A Formula for the Solution of Algebraic or Transcendental Equations.

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§ 1. *Statement of the formula and numerical examples of it.*

The object of the present note is to communicate the following formula for the solution of algebraic or transcendental equations:

The root of the equation

$$0 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots, \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

which is the smallest in absolute value, is given by the series

$$-\frac{a_0}{a_1} - \frac{a_0^2 a_2}{a_1 \begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}} - \frac{a_0^3 \begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}} - \frac{a_0^4 \begin{vmatrix} a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \end{vmatrix}} - \ldots \qquad (2)$$

As a numerical example, consider the equation

$$x^3 - 4x^2 - 321x + 20 = 0.$$

Here $a_0 = 20, \quad a_1 = -321, \quad a_2 = -4, \quad a_3 = 1,$

$$\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix} = 103{,}121. \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix} = -33{,}127{,}121. \qquad \begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix} = 337.$$

The smallest root of the equation is therefore

$$\frac{20}{321} - \frac{20^2 \times 4}{321 \times 103,\,121} + \frac{20^3 \times 337}{103,\,121 \times 33,\,127,\,121} + \ldots$$

or $\quad 0{\cdot}062{,}305{,}3 - 0{\cdot}000{,}048{,}3 + 0{\cdot}000{,}000{,}7$

or $\quad 0{\cdot}062{,}257{,}7$ correctly to 7 decimal places.

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The series converges rapidly when the ratio of the smallest root to every one of the other roots is small. In calculating a root of a given equation by the formula, it is therefore advisable in many cases first to transform the given equation by two or three Lobatchevsky-Graeffe operations, each of which replaces the equation operated on by an equation whose roots are the squares of its roots: or else, in those cases where an approximate value of the root is already known, to transform the given equation by a substitution of the type

$$x = a + y,$$

where a is the known approximate value of the required root, so that the required root of the equation in y is small compared with any of the other roots.

§ 2. *Proof of the formula.*

Let the roots of the equation (1), supposed for the present to be of degree n, be $x_1, x_2, \ldots x_n$. Then if z be any number whose modulus is smaller than each of the moduli of the roots, we have

$$\frac{a_0}{a_0 + a_1 z + a_2 z^2 + \ \ldots \ + a_n z^n} = \frac{1}{\left(1 - \frac{z}{x_1}\right)\left(1 - \frac{z}{x_2}\right)\ldots\left(1 - \frac{z}{x_n}\right)}$$

$$= \left(1 + \frac{z}{x_1} + \frac{z^2}{x_1^2} + \ldots\right)\left(1 + \frac{z}{x_2} + \frac{z^2}{x_2^2} + \ldots\right)\ldots\left(1 + \frac{z}{x_n} + \frac{z^2}{x_n^2} + \ldots\right)$$

$$= 1 + P_1 z + P_2 z^2 + P_3 z^3 + \ldots$$

where P_r denotes the sum of the homogeneous powers and products of the reciprocals of the roots taken r at a time.

$$\therefore \quad a_0 = (a_0 + a_1 z + a_2 z^2 + \ldots)\,(1 + P_1 z + P_2 z^2 + P_3 z^3 + \ldots).$$

Equating coefficients of powers of z, we have

$$\begin{cases} 0 = a_1 + a_0 P_1 \\ 0 = a_2 + a_1 P_1 + a_0 P_2 \\ 0 = a_3 + a_2 P_1 + a_1 P_2 + a_0 P_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

whence*

* The formulae of equation (3) were known to Wronski, *Introd. a la philos. des math.*, Paris, 1811.

$$P_1 = -\frac{a_1}{a_0}, \quad P_2 = \frac{1}{a_0^2}\begin{vmatrix} a_1 & a_0 \\ a_2 & a_1 \end{vmatrix}, \quad P_3 = -\frac{1}{a_0^3}\begin{vmatrix} a_1 & a_0 & 0 \\ a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 \end{vmatrix}, \text{ etc.} \quad \ldots\ldots(3)$$

Now, since $\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix} = a_1^2 - a_0 a_2$, we see that the first two terms of the series (2) are equivalent to the single term

$$-\frac{a_0 a_1}{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}} \quad \text{or} \quad \frac{P_1}{P_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Moreover, by Jacobi's theorem on the minors of the adjugate, we have

$$\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}^2 - a_0^2 \begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix} = a_1 \begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix},$$

and this shows that the term (4), together with the third term of the series (2), is equal to .

$$-a_0 \frac{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}} \quad \text{or} \quad \frac{P_2}{P_3}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Again, by Jacobi's theorem we have

$$\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}^2 - a_0^3 \begin{vmatrix} a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \end{vmatrix},$$

and this shows that the term (5), together with the fourth term of the series (2) is equal to

$$-a_0 \frac{\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \end{vmatrix}} \quad \text{or} \quad \frac{P_3}{P_4},$$

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which is therefore equal to the sum of the first four terms of the series (2).

Proceeding in this way, we see that the sum of the first s terms of the series (2) is equal to P_{s-1}/P_s.

If now for simplicity we consider the case when $n=2$, so that there are only two roots x_1 and x_2, of which we shall suppose x_1 to have the smaller modulus, we have

$$\frac{P_{s-1}}{P_s} = \frac{\dfrac{1}{x_1^{s-1}} + \dfrac{1}{x_1^{s-2}\,x_2} + \dfrac{1}{x_1^{s-3}\,x_2^2} + \ldots + \dfrac{1}{x_2^{s-1}}}{\dfrac{1}{x_1^{s}} + \dfrac{1}{x_1^{s-1}\,x_2} + \dfrac{1}{x_1^{s-2}\,x_2^2} + \ldots + \dfrac{1}{x_2^{s}}}$$

$$= x_1 \frac{1 + \dfrac{x_1}{x_2} + \dfrac{x_1^2}{x_2^2} + \ldots + \dfrac{x_1^{s-1}}{x_2^{s-1}}}{1 + \dfrac{x_1}{x_2} + \dfrac{x_1^2}{x_2^2} + \ldots + \dfrac{x_1^{s-1}}{x_2^{s-1}} + \dfrac{x_1^s}{x_2^s}},$$

and since $\left|\dfrac{x_1}{x_2}\right| < 1$, this gives at once

$$Lt_{s\to\infty} \frac{P_{s-1}}{P_s} = x_1 .$$

Similar reasoning leads to the same result when $n > 2$.

Thus the sum of the first s terms of the series is equal to P_{s-1}/P_s, which, as s increases indefinitely, tends to the limit x_1, where x_1 is that root of equation (1) which has the smallest modulus: which establishes the theorem.