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2-SUBNORMAL QUADRATIC OFFENDERS AND OLIVER'S p-GROUP CONJECTURE

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Abstract Bob Oliver conjectures that if p is an odd prime and S is a finite p-group, then the Oliver subgroup $\mathfrak{X}(S)$ contains the Thompson subgroup $J_{\mathrm{e}}(S)$. A positive resolution of this conjecture would give the existence and uniqueness of centric linking systems for fusion systems at odd primes. Using the ideas and work of Glauberman, we prove that if $p \geqslant 5$, G is a finite p-group, and V is an elementary abelian p-group which is an F-module for G, then there exists a quadratic offender which is 2-subnormal (normal in its normal closure) in G. We apply this to show that Oliver's Conjecture holds provided that the quotient $G = S/\mathfrak{X}(S)$ has class at most $\log_2(p-2)+1$, or $p \geqslant 5$ and G is equal to its own Baumann subgroup.

Keywords: p-group; quadratic offender; p-local finite group

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1. Introduction

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. If $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G, then Broto *et al*. show [**1**] that one can associate a classifying space to $\mathcal{F}$ whose completion at p has the homotopy type of the completion of BG at p. The construction of such a classifying space is based upon a centric linking system associated to $\mathcal{F}$. But, for arbitrary $\mathcal{F}$, it is an open question as to whether an associated centric linking system (and thus a suitable classifying space for $\mathcal{F}$) exists.† See the survey article [**2**] for more details.

In the course of proving the Martino–Priddy Conjecture for odd primes [**11**], Oliver defines a certain subgroup $\mathfrak{X}(S)$ of S, now called the Oliver subgroup. He shows that if p is odd and the Thompson subgroup $J_{\mathrm{e}}(S)$ is contained in $\mathfrak{X}(S)$, then the homological obstruction to the existence and uniqueness of a centric linking system for $\mathcal{F}$ vanishes.

In this paper we investigate Oliver's (purely group theoretic) conjecture that this inclusion $J_{\mathrm{e}}(S) \leqslant \mathfrak{X}(S)$ always holds for odd p and prove it for p-groups whose quotient $G = S/\mathfrak{X}(S)$ has small nilpotence class.

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† Note added in proof: this question has now been resolved in the affirmative by Andrew Chermak [**3**].

Theorem A. *Let p be an odd prime, and suppose S is a finite p-group such that $G = S/\mathfrak{X}(S)$ has nilpotence class at most $\log_2(p-2)+1$. Then $J_e(S) \leqslant \mathfrak{X}(S)$, so Oliver's Conjecture holds for S.*

Green *et al*. show in [**7**] that $J_e(S) \leqslant \mathfrak{X}(S)$ if the quotient G is metabelian, of maximal class or of p-rank at most p. They also settle the case where G has class at most 4, so the above result is known for $p \leqslant 17$. To our knowledge though, Theorem A provides the first non-constant lower bound on the class of a counter-example to Oliver's Conjecture.

The *Baumann subgroup* (see [**10**]) of a finite p-group G is defined as $\mathrm{Baum}(G) = C_G(\Omega_1 Z(J_e(G)))$. In particular, $\mathrm{Baum}(G) \geqslant J_e(G)$. We define in § 2 the *$k$th Oliver subgroup* for $3 \leqslant k \leqslant p$ in direct analogy with $\mathfrak{X}(S)$, which gives a chain of characteristic, self-centralizing subgroups $\mathfrak{X}_3(S) \leqslant \mathfrak{X}_4(S) \leqslant \cdots \leqslant \mathfrak{X}_p(S)$ of S. The last one, $\mathfrak{X}_p(S)$, is the original Oliver subgroup $\mathfrak{X}(S)$. We prove the following.

Theorem B. *Let $p \geqslant 5$ and $3 \leqslant k \leqslant p$. Suppose S is a finite p-group, and set $G = S/\mathfrak{X}_k(S)$. If $\mathrm{Baum}(G) = G$, then $J_e(S) \leqslant \mathfrak{X}_k(S)$. In particular, taking $k = p$, Oliver's Conjecture holds for S.*

Perhaps one reason there has not been a general attack on Oliver's Conjecture to date lies in its resistance to any kind of inductive argument. The strategy for proving Theorem B gives some hint of how such an inductive argument may possibly proceed, at least for $p \geqslant 5$. See Lemma 5.2 and § 6 for speculation on this matter.

In general, we take the point of view of Green *et al*. in [**6**], where they reformulate Oliver's Conjecture in terms of a statement about representations of the quotient group G over $\mathbb{F}_p$. In particular, in a putative counter-example to Oliver's Conjecture, $V = \Omega_1 Z(\mathfrak{X}(S))$ is an F-module for G.

Our methods rely on ideas of Glauberman (G. Glauberman, personal communication, 2008) and a modification of Glauberman's generalization of Thompson's Replacement Theorem in [**4**]. For $p \geqslant 5$ and V an F-module for G, the modification gives rise to the existence of 2-subnormal quadratic offenders in G, and we use this to prove Theorems A and B. Recall that a subgroup H of G is 2-*subnormal* if it is normal in a normal subgroup of G or, equivalently, normal in its normal closure in G.

Theorem C. *Suppose $p \geqslant 5$. Let V be an elementary abelian p-group which is an F-module for the p-group G. Then there exists a quadratic offender on V in G which is normal in its normal closure in G.*

The organization of the paper is as follows. In § 2, we recap definitions and outline the reformulation of Oliver's Conjecture due to Green *et al*. Section 3 contains a couple of elementary lemmas needed for § 4, where we prove Theorem C. We use Theorem C in § 5 to produce in a counter-example an elementary abelian subgroup generated by quadratic elements, and we leverage this subgroup to prove Theorems A and B. In the final section we record an observation which could allow for an inductive approach to the conjecture.

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2. Definitions and preliminaries

Let p be an odd prime. We outline in this section the module-theoretic version of Oliver's Conjecture from [**6**].

Definition 2.1. Let $3 \leqslant k \leqslant p$. The *kth Oliver subgroup* of S is the largest subgroup $\mathfrak{X}_k(S)$ such that there exists a series

$$1 = Q_0 < Q_1 < \cdots < Q_{n-1} < Q_n = \mathfrak{X}_k(S)$$

with each Q_i normal in S and with the property that for all $1 \leqslant i \leqslant n$,

$$[\Omega_1(C_S(Q_{i-1})), Q_i; k-1] = 1.$$

Taking $k = p$ in the definition, we get Oliver's original definition, and so $\mathfrak{X}_p(S) = \mathfrak{X}(S)$. For each k, $\mathfrak{X}_k(S)$ is a characteristic subgroup of S. Furthermore, as is clear from the definition, $\mathfrak{X}_k(S)$ contains every normal subgroup of S of nilpotence class at most $k-2$. In particular, it contains every normal abelian subgroup of S, and so $C_S(\mathfrak{X}_k(S)) = Z(\mathfrak{X}_k(S))$ (that is, $\mathfrak{X}_k(S)$ is self-centralizing). A recent preprint from Green *et al*. [**8**] independently calls attention to $\mathfrak{X}_3(S)$, which is their $\mathcal{Y}(S)$.

Recall that the *Thompson subgroup* is the subgroup $J_{\mathrm{e}}(S)$ of S generated by the elementary abelian subgroups of maximum order. We will keep the e subscript throughout, since in § 4 we will be working with (not necessarily elementary) abelian subgroups of maximum order. In [**11**, Conjecture 3.9], Oliver poses the following.

Conjecture 2.2 (Oliver). *Let S be a finite p-group with p odd. Then $J_{\mathrm{e}}(S) \leqslant \mathfrak{X}(S)$.*

Perhaps the most effective way of viewing these Oliver subgroups is by way of the action of the quotient group $G := S/\mathfrak{X}_k(S)$ on $V := \Omega_1 Z(\mathfrak{X}_k(S))$. If the kth Oliver subgroup is a proper subgroup of S, then, for each element $z \in S$ such that $z\mathfrak{X}_k(S)$ is an element of order p in $Z(G)$, the subgroup $\langle z\rangle\mathfrak{X}_k(S)$ is normal in S and properly contains $\mathfrak{X}_k(S)$. Thus, by maximality of the Oliver subgroup,

$$[V, \mathfrak{X}_k(S)\langle z\rangle; k-1] \neq 1.$$

Since $\mathfrak{X}_k(S)$ and V commute, this means that $[V, z; k-1] \neq 1$. In other words, $z\mathfrak{X}_k(S)$ acts on V with minimum polynomial of degree at least k.

This motivates the following definition.

Definition 2.3. Suppose that G is a finite p-group and V is an elementary abelian p-group on which G acts. Then V is said to be a *PS-module of degree* k for G if each $1 \neq z \in \Omega_1 Z(G)$ has minimum polynomial of degree at least k. We drop the degree qualifier if $k = p$, and say V is a PS-module if it is a PS-module of degree p.

Thus, whenever $\mathfrak{X}_k(S)$ is a proper subgroup of S, V is a PS-module of degree k for G. Conversely, if G is any finite p-group and V is a PS-module of degree k for G, then $\mathfrak{X}_k(V \rtimes G) = V$.

Let us examine what happens if Oliver's Conjecture fails. If $J_{\mathrm{e}}(S)$ is not contained in $\mathfrak{X}_k(S)$, then there is an elementary abelian subgroup $A \leqslant S$ of maximum order not

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contained in $\mathfrak{X}_k(S)$. Setting $E = A\mathfrak{X}_k(S)/\mathfrak{X}_k(S) \leqslant G$, this means by maximality of A that $|E|\,|C_V(E)|$ has order at least $|V|$. This leads us to the next definition and to the reformulation of Oliver's Conjecture due to Green *et al*. in [**6**, Theorem 1.2].

Definition 2.4. Let G be a finite p-group and V an elementary abelian p-group on which G acts faithfully. Then V is an *F-module* for G if there exists a non-trivial elementary abelian subgroup E of G such that $|E|\,|C_V(E)| \geqslant |V|$. In this case, we call E an *offender* and say that E *offends* on V.

Theorem 2.5. *Let p be an odd prime. Oliver's Conjecture holds if and only if for every finite p-group G, no PS-module for G is an F-module.*

First appearing in John Thompson's N-group paper as obstructions to factorizations in 2-constrained groups, the properties of F-modules are by now well known. In particular, as proved by Thompson in his Replacement Theorem, if there exists an offender $E \leqslant G$ on V, then one may choose E to act *quadratically*, that is, satisfying $[V, E, E] = 1$ but not centralizing V. In particular, the nonidentity elements of E are *quadratic elements*, that is, they have quadratic minimum polynomial on V. Recall that because of the identity $(x-1)^p = x^p - 1$ in characteristic p, a quadratic element is of order p provided G is faithful on V.

When $3 \leqslant k \leqslant p$ and $J_{\mathrm{e}}(S)$ is not contained in $\mathfrak{X}_k(S)$, the existence of quadratic offenders in $G = S/\mathfrak{X}_k(S)$ on $V = \Omega_1 Z(\mathfrak{X}_k(S))$ portends the presence of quadratic elements in $Z(G)$ and leads the way to a possible contradiction. Indeed, many of the proofs for special classes of p-groups have gone this way. So it is quite possible that if Oliver's original conjecture holds at all, it holds in the stronger form: $J_{\mathrm{e}}(S) \leqslant \mathfrak{X}_3(S)$ for all S.

In §§ 3 and 4, where the focus is on *abelian* subgroups of S of maximal order, we follow Huppert and Blackburn's treatment [**9**, pp. 19–21] of Thompson's Replacement Theorem. Then in the proof of Theorem B, we use inside the quotient group G a version [**5**, Theorem 25.2] of the Thompson Replacement Theorem stated in terms of *elementary abelian* subgroups. Here we state the following well-known preliminary lemma to the Thompson Replacement Theorem from [**9**] for use in § 4.

Lemma 2.6. *Suppose that S is a p-group and A is an abelian subgroup of S. Let v be an element of S for which $N = [v, A]$ is abelian, and put $A^* = NC_A(N)$. Then*

(a) *A^* is an abelian subgroup and $|A \cap M| \leqslant |A^* \cap M|$ for every normal subgroup M of S. In particular, $|A| \leqslant |A^*|$;*

(b) *if also $1 = Z_0 \leqslant Z_1 \leqslant \cdots \leqslant Z_n = S$ is a central series of S such that $|A \cap Z_i| = |A^* \cap Z_i|$ for all $i = 1, \ldots, n$, then $A^* = A$.*

3. Product subgroups

Let V be an elementary abelian p-group. Suppose G is a p-group which acts faithfully on V, and set $S = V \rtimes G$. Let $\pi\colon S \to G$ denote the canonical projection.

In this section we make a definition and collect a couple of elementary lemmas needed for carrying out a modification of Glauberman's generalization of the Thompson Replacement Theorem in the next section.

Definition 3.1. Let B be a subgroup of S. Define $B_\times$ to be the subgroup $(B \cap V)\pi(B)$ of S. We say B is a *product subgroup* if $B = B_\times$.

The following lemma is clear from the definitions, but is included here for reference.

Lemma 3.2. *Let B be a subgroup of S. Then we have the following:*

(a) $(B_\times)_\times = B_\times$;

(b) $|B| = |B_\times|$;

(c) $B = B_\times$ *if and only if* $\pi(B) \leqslant B$;

(d) *if* $B \leqslant V$ *or* $V \leqslant B$, *then* $B = B_\times$;

(e) *if* B *is abelian, then* $B_\times$ *is abelian.*

Lemma 3.3. *Let A and M be subgroups of S and suppose $M \leqslant V$ or $V \leqslant M$. Then $(A \cap M)_\times = A_\times \cap M$.*

Proof. Since $\pi(A \cap M) \leqslant \pi(A) \cap \pi(M)$, we always have $(A \cap M)_\times \leqslant A_\times \cap M_\times$, and by Lemma 3.2 (d), the right-hand side is $A_\times \cap M$, so it suffices to show the reverse inclusion.

If $M \leqslant V$, then $A_\times \cap M = A_\times \cap V \cap M = A \cap V \cap M = (A \cap M)_\times$, so we may assume $V \leqslant M$.

Let $s \in A_\times \cap M$, and write $s = uh$ with $u \in A \cap M \cap V$ and $h \in \pi(A) \cap \pi(M) \leqslant M$, the inclusion by Lemma 3.2 (c). For s to be in $(A \cap M)_\times$, it is enough to show that $h \in \pi(A \cap M)$. But as $h \in \pi(A)$, there exists $v \in V \leqslant M$ with $vh \in A$. So $vh \in M$ as well. Therefore, $h \in \pi(A \cap M)$, completing the proof. $\square$

4. Offenders

In this section, we shall investigate properties of offending subgroups of G, using ideas (G. Glauberman, personal communication (2008)) and results [**4**] of Glauberman, and work towards the proof of Theorem C.

Given an elementary abelian p-group V and a p-group G acting faithfully on V, we form the semidirect product $S = V \rtimes G$, as before.

Denote by $\mathcal{A}(S)$ the set of abelian subgroups of S of maximum order, and set

$$\mathcal{A}_\times(S) = \{A \in \mathcal{A}(S) \mid A = A_\times \text{ and } A \nleqslant V\}.$$

Definition 4.1. Let

$$\mathcal{S} : 1 = Z_0 \leqslant Z_1 \leqslant \cdots \leqslant Z_n = S$$

be a central series of S. For abelian subgroups A and B of S, we say that $A \leqslant_{\mathcal{S}} B$ if

$$|A| = |B| \quad \text{and} \quad |A \cap Z_i| \leqslant |B \cap Z_i| \quad \text{for all } 1 \leqslant i \leqslant n.$$

If $A \leqslant_{\mathcal{S}} B$ and, furthermore, $|A \cap Z_i| < |B \cap Z_i|$ for some i, we say that $A <_{\mathcal{S}} B$. Let $\mathcal{C}$ be a collection of abelian subgroups of S. We say that A is *maximal in $\mathcal{C}$ with respect to $\mathcal{S}$* if $A \in \mathcal{C}$ and there does not exist $B \in \mathcal{C}$ with $A <_{\mathcal{S}} B$.

As a consequence of the Thompson Replacement Theorem, there exists $A \in \mathcal{A}(S)$ such that, for any abelian subgroup B of S which is normalized by A, we have that A in turn is normalized by B. The corollary to [**4**, Theorem 1] extends this statement for odd primes to include B of nilpotence class at most 2 by choosing A to be maximal in $\mathcal{A}(S)$ with respect to any fixed central series of S. Furthermore, for $p \geqslant 5$, Glauberman obtains with the same choice of A that

($*$) A is normalized by any subgroup B of S (not necessarily normalized by A) for which $A \trianglelefteq \langle A^B \rangle$ and $[A, u; 3] = 1$ for every $u \in B$.

To get a statement modulo V about offenders, we must take A not contained in V, and make sure the replacement still lies outside V. To do this, we modify Glauberman's proof of his Theorem 1 in a couple of ways to obtain a version of this last result ($*$) for $p \geqslant 5$. Firstly, because of difficulties retaining the commutator condition $[A, u; 3] = 1$ after mapping u into G, we restrict the discussion and replacement process only to elements of $\mathcal{A}_\times(S)$ and require that B be a product subgroup of S. Secondly, we require that A is maximal with respect to a central series which contains our distinguished subgroup V as a member. Since we are just concerned with what happens modulo V, this is harmless and facilitates verification that our replacement subgroup A^* truly is greater than A in the above ordering.

We begin by showing that $\mathcal{A}_\times(S)$ is non-empty and proving a product subgroup version of a consequence of the Thompson Replacement Theorem.

Lemma 4.2. *The following hold.*

(a) *If V is an F-module for G, then $\mathcal{A}_\times(S)$ is not empty.*

(b) *Let $\mathcal{S}$ be a central series of S. Suppose A is maximal in $\mathcal{A}_\times(S)$ with respect to $\mathcal{S}$. Then V normalizes A, and so $[V, A, A] = 1$.*

Proof. Suppose V is not an element of $\mathcal{A}(S)$. If $A \in \mathcal{A}(S)$, then in this case $|A_\times| = |A| > |V|$, and $A_\times$ is a product subgroup by Lemma 3.2. Thus, $A_\times \in \mathcal{A}_\times(S)$.

Suppose that $V \in \mathcal{A}(S)$. By assumption, $S = V \rtimes G$ and V is an F-module for G. Choose a non-trivial elementary abelian subgroup $E \leqslant G$ such that $|E|\,|C_V(E)| \geqslant |V|$. Then $A := C_V(E)E \in \mathcal{A}(S)$ by maximality of $|V|$, and A is a product subgroup. Since $E \neq 1$, $A \in \mathcal{A}_\times(S)$, proving (a).

Suppose A is maximal in $\mathcal{A}_\times(S)$ with respect to $\mathcal{S}$. Let $E = A \cap G$, and note that $E \neq 1$ by definition. Set $D = A \cap V = C_V(A)$.

Suppose V does not normalize A and let $M = N_V(A)$, which is a proper subgroup of V. Then M and V are normal subgroups of VA. Thus, $1 \neq V/M \trianglelefteq VA/M$ and so $V/M \cap Z(VA/M) \neq 1$. Let $v \in V - M$ such that v maps into $Z(VA/M)$ in the quotient VA/M. Then $N := [v, A] \leqslant M$ is abelian and normalizes A. Furthermore, N is not contained in $D = C_V(A) < A$ since v does not normalize A.

Thus, $D < ND \leqslant M < V$. Set $A^* = NC_A(N)$. Since $[D, N] = 1$ and $A = D \times E$, we get $A^* = NDC_E(N)$ so that A^* is a product subgroup. Then, as $A^* \cap V = ND > D = A \cap V$, we have that $A^* \neq A$. By Lemma 2.6 (a), A^* is abelian and $|A^*| \geqslant |A|$. First, this means that $A^* \in \mathcal{A}(S)$. Second, we get $A^* \cap G \neq 1$ and therefore $A^* \in \mathcal{A}_\times(S)$, since $A^* \cap V = ND$ is proper in V and $|V| \leqslant |A| = |A^*|$. Also, $A \leqslant_{\mathcal{S}} A^*$ by Lemma 2.6 (a). But $|A \cap Z_i| < |A^* \cap Z_i|$ for some i by Lemma 2.6 (b). This contradicts the maximality of A, and shows that V does indeed normalize A. Thus, we also have $[V, A, A] \leqslant [A, A] = 1$. □

We now assume for the remainder of this section that $p \geqslant 5$.

Definition 4.3. Given a subgroup A of S, let $\mathcal{B}_A$ be the set of all $B \leqslant S$ such that $B = B_\times$, $A \trianglelefteq \langle A^B \rangle$ and $[A, u; 3] = 1$ for all $u \in B$.

Theorem 4.4. *Suppose $\mathcal{S}_V$ is a central series of S which passes through V and A is maximal in $\mathcal{A}_\times(S)$ with respect to $\mathcal{S}_V$. Then A is normalized by every member of $\mathcal{B}_A$.*

Proof. Suppose the contrary, and choose $B \in \mathcal{B}_A$ such that B does not normalize A. By Lemma 4.2 (b), V normalizes A. As $B = B_\times = (B \cap V)(B \cap G)$, there exists $b \in B \cap G$ which does not normalize A. Set $T = \langle A, b \rangle$ and $\hat{A} = \langle A^T \rangle$. Under this set-up, Glauberman shows (see the last paragraph of step 1 in [**4**, p. 320]) that

$$\hat{A} = \langle A, A^b, A^{b^2} \rangle \quad \text{and} \quad \hat{A} \text{ has class at most 3.}$$

Since $p \geqslant 5$ and the class of $\hat{A}$ is at most 3, we view $\hat{A}$ as a Lie ring by the Lazard Correspondence [**4**, Theorem 3.4]. Let α be the map on $\hat{A}$ which is conjugation by b, and let $\delta = \log(\alpha)$. By [**4**, pp. 320–323], we have that $\delta^3 = 0$,

$$\delta = (\alpha - 1) - \tfrac{1}{2}(\alpha - 1)^2 \text{ is a derivation of } \hat{A},$$
$$\hat{A} = A + \delta(A) + \delta^2(A),$$

and

$$A \geqslant Z(\hat{A}).$$

Set

$$A_1 = \{a_1 - 2\delta(a_2) \mid a_1, a_2 \in A \text{ and } \delta(a_1) - \delta^2(a_2) \in Z(\hat{A})\}$$

and

$$A^* = A_1 + \delta^2(A).$$

Then Glauberman shows by way of a Lie ring theoretic result [**4**, Theorem 3.2] that, as groups, A^* is abelian, $|A| = |A^*|$ and $A <_{\mathcal{S}} A^*$ for every central series $\mathcal{S}$ of S. Set $A^*_\times = (A^*)_\times$. Then, by Lemma 3.2 (b), $|A^*_\times| = |A^*| = |A|$ and by Lemma 4.5, below, $A^* \nleqslant V$ so $A^*_\times \nleqslant V$. Thus, $A^*_\times \in \mathcal{A}_\times(S)$. Since every member M of $\mathcal{S}_V$ satisfies $M \leqslant V$ or $V \leqslant M$, we have by Lemma 3.3 that $A^* \leqslant_{\mathcal{S}_V} A^*_\times$. Therefore, $A <_{\mathcal{S}_V} A^*_\times$. This contradicts the assumption that A is maximal in $A_\times(S)$ with respect to $\mathcal{S}_V$ and completes the proof. □

Lemma 4.5. *Let A, δ, $\hat{A}$ and A^* be as in the proof of Theorem 4.4, with $\hat{A}$ and its subgroups viewed as Lie rings. Let $D = A \cap G$, and set $\hat{D} = D + \delta(D) + \delta^2(D)$. Then $A^* \cap \hat{D} \neq \{0\}$, and therefore $A^* \cap G \neq 1$.*

Proof. Note that for the proof we do not need to knowledge that $\hat{D}$ is closed under the Lie bracket, but only that $\hat{D}$ is a abelian group (as a subspace of a Lie ring).

Recall that $A^* = A_1 + \delta^2(A)$, where

$$A_1 = \{a_1 - 2\delta(a_2) \mid a_1, a_2 \in A \text{ and } \delta(a_1) - \delta^2(a_2) \in Z(\hat{A})\}.$$

As $D = A \cap G$ is non-trivial, we have that $\hat{D}$ is non-trivial. Since $\delta^2(D)$ is contained in A^*, the moment $\delta^2(a) \neq 0$ for some $a \in D$, we are finished. So assume that $\delta^2(D) = 0$.

Let a be a non-zero element of D. Suppose first that $\delta(a) = 0$. Let $a_2 = 0$. Then $\delta(a) - \delta^2(a_2) = 0 \in Z(\hat{A})$, and so $a = a - 2\delta(a_2) \in A_1 \leqslant A^*$, and $a \in \hat{D}$ as well.

Suppose $\delta(a) \neq 0$. If $a \in Z(\hat{A})$, then $\delta(a) \in Z(\hat{A})$ as well, by [**4**, Theorem 3.2 (b)]. Taking $a_2 = 0$ again, we have $\delta(a) - \delta^2(a_2) \in Z(\hat{A})$, giving $a \in A^* \cap \hat{D}$. Suppose $a \notin Z(\hat{A})$. By [**4**, Theorem 3.2 (c)], either $\delta^2(a) \in A^* - A$ or $\delta(a) \in A^* - A$. The former cannot hold as $\delta^2(a) = 0$, and the latter gives $\delta(a) \in A^* \cap \hat{D}$.

This shows that $A^* \cap \hat{D} \neq \{0\}$. However, recall that δ is a polynomial in α, which is an automorphism induced by conjugation by an element of G. It follows that $\hat{D} \leqslant G$, and $A^* \cap G$ is non-trivial as claimed. □

The following is basically the content of [**4**, Theorem 4.1]. However, since we have restricted the collections from which A and B are chosen in proving Theorem 4.4, the statement and argument have to be modified. We present a complete proof in our case for the convenience of the reader.

Theorem 4.6. *Suppose A and B are abelian product subgroups of S, both normalized by V. Assume that A is normalized by every member of $\mathcal{B}_A$ and B is normalized by every member of $\mathcal{B}_B$. Then A and B normalize each other.*

Proof. We proceed by induction on the order of G. If $G = 1$, then $S = V$, and the statement is trivial. Suppose that $G > 1$.

Assume first that $A \cap G = G$. Then G is abelian and so the conjugation action of S is trivial modulo V. Hence, $\langle A^B \rangle \leqslant VA$, so A is normal in $\langle A^B \rangle$ because V normalizes A. As $[S, S] \leqslant V$ and V normalizes B, we also have $[A, u, u, u] \leqslant [V, u, u] \leqslant [B, u] = 1$ for all $u \in B$. So $B \in \mathcal{B}_A$. Because G is abelian, we still have $\langle B^A \rangle \leqslant VB$, and the same argument applies to give $A \in \mathcal{B}_B$. Thus, A and B normalize each other. By symmetry, the same conclusion holds in the case when $B \cap G = G$.

So, we may assume that $A \cap G$ and $B \cap G$ are proper in G. Let M_0 be a maximal subgroup of G containing $A \cap G$, and set $M = VM_0$. Then $M \trianglelefteq S$, so

$$A \leqslant \langle A^B \rangle \leqslant \langle M^S \rangle = M.$$

If $x \in B$, then $A^x = A^{\pi(x)}$, since V normalizes A. Hence, A^x is a product subgroup. Furthermore, it is easy to see that A^x is normalized by every member of $\mathcal{B}_{A^x} = (\mathcal{B}_A)^x$.

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By the inductive hypothesis, A^x normalizes A. It follows that $A \trianglelefteq \langle A^B \rangle$ and

$$[B, A; 3] \leqslant [\langle A^B \rangle, A, A] \leqslant [A, A] = 1.$$

By symmetry $[A, B; 3] = 1$. This means that $B \in \mathcal{B}_A$. Hence, B normalizes A by hypothesis. By symmetry, A normalizes B. □

Now we are in a position to prove Theorem C.

Proof of Theorem C. Let $S = V \rtimes G$. By Lemma 4.2 (a), $\mathcal{A}_\times(S)$ is non-empty. Let $\mathcal{S}$ be a central series which passes through V and let A be an element of $\mathcal{A}_\times(S)$ maximal in $\mathcal{A}_\times(S)$ with respect to $\mathcal{S}$. Set $E := A \cap G \neq 1$. As $A \in \mathcal{A}(S)$, we have $|V| \leqslant |A|$. Since A is a product subgroup, $|V| \leqslant |A| = |A \cap V|\,|A \cap G| = |C_V(E)|\,|E|$. So E offends on V. By Lemma 4.2 (b), V normalizes A, and E acts quadratically on V. Let B be a conjugate of A. Then B is a product subgroup because V normalizes A. Thus, B is maximal in $\mathcal{A}_\times(S)$ with respect to $\mathcal{S}$. By Theorem 4.4, B is normalized by each element in $\mathcal{B}_B$. By Theorem 4.6, A is normalized by its S-conjugates. Therefore, E is normalized by its G-conjugates. This is another way of saying that $E \trianglelefteq \langle E^G \rangle$. □

5. Oliver's Conjecture

In this section we leverage the existence of a 2-subnormal quadratic offender guaranteed in Theorem C to prove Theorems A and B. First we state the key lemma of Green *et al.* in [**6**, Lemma 4.1].

Lemma 5.1. *Suppose that V is an elementary abelian p-group and G is a p-group acting faithfully on V. Let a be an element of G acting quadratically on V. Let $x \in G$ and suppose that $c := [a, x] \neq 1$ centralizes both a and x. Then c acts quadratically on V.*

Lemma 5.1 allows one, under certain conditions, to begin with a quadratic element and descend the central series of G, finding quadratics further and further down. This process stops when at some point, and, having some quadratic element a, one can find no element $x \in G$ for which $[a, x] \neq 1$ centralizes a, and $[a, x, x] = 1$. The proof of the next lemma uses the existence of a 2-subnormal offender to produce normal abelian subgroups of G that act as a container for descent, and identifies a normal abelian subgroup W at the bottom when the descent process terminates.

Lemma 5.2. *Suppose that $p \geqslant 5$ and $3 \leqslant k \leqslant p$. Suppose that S is a finite p-group such that $J_e(S)$ is not contained in $\mathfrak{X}_k(S)$. Set $G = S/\mathfrak{X}_k(S)$, which acts faithfully on $V = \Omega_1 Z(\mathfrak{X}_k(S))$. Then there exists a normal elementary abelian subgroup $W = \langle a^G \rangle$ of G, generated by the G-conjugates of a quadratic element, such that for each $x \in G$ either $[W, x] = 1$ or $[W, x, x] \neq 1$.*

Proof. Note that $J_e(V \rtimes G)$ is not contained in $\mathfrak{X}_k(V \rtimes G) = V$, and so we assume that $S = V \rtimes G$. Furthermore, V is both a PS-module of degree k and an F-module for G. Choose by Theorem C a quadratic offender $E \leqslant G$ which is normalized by its

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conjugates in G. Let N be the normal closure $\langle E^G \rangle$ of E in G, so that $E \trianglelefteq N$. Then $E \cap \Omega_1 Z(N) \neq 1$, and hence there are quadratic elements in $\Omega_1 Z(N)$. For each $i \geqslant 0$, set $W_i = [\Omega_1 Z(N), G; i]$. Let j be the greatest integer such that there exists a quadratic element in W_j. Choose such a quadratic element $a \in W_j$. Note that W_j is not in the centre of G, as $Z(G)$ has no quadratic elements. Set $W = \langle a^G \rangle$. Then W is a normal elementary abelian subgroup of G generated by quadratic elements.

Let $x \in G$. Suppose that x does not centralize W, but that $[W, x, x] = 1$. Then there exists a G-conjugate b of a such that $[b, x] \neq 1$, but $[b, x, x] = 1$. Setting $c = [b, x]$, we have that c centralizes both b and x. Since b is quadratic, c is as well, by Lemma 5.1. Thus, $c \in W_{j+1}$ and quadratic. This contradicts the maximality of j. Therefore, for every $x \in G$, we have either $[W, x] = 1$ or $[W, x, x] \neq 1$. $\square$

We have recently learned that Green *et al*. have independently obtained a similar subgroup (generated by *last quadratics* in the terminology of their paper) as the W in Lemma 5.2 under only the assumption that G has quadratic elements on V; see [**8**, § 8].

Proof of Theorem B. Suppose the contrary and, as in the proof of Lemma 5.2, let $S = V \rtimes G$ be a counter-example. Let $W = \langle a^G \rangle$ be as in Lemma 5.2. Let $\mathcal{A}_{\mathrm{e}}(G)$ denote the set of elementary abelian subgroups of G of maximum order.

By the Thompson Replacement Theorem [**5**, Theorem 25.2], there exists $A^* \in \mathcal{A}_{\mathrm{e}}(G)$ acting quadratically on W, or else $[W, A] = 1$ for all $A \in \mathcal{A}_{\mathrm{e}}(G)$. The first possibility is ruled out by Lemma 5.2. It follows that $[W, A] = 1$, whence $W \leqslant A$ for all $A \in \mathcal{A}_{\mathrm{e}}(G)$. So $W \leqslant \Omega_1 Z(J_{\mathrm{e}}(G)) = \Omega_1 Z(G)$ as $\mathrm{Baum}(G) = G$. Since W contains quadratic elements and V is a PS-module of degree $k \geqslant 3$ for G, this is a contradiction. $\square$

We need the following basic lemma for the proof of Theorem A.

Lemma 5.3. *Suppose G is a p-subgroup of $\mathrm{GL}(n, p)$. Assume that a and b are commuting elements of G whose minimum polynomials on V have degree at most k and l, respectively. Then ab has minimum polynomial of degree at most $k + l - 1$. In particular, if $a_1, \ldots, a_k$ are commuting quadratic elements, then $a_1 \cdots a_k$ has minimum polynomial of degree at most $k + 1$.*

Proof. We have $ab - 1 = (a - 1) + (b - 1) + (a - 1)(b - 1)$. Raise the right-hand side to the $(k + l - 1)$th power, and distribute. Since a and b commute, each term is of the form $(a - 1)^i (b - 1)^j$ with $i + j \geqslant k + l - 1$. As $(a - 1)^k = 0$ and $(b - 1)^l = 0$, either $(a - 1)^i$ or $(b - 1)^j$ must be zero in the respective term. Therefore, $(ab - 1)^{k+l-1} = 0$. The last statement follows by induction on k. $\square$

Proof of Theorem A. The conjecture has been shown in [**7**, Theorem 5.2] to hold when $S/\mathfrak{X}(S)$ has nilpotence class at most 4, so the result is known for small primes. Therefore, we may assume that $p \geqslant 5$.

Suppose the contrary and choose a counter-example $S = V \rtimes G$ with V both a PS-module and an F-module for G. Let $W = \langle a^G \rangle$ be the normal elementary abelian subgroup of G guaranteed by Lemma 5.2.

As $W \trianglelefteq G$, there exists an integer $k > 0$ such that $1 \neq [W, G; k] \leqslant \Omega_1 Z(G)$. Hence, there exists a G-conjugate b of a, and elements $x_1, \ldots, x_k$ of G, such that

$$1 \neq [b, x_1, \ldots, x_k] \in \Omega_1 Z(G).$$

Applying (k times) the standard identity $[x, y] = x^{-1}x^y$, one sees that

$$z := [b, x_1, \ldots, x_k] = \prod_{j=0}^{k} \prod_{(i_1,\ldots,i_j)\in X_j} (b^{(-1)^{j+1}})^{x_{i_1}\cdots x_{i_j}},$$

where X_j is the set of strictly increasing sequences of length j from $\{1, \ldots, k\}$. It follows that z is a product of

$$\sum_{j=0}^{k} \binom{k}{j} = 2^k$$

commuting conjugates of b and b^{-1}. By Lemma 5.3, the degree of the minimum polynomial of z on V is at most $2^k + 1$. On the other hand, the degree of the minimum polynomial of z is p, because V is a PS-module for G. Thus, $p < 2^k + 2$, and hence $\log_2(p-2) < k$. By definition of k, we have that the class of G is at least $k + 1$. Therefore, the class of G is strictly larger than $\log_2(p-2)+1$, and this completes the proof. □

6. Speculation regarding an inductive approach

Lemma 5.2 applied with $k = 3$ gives some hint as to how an inductive approach to the conjecture might proceed, at least for $p \geqslant 5$. (In view of the subgroup generated by last quadratics in [**8**, § 8], the approach applies for all odd p.) We record here an observation that forms the main motivation for this idea.

Proposition 6.1. *Suppose that $p \geqslant 5$ and S is a p-group such that $J_{\mathrm{e}}(S)$ is not contained in $\mathfrak{X}_3(S)$. Let $G = S/\mathfrak{X}_3(S)$. Then $\mathfrak{X}_3(G)$ is a proper subgroup of G.*

Proof. As $J_{\mathrm{e}}(S)$ is not contained in $\mathfrak{X}_3(S)$, $\mathfrak{X}_3(S)$ is a proper subgroup of S. Let W be as in Lemma 5.2. Thus, W is a normal elementary abelian subgroup of G generated by quadratic elements. Set $X := C_G(W)$, and note that X is a proper normal subgroup of G. We will show that $\mathfrak{X}_3(G) \leqslant X$, and this will settle the claim. Suppose to the contrary that $\mathfrak{X}_3(G)$ is not contained in X. Let $1 = Q_0 < Q_1 < \cdots < Q_n = \mathfrak{X}_3(G)$ be a series of normal subgroups of G such that

$$[\Omega_1(C_G(Q_{i-1})), Q_i, Q_i] = 1$$

for each i. Let j be the greatest integer such that $Q_{j-1} \leqslant X$. Let $x \in Q_j - X$. Then as $W \leqslant \Omega_1 Z(X) \leqslant \Omega_1(C_G(Q_{j-1}))$, we have

$$[W, x, x] \leqslant [\Omega_1(C_G(Q_{j-1})), x, x] = 1,$$

a contradiction to Lemma 5.2 because $x \notin X$. Therefore, $\mathfrak{X}_3(G) \leqslant X$, and $\mathfrak{X}_3(G)$ is a proper subgroup of G. □

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For an inductive approach, we would choose S to be minimal subject to the condition that $J_{\mathrm{e}}(S)$ is not contained in $\mathfrak{X}_3(S)$. We would then like to show that $J_{\mathrm{e}}(G)$ is not contained in $\mathfrak{X}_3(G)$ to contradict the minimality of S. By the proof of Proposition 6.1, we have that $\mathfrak{X}_3(S) \leqslant C_G(W)$. However, the content of the proof of Theorem B is that $J_{\mathrm{e}}(G)$ is also contained in $C_G(W)$. In fact, one can see this explicitly in the example of Green *et al.* in [**6**, § 5]. In this case $G \cong C_3 \wr C_3$ acting on an eight-dimensional module V, and we have that $X = W = J_{\mathrm{e}}(G) = \mathfrak{X}_3(G)$ is the base subgroup of G. Note that G is not a counter-example to Oliver's Conjecture, as can be seen from subsequent work (see [**7**, Theorem 1.1] or [**8**]). But this example shows that, taken alone, the existence of a normal elementary abelian subgroup of G generated by quadratic elements is not enough to carry out this inductive argument.

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