by any rotation of $\mathfrak{G}$. Proper choice of the direction numbers can easily be made in order that the above expression be an invariant.

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TABLE III

INVARIANCE OF OPERATING POLYNOMIALS FOR $\mathfrak{C}n$ WITH RESPECT TO DERIVED GROUPS

$$u = 1, 3, 5, \ldots$$
$$v = 2, 4, 6, \ldots$$

n even	z^2	z	C_{un}	C'_{un}	zC_{un}	zC'_{un}	C_{vn}	C'_{vn}	zC_{vn}	zC'_{vn}	n odd
$\mathfrak{C}_n$	+	+	+	+	+	+	+	+	+	+	$\mathfrak{C}_n$
$\mathfrak{D}_n$	+	−	+	−	−	+	+	−	−	+	$\mathfrak{D}_n$
$\mathfrak{C}_n[\mathfrak{D}_n$	+	+	+	−	+	−	+	−	+	−	$\mathfrak{C}_n[\mathfrak{D}_n$
$\mathfrak{C}_{ni}$	+	−	+	+	−	−	+	+	−	−	$\mathfrak{C}_n[\mathfrak{C}_{2n}$
$\mathfrak{D}_{ni}$	+	−	+	−	−	−	+	−	−	−	$\mathfrak{D}_n[\mathfrak{D}_{2n}$
$\mathfrak{C}_n[\mathfrak{C}_{2n}$	+	−	−	−	+	+	+	+	−	−	$\mathfrak{C}_{ni}$
$\mathfrak{D}_n[\mathfrak{D}_{2n}$	+	−	−	−	−	+	+	−	−	−	$\mathfrak{D}_{ni}$

The regular tetrahedron is placed in the position described in §I, inscribed in a cube. Then it is clear that the 13 axes of rotation separate into three disjoint subsets—the three axes through the centroids of the faces of the cube, the four diagonals of the cube, and the six axes through the midpoints of the edges of the cube. Corresponding to these three sets of axes are the polynomials

$$O_3 = xyz$$
$$O^*_4 = (x + y + z)(-x + y + z)(x - y + z)(x + y - z)$$
$$O_6 = (x^2 - y^2)(y^2 - z^2)(z^2 - x^2).$$

Since $O^*_4 \equiv -2(x^4 + y^4 + z^4)$, mod $(x^2 + y^2 + z^2)$, in the following we shall use instead

$$O_4 = x^4 + y^4 + z^4.$$

It can be easily shown that O_3, O_4, and O_6 are invariants of $\mathfrak{T}$. It now seems reasonable to conjecture that the set of invariant operating polynomials for $\mathfrak{T}$ is represented by the expression

$$(1 - O_3)^{-1}(1 - O_4)^{-1}(1 + O_6),$$

in the sense illustrated in the foregoing section by the discussions of (3) and (12). To prove the above conjecture it remains to show that all terms of a given

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degree m are linearly independent mod $(x^2 + y^2 + z^2)$. That is, we wish to show that

$$(13) \quad \sum_i a_i \cdot O_3^{\alpha_i} O_4^{\beta_i} + O_6 \sum_j c_j \cdot O_3^{\gamma_i} O_4^{\delta_i} \equiv 0 \qquad \mathrm{mod}(x^2 + y^2 + z^2),$$
$$3\alpha_i + 4\beta_i = 6 + 3\gamma_j + 4\delta_j = m \text{ (for all } i \text{ and } j),$$

only if all a_i and c_j are zero.

Case I. If m is odd,

$$3\alpha_i \equiv \alpha_i \equiv m \equiv 1 \qquad (\mathrm{mod}\ 2) \quad \text{for all } i,$$

$$3\gamma_j \equiv \gamma_j \equiv m \equiv 1 \qquad (\mathrm{mod}\ 2) \quad \text{for all } j.$$

Hence, O_3 is a common factor of the left member of (13), and we may divide the congruence by it since

$$O_3 \not\equiv 0 \qquad \mathrm{mod}\ (x^2 + y^2 + z^2).$$

This reduces the problem to

Case II. If m is even,

$$\alpha_i \equiv m \equiv 0 \qquad (\mathrm{mod}\ 2) \quad \text{for all } i,$$

$$\gamma_j \equiv m \equiv 0 \qquad (\mathrm{mod}\ 2) \quad \text{for all } j.$$

Hence, the left member of (13) is a polynomial in O_3^2, O_4, and O_6. If now we make the substitution $z^2 = -x^2 - y^2$ in (13), the resulting expression must vanish identically. Upon making this substitution and absorbing the constant factors in the a_i and c_j, (13) becomes

$$(14) \quad \sum_i a_i(x^4y^2 + x^2y^4)^{\frac{1}{2}\alpha_i}(x^4 + x^2y^2 + y^4)^{\beta_i}$$
$$+ (x^2 - y^2)(x^2 + 2y^2)(2x^2 + y^2) \sum_j c_j(x^4y^2 + x^2y^4)^{\frac{1}{2}\gamma_i}(x^4 + x^2y^2 + y^4)^{\delta_i}$$
$$= 0,$$

identically. Our assertion is that all the a_i and c_j vanish. We assume the contrary and obtain a contradiction.

We may assume that all the a_i, c_j in (14) are different from zero; otherwise, we could simply omit the terms with vanishing coefficients. We may assume that $\alpha_1 < \alpha_i$, $\gamma_1 < \gamma_j$ for $i = 2, 3, \ldots$ and $j = 2, 3, \ldots$; this is a matter of notation. Finally, we may assume that one of the numbers α_1 and γ_1 is equal to zero; otherwise, we could divide by a suitable power of $x^4y^2 + x^2y^4$. Both α_1 and γ_1 cannot vanish; otherwise, we should have $4\beta_1 = 6 + 4\delta_1$, $2(\beta_1 - \delta_1) = 3$. But 3 is not an even number.

Now we set $y = 0$ and obtain:

$$a_1 x^{4\beta_1} = 0, \quad \text{so that } a_1 = 0 \text{ if } \alpha_1 = 0;$$

$$c_1 x^{6+4\delta_1} = 0, \quad \text{so that } c_1 = 0 \text{ if } \gamma_1 = 0;$$

a contradiction in either case. This proves our assertion.

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For the other groups of the Cubic System, the series representing the invariant operating polynomials are derived in the manner shown in the last section. Tables I, II, IV, and V summarize these derivations completely.

TABLE IV

SERIES REPRESENTING OPERATING POLYNOMIALS OF THE FINITE GROUPS

n even	Polynomials	n odd
$\mathfrak{C}_n$	$(1-z)^{-1}\,(1+C_n+C'_n+C_{2n}+C'_{2n}+\ldots)$	$\mathfrak{C}_n$
$\mathfrak{D}_n$	$(1-z^2)^{-1}\,(1+C_n+zC'_n+C_{2n}+zC'_{2n}+\ldots)$	$\mathfrak{D}_n$
$\mathfrak{C}_n[\mathfrak{D}_n$	$(1-z)^{-1}\,(1+C_n+C_{2n}+\ldots)$	$\mathfrak{C}_n[\mathfrak{D}_n$
$\mathfrak{C}_{n\,i}$	$(1-z^2)^{-1}\,(1+C_n+C'_n+C_{2n}+C'_{2n}+\ldots)$	$\mathfrak{C}_n[\mathfrak{C}_{2n}$
$\mathfrak{D}_{n\,i}$	$(1-z^2)^{-1}\,(1+C_n+C_{2n}+\ldots)$	$\mathfrak{D}_n[\mathfrak{D}_{2n}$
$\mathfrak{C}_n[\mathfrak{C}_{2n}$	$(1-z^2)^{-1}\,(1+zC_n+zC'_n+C_{2n}+C'_{2n}+\ldots)$	$\mathfrak{C}_{n\,i}$
$\mathfrak{D}_n[\mathfrak{D}_{2n}$	$(1-z^2)^{-1}\,(1+zC'_n+C_{2n}+\ldots)$	$\mathfrak{D}_{n\,i}$
$\mathfrak{T}$	$(1-O_3)^{-1}\,(1-O_4)^{-1}\,(1+O_6)$	$\mathfrak{T}$
$\mathfrak{O}$	$(1-O_3^2)^{-1}\,(1-O_4)^{-1}\,(1+O_3O_6)$	$\mathfrak{O}$
$\mathfrak{T}[\mathfrak{O}$	$(1-O_3)^{-1}\,(1-O_4)^{-1}$	$\mathfrak{T}[\mathfrak{O}$
$\mathfrak{T}_i$	$(1-O_3^2)^{-1}\,(1-O_4)^{-1}\,(1+O_6)$	$\mathfrak{T}_i$
$\mathfrak{O}_i$	$(1-O_3^2)^{-1}\,(1-O_4)^{-1}$	$\mathfrak{O}_i$
$\mathfrak{J}$	$(1-I_6)^{-1}\,(1-I_{10})^{-1}\,(1+I_{15})$	$\mathfrak{J}$
$\mathfrak{J}_i$	$(1-I_6)^{-1}\,(1-I_{10})^{-1}$	$\mathfrak{J}_i$

5. The icosahedral groups. In §III we found that

$$h(t,\mathfrak{J}) = \frac{1+t^{15}}{(1-t^6)(1-t^{10})}\,.$$

With this in mind we wish to find an invariant of $\mathfrak{J}$ for each of the degrees six, ten, and fifteen. Placing the icosahedron as explained in §I, we find these in-

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variants in a manner analogous to that used for the groups of the Cubic System. In the following, $\tau = \frac{1}{2}(1 + \sqrt{5})$. The three invariants are

$$I_6 = (\tau^2x^2 - y^2)(\tau^2y^2 - z^2)(\tau^2z^2 - x^2)$$

$$I_{10} = (x^4 + y^4 + z^4)(\tau^{-2}x^2 - \tau^2y^2)(\tau^{-2}y^2 - \tau^2z^2)(\tau^{-2}z^2 - \tau^2x^2)$$

$$\begin{aligned} I_{15} = xyz&(\tau x + \tau^{-1}y + z)(-\tau x + \tau^{-1}y + z)(\tau x - \tau^{-1}y + z)(\tau x + \tau^{-1}y - z) \\ &(x + \tau y + \tau^{-1}z)(-x + \tau y + \tau^{-1}z)(x - \tau y + \tau^{-1}z)(x + \tau y - \tau^{-1}z) \\ &(\tau^{-1}x + y + \tau z)(-\tau^{-1}x + y + \tau z)(\tau^{-1}x - y + \tau z)(\tau^{-1}x + y - \tau z). \end{aligned}$$

We now conjecture that the set of invariant operating polynomials for $\mathfrak{J}$ is represented by the series $(1 - I_6)^{-1}(1 - I_{10})^{-1}(1 + I_{15})$.

To prove this conjecture it remains to show that all terms of a given degree m are linearly independent mod $(x^2 + y^2 + z^2)$.

Case I. m even. We wish to show that

$$\sum_i a_i I_6^{\alpha_i} I_{10}^{\beta_i} \equiv 0 \quad \mathrm{mod}(x^2 + y^2 + z^2),\ 6\alpha_i + 10\beta_i = m \tag{15}$$

only if all a_i are zero.

Case II. m odd. We wish to show that

$$I_{15}\sum_i a_i I_6^{\alpha_i} I_{10}^{\beta_i} \equiv 0 \qquad \mathrm{mod}(x^2 + y^2 + z^2),\ 15 + 6\alpha_i + 10\beta_i = m$$

only if all a_i are zero. Since it can be shown that $I_{15} \not\equiv 0$, mod $(x^2 + y^2 + z^2)$, Case II reduces to Case I.

TABLE V

INVARIANCE OF OPERATING POLYNOMIALS FOR $\mathfrak{T}$ WITH RESPECT TO DERIVED GROUPS

Groups	O_3^2	O_3O_6	O_3	O_4	O_6
$\mathfrak{D}_2$	+	+	+	+	+
$\mathfrak{T}$	+	+	+	+	+
$\mathfrak{O}$	+	+	−	+	−
$\mathfrak{T}[\mathfrak{O}$	+	−	+	+	−
$\mathfrak{T}_i$	+	−	−	+	+
$\mathfrak{O}_i$	+	−	−	+	−

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The remainder of the proof is similar to that of the preceding section and will be briefly indicated. As before, we assume all a_i different from zero and $\alpha_1 < \alpha_i$ for $i = 2, 3, \ldots$. Since

$$I_6 \not\equiv 0 \qquad \mod(x^2 + y^2 + z^2),$$

we may divide (15) by $I_6^{\alpha_1}$, obtaining an equivalent congruence in which $\alpha_1 = 0$. If in this congruence we first let $z^2 = -x^2 - y^2$ and then let $y = \tau x$ we obtain

$$-32\tau^2(\tau^4 + \tau^2 + 1)(\tau^{-2} - \tau^4)\, a_1 x^{4\beta_1} = 0$$

identically. Hence $a_1 = 0$, a contradiction to the assumption that there were any non-zero coefficients in (15).

It is easily seen that I_6 and I_{10} are invariants of $\mathfrak{J}_i$ but I_{15} is not. Hence, the set of invariant operating polynomials for $\mathfrak{J}_i$ is $(1 - I_6)^{-1}\,(1 - I_{10})^{-1}$.

6. Invariant harmonic bases using the associated Legendre functions. For the simpler groups, the cyclic, dihedral and related groups, representation of the invariant harmonic basis, in terms of the associated Legendre functions, is somewhat simpler to derive than in the Maxwell representation. It will be shown later that the same basis is obtained, except for constant factors, in both representations.

It is well known that

$$(16) \qquad \begin{array}{l} P_m(\cos\theta), \\ P_{m,1}(\cos\theta)\cos\phi,\ P_{m,2}(\cos\theta)\cos 2\phi, \ldots,\ P_{m,m}(\cos\theta)\cos m\phi, \\ P_{m,1}(\cos\theta)\sin\phi,\ P_{m,2}(\cos\theta)\sin 2\phi, \ldots,\ P_{m,m}(\cos\theta)\sin m\phi \end{array}$$

are a set of linearly independent surface harmonics of order m and all others are expressible as linear combinations of them. By multiplying each of the elements of the set (16) by r^m, we obtain an invariant harmonic basis of degree m for $\mathfrak{C}_1$.

The cyclic group $\mathfrak{C}_n$ is generated by the transformation $\phi \to \phi + 2\pi/n$. Of the set (16), the elements

$$(17) \qquad \begin{array}{l} P_m(\cos\theta), \\ P_{m,n}(\cos\theta)\cos n\phi,\ P_{m,2n}(\cos\theta)\cos 2n\phi, \ldots,\ P_{m,sn}(\cos\theta)\cos sn\phi, \\ P_{m,n}(\cos\theta)\sin n\phi,\ P_{m,2n}(\cos\theta)\sin 2n\phi, \ldots,\ P_{m,sn}(\cos\theta)\sin sn\phi \\ \qquad\qquad (sn \leqslant m < (s+1)n) \end{array}$$

are clearly invariants of $\mathfrak{C}_n$. There are $2[m/n] + 1$ elements in the set (17), and, from III, $h_m = 2[m/n] + 1$ for $\mathfrak{C}_n$. Hence, (17) is an invariant basis of surface harmonics of order m for $\mathfrak{C}_n$.

The bases for the derived groups are found by selecting those elements of (17) which are invariants of the transformation generating the group from

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$\mathfrak{C}_n$. It may be verified that in each case the basis obtained has the correct number of elements (that is, h_m of the group in question). These derivations are summarized in Table VI.

TABLE VI

INVARIANT HARMONIC BASES IN TERMS OF THE ASSOCIATED LEGENDRE FUNCTIONS

(In all formulas below, the argument "cos θ" is omitted in the associated Legendre functions.)

$w = 1, 2, \ldots, [m/n]$
$u = 1, 3, \ldots, [m/n]$ or $[m/n] - 1$ (whichever is odd)
$v = 2, 4, \ldots, [m/n]$ or $[m/n] - 1$ (whichever is even)

Group			Basis
$\mathfrak{C}_n$			P_m, $P_{m,wn} \cos wn\phi$, $P_{m,wn} \sin wn\phi$
$\mathfrak{D}_n$	(n even)	m even	P_m, $P_{m,wn} \cos wn\phi$
		m odd	$P_{m,wn} \sin wn\phi$
	(n odd)	m even	P_m, $P_{m,vn} \cos vn\phi$, $P_{m,un} \sin un\phi$
		m odd	$P_{m,un} \cos un\phi$, $P_{m,vn} \sin vn\phi$
$\mathfrak{C}_n[\mathfrak{D}_n$			P_m, $P_{m,wn} \cos wn\phi$
$\mathfrak{C}_{ni}$		m even	P_m, $P_{m,wn} \cos wn\phi$, $P_{m,wn} \sin wn\phi$
		m odd	None
$\mathfrak{D}_{ni}$	(n even)	m even	P_m, $P_{m,wn} \cos wn\phi$
		m odd	None
	(n odd)	m even	P_m, $P_{m,vn} \cos vn\phi$, $P_{m,un} \sin un\phi$
		m odd	None
$\mathfrak{C}_n[\mathfrak{C}_{2n}$		m even	P_m, $P_{m,vn} \cos vn\phi$, $P_{m,vn} \sin vn\phi$
		m odd	$P_{m,un} \cos un\phi$, $P_{m,un} \sin un\phi$
$\mathfrak{D}_n[\mathfrak{D}_{2n}$	(n even)	m even	P_m, $P_{m,vn} \cos vn\phi$
		m odd	$P_{m,un} \sin un\phi$
	(n odd)	m even	P_m, $P_{m,vn} \cos vn\phi$
		m odd	$P_{m,un} \cos un\phi$

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7. Equivalence of the two representations. For the groups $\mathfrak{C}_n$, $\mathfrak{D}_n$ and those of Types 2 and 3 derived from them, we have found invariant harmonic bases in two different representations—in terms of the differential operators and in terms of the associated Legendre functions. In all cases, however, the bases obtained are the same, except for constant factors.

This follows from the relationship (**5**, p. 134).

$$\frac{\partial^{n-m}}{\partial z^{n-m}}\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right)^m \frac{1}{r} = \frac{(-1)^{n-m}(n-m)!}{r^{n+1}} (\cos m\phi \pm i \sin m\phi)\, P_{n,m}(\cos\theta).$$

Now, set $k = m$, $j = n - m$, multiply the above equation by $r^{2(j+k)+1}$, and separate into real and imaginary parts:

$$r^{2(j+k)+1} \frac{\partial^j}{\partial z^j} C_k \frac{1}{r} = (-1)^j j!\, r^{j+k} P_{j+k,k}(\cos\theta) \cos k\phi, \tag{18}$$

and

$$r^{2(j+k)+1} \frac{\partial^j}{\partial z^j} C'_k \frac{1}{r} = (-1)^j j!\, r^{j+k} P_{j+k,k}(\cos\theta) \sin k\phi, \tag{19}$$

in which C_k and C'_k are the differential operators obtained by substituting

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$$

for x, y, z in the operating polynomials previously defined.

The equivalence of the two representations can easily be shown by use of (18), (19), and Tables IV and VI.

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University of Arizona

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