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ON THE RESIDUAL FINITENESS OF CERTAIN POLYGONAL PRODUCTS

BY

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ABSTRACT. We give examples to show that unlike generalized free products of groups (g.f.p.) polygonal products of finitely generated (f.g.) nilpotent groups with cyclic amalgamations need not be residually finite (${}_R\mathcal{F}$) and polygonal products of finite p-groups with cyclic amalgamations need not be residually nilpotent. However, polygonal products f.g. abelian groups are ${}_R\mathcal{F}$, and under certain conditions polygonal products of finite p-groups with cyclic amalgamations are ${}_R\mathcal{F}$.

1. **Introduction.** Polygonal products of groups were introduced by A. Karrass, A. Pietrowski and D. Solitar [6]. They studied the subgroups of these products and applied their results to the study of the Picard group $PSL(2, Z(i))$. Brunner, Frame, Lee and Wielenberg [2] made use of their results to determine all the torsion-free subgroups of finite index in the Picard group. These products are defined as follows. Let P be a polygon. Let there be given, to each vertex v of P a vertex group G_v and to each edge e an edge group G_e together with monomorphisms λ_e and ρ_e embedding G_e as a subgroup of the two vertex groups at the ends of the edge e. The polygonal product of this system is the group G with generators and relations those of the vertex groups together with the extra relations obtained by identifying $g_e\lambda_e$ and $g_e\rho_e$ for each $g_e \in G_e$. The case where the embedded subgroups at each vertex are permitted to have non-trivial intersections can be quite unpleasant (see B. H. Neumann [7], p. 525, also [8]). So we restrict ourselves to the case where at each vertex the pair of embedded subgroups has trivial intersection. Even with this restriction, the case when the polygon is a triangle can be troublesome since the polygonal product so formed may not contain the vertex groups isomorphically ([7], p. 525). Thus we deal with polygons with four or more vertices. In this case it is not difficult to see that a polygonal product may be regarded as a generalized free product. In particular, such a free product embeds the original polygon of groups isomorphically. In this paper we study the residual finiteness (${}_R\mathcal{F}$) of

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certain polygonal products. For simplicity of presentation we restrict ourselves to the case where the polygon is a square and the embedded subgroups are cyclic with trivial intersections.

In Section 3, we prove that the polygonal products of free abelian groups of finite ranks with disjoint, cyclic embedded subgroups are $_R\mathcal{F}$. We also give an example of a polygonal product of four finitely generated torsion-free nilpotent groups embedding disjoint cyclic subgroups that is not $_R\mathcal{F}$. Similar examples can be constructed for polygonal products of free groups. This contrasts with the results of G. Baumslag [1] who proved that the generalized free products of finitely generated torsion-free nilpotent groups (and also free groups) with cyclic amalgamations are always $_R\mathcal{F}$.

In Section 4, we prove that polygonal products of four finite p-groups embedding dispoint cyclic groups such that a pair of the opposite edge groups are of order p, are $_R\mathcal{F}$. In particular, polygonal products of finite p-groups embedding disjoint subgroups of order p are $_R\mathcal{F}$. In fact we conjecture that polygonal products of finite p-groups embedding disjoint cyclic groups are $_R\mathcal{F}$. So far we have been unable to prove this. Unlike generalized free products of finite p-groups amalgamating a cyclic subgroup, which are residually finite p-groups [5], whence residually nilpotent, we construct an example of a polygonal product of four dihedral groups of order 8 embedding disjoint cyclic groups which is not even residually nilpotent. This is Example 4.1. The question whether polygonal products of four finite groups embedding disjoint subgroups necessarily have a proper subgroup of finite index seems quite hard to resolve.

2. **Preliminaries.** Let P be the polygonal product of the groups A_1, $A_2, \ldots, A_n$ (vertex groups) with amalgamated subgroups $A_{i,i+1} = A_{i+1,i}$ (edge groups). Then P is said to have disjoint amalgamations if $A_{i,i-1} \cap A_{i,i+1} = 1$ where $i = 1, \ldots, n$ with $A_{n,n+1} = A_{n1}$.

We write g.f.p. to denote generalized free product and use the usual notation $A *_U B$ to denote the g.f.p. of A and B amalgamating the subgroup U.

If $G = A *_U B$ and $x \in g$, then $\|x\|$ denotes the free product length of x in G.

We let $N \triangleleft_f G$ denote that N is a normal subgroup of finite index in G.

If $\bar{G}$ is a homomorphic image of G then we use $\bar{x}$ to denote the image of $x \in G$ in $\bar{G}$.

Let G be a group and S be a subgroup of G. Then we say that G is S-separable if to each $x \in G \backslash S$ there exists $N \triangleleft_f G$ such that $x \notin SN$, where $G \backslash S$ denotes the set of elements of G deleting the elements of S.

The following theorem, which was proved implicitly by M. Hall [4], see also Burns [3], will be needed.

THEOREM 2.1. *Let G be a finite extension of a free group of finite rank and let S be a finitely generated subgroup of G. Then G is S-separable. In particular, by* [1], *this holds for the* g.f.p. *of finite groups.*

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3. **Abelian and nilpotent groups.** In this section we shall study the residual finiteness of polygonal products of finitely generated torsion free abelian groups and nilpotent groups. We show that if P is a polygonal product of four or more free abelian groups amalgamating disjoint cyclic subgroups then P is ${}_R\mathcal{F}$. On the other hand this is not true for finitely generated torsion-free nilpotent groups. Example 3.1 illustrates this.

LEMMA 3.1. *Let A be a finitely generated free abelian group and U be a subgroup of a group B, where B is U-separable. Let $G = A *_U B$. If S is a finitely generated subgroup of B such that B is S-separable then G is S-separable.*

PROOF. It is well-known that we can choose a basis $\{a_1, a_2, \ldots, a_s\}$ for A such that there exist integers $\alpha_1, \alpha_2, \ldots, \alpha_r$ such that $\{a_1^{\alpha_1}, a_2^{\alpha_2}, \ldots, a_r^{\alpha_r}\}$ is a basis for U. Choose $a_1^{t_1}a_2^{t_2} \ldots a_r^{t_r}a_{r+1}^{t_{r+1}} \ldots a_s^{t_s}$, $0 \leqq t_i < \alpha_i$ for $1 \leqq i < r$ and $t_i \in Z$ for $r + 1 \leqq i \leqq s$, as coset representatives of U in A. Let $x \in G \backslash S$.

CASE 1. $\|x\| = 0$. Then $x \in U \backslash S$. Since B is S-separable there exists $M \lhd_f B$ such that $x \notin SM$. Let $L = M \cap U$ and $N = (M \cap U) \times \langle a_{r+1}, \ldots, a_s \rangle$. Then G can be mapped onto $\bar{G} = A/N *_{U/L} B/M$. Clearly $\bar{x} \notin \bar{S} = SM/M$. Since $\bar{G}$ is a g.f.p. of finite groups, $\bar{G}$ is $\bar{S}$-separable by Theorem 2.1. It follows immediately that there exists a finite image G^* of G in which $x^* \notin S^*$. Hence G is S-separable.

CASE 2. $\|x\| = 1$. If $x \in A \backslash U$, then consider $\bar{G} = G/B^G$. Clearly $\bar{x} \neq 1$ and $\bar{S} = 1$. Since $\bar{G}$ is finitely generated abelian it follows that $\bar{G}$ is ${}_R\mathcal{F}$. Thus $\bar{G}$ is $\bar{S}$-separable. Hence, as in Case 1, G is S-separable. If $x \in B \backslash U$ we proceed as in Case 1.

CASE 3. $\|x\| > 1$. Suppose $x = y_1b_1 \ldots y_nb_n$ (other cases being almost identical) where $y_i \in A \backslash U$, $b_i \in B \backslash U$. Let $y_i = c_iu_i$ where c_i is one of the chosen coset representatives of U in A and $u_i \in U$. For each c_i choose $N_i \lhd_f \langle a_{r+1}, \ldots, a_s \rangle$ such that $c_i \notin N_iU$. Let $N = \cap_{i=1}^n N_i$. Since B is U-separable, there exists $M_i \lhd_f B$ such that $b_i \notin M_iU$ for each i. Let $M = \cap_{i=1}^n M_i$. Let $L = M \cap U$. Then G can be mapped onto $\bar{G} = A/(N \times L) *_{U/L} B/M$. Now $\bar{G}$ is a g.f.p. of finite groups in which $\|\bar{x}\| = \|x\| > 1$. This implies $\bar{x} \notin \bar{S}$. Since $\bar{G}$ is $\bar{S}$-separable, it follows, as in Case 1, that G is S-separable.

Now a group G is ${}_R\mathcal{F}$ if and only if G is S-separable for $S = 1$. Thus we have:

COROLLARY 3.2. *Let $G = A *_U B$ as given in Lemma 3.1. If B is ${}_R\mathcal{F}$ then G is ${}_R\mathcal{F}$.*

LEMMA 3.3. *Let $A = \langle a, b \rangle$ and $B = \langle c, d \rangle$ be free groups of rank 2. Let $G = A \times B$ and $S = \langle a \rangle \times \langle c \rangle$. Then G is S-separable.*

PROOF. Let $g \in G$. Then $g = w_1 w_2$ where w_1 is a word on a, b and w_2 is a word on c, d. Let $s \in S$. Then $s = a^i c^j$. Thus if $g \notin S$ then we must have one of the followings: (i) $w_1 \notin \langle a \rangle$, or (ii) $w_2 \notin \langle c \rangle$. Suppose we have Case (i). Since A is free, by Theorem 2.1, A is $\langle a \rangle$-separable. Therefore there exists a finite image $\bar{A}$ of A such that $\bar{w}_1 \notin \langle \bar{a} \rangle$. Extend this map to G by mapping B onto 1. Then clearly $\bar{g} \notin \bar{S}$. Since $\bar{G}$ is finite, it follows that G is S-separable. Case (ii) is similar.

THEOREM 3.4. *Let P be the polygonal product of the free abelian groups of finite ranks A, B, C, D with $A \cap B = U$, $B \cap C = K$, $C \cap D = V$, $D \cap A = H$, where U, K, V, H are cyclic and such that $H \cap U = U \cap K = K \cap V = V \cap H = 1$. Then P is ${}_R\mathscr{F}$.*

PROOF. Let P_0 be the "reduced" polygonal product of A_0, B_0, C_0, D_0 amalgamating the subgroups $U = \langle u \rangle$, $K = \langle k \rangle$, $V = \langle v \rangle$ and $H = \langle h \rangle$, where $A_0 = \langle h, u \rangle = H \times U$, $B_0 = \langle u, k \rangle = U \times K$, $C_0 = \langle k, v \rangle = K \times V$ and $D_0 = \langle v, h \rangle = V \times H$. Clearly $P_0 = \langle u, v \rangle \times \langle h, k \rangle$, where $\langle u, v \rangle$ and $\langle h, k \rangle$ are both free of rank 2. Since direct products of residually finite groups are again residually finite, it follows P_0 is ${}_R\mathscr{F}$. Moreover, by Lemma 3.3, P_0 is $\langle h, u \rangle$-separable. Now consider the polygonal product P_1 of A, B_0, C_0, D_0 amalgamating U, K, V, H. Then $P_1 = A *_{\langle h,u \rangle} P_0$. Since $P_0 \in {}_R\mathscr{F}$, by Corollary 3.2, $P_1 \in {}_R\mathscr{F}$. Also, by Lemma 3.3, P_0 is $\langle u, k \rangle$-, $\langle k, v \rangle$-, and $\langle v, h \rangle$-separable. Thus, by Lemma 3.1, P_1 is $\langle u, k \rangle$-, $\langle k, v \rangle$-, and $\langle v, h \rangle$-separable. Let $P_2 = B *_{\langle u,k \rangle} P_1$. Applying a similar argument as in the case of P_1, we have $P_2 \in {}_R\mathscr{F}$, and $\langle k, v \rangle$- and $\langle v, h \rangle$-separable. In the same way, $P_3 = C *_{\langle k,v \rangle} P_2$ is ${}_R\mathscr{F}$ and $\langle v, h \rangle$-separable. Finally $P_4 = D *_{\langle v,h \rangle} P_3$ is ${}_R\mathscr{F}$. But $P_4 = P$. This proves the theorem.

It is quite easy to extend Theorem 3.4 to the case where U, K, V, H are finitely generated free abelian subgroups of A, B, C, D provided $H \cap U = U \cap K = K \cap V = V \cap H = 1$.

The proof for extension to five or more groups is omitted as the details are more tedious. In the case of three abelian groups A, B, C with disjoint cyclic amalgamated subgroups, say, U, V, L, the "reduced" polygonal product P_0 given by the subgroups U, V, L turns out to be a free abelian group of rank 3 whence it is ${}_R\mathscr{F}$. Then, as above, one easily proves that the polygonal product of A, B, C with disjoint amalgamations is ${}_R\mathscr{F}$.

In [1] Baumslag showed that the g.f.p. of two finitely generated torsion-free nilpotent groups amalgamating a cyclic subgroup is ${}_R\mathscr{F}$. That Theorem 3.4 cannot be extended even to polygonal products of finitely generated torsion-free nilpotent groups of class 2 with disjoint cyclic amalgamated subgroups is surprising. This is shown by the following example.

EXAMPLE 3.1. Let $A_i = \langle a_i, b_i; [a_i, b_i, a_i] = [a_i, b_i, b_i] = 1\rangle$ for $i = 1, 2, 3, 4$. Clearly A_i is a free nilpotent group of class 2. Form the polygonal product P by letting $a_{i+1}^2 = b_i^{-1}a_i^3b_i$ for $i = 1, 2, 3, 4$ with $a_5 = a_1$. Let $x \sim y$ denote x being conjugate to y. Then $a_1^3 \sim a_2^2$, $a_2^3 \sim a_3^2$, $a_3^3 \sim a_4^2$, $a_4^3 \sim a_1^2$. This implies $a_1^{81} \sim a_1^{16}$. Let $\bar{P}$ be a finite homomorphic image of P. If $\bar{a}_1$ is of even order, say $2n$ exactly, then $\bar{a}_1^{81n} = \bar{a}_1^{16n} = 1$. But this implies that $2n|81n$ which is clearly impossible. Thus $\bar{a}_1$ must be of odd order in $\bar{P}$. Now P contains $A_1 *_{a_1^2=a_4^3} A_4$. This implies $[a_1, b_4^{-1}a_4b_4] \neq 1$. So, if P were ${}_R\mathcal{F}$ then there would exist a finite homomorphic image $\bar{P}$ in which $[\bar{a}_1, \bar{b}_4^{-1}\bar{a}_4\bar{b}_4] \neq 1$. Since $\bar{a}_1$ must be of odd order in $\bar{P}$, we must have $\bar{a}_1 \in \langle \bar{a}_1^2\rangle = \langle \bar{b}_4^{-1}\,\bar{a}_4^3\bar{b}_4\rangle$. But this implies $[\bar{a}_1, \bar{b}_4^{-1}\bar{a}_4\bar{b}_4] = 1$, a contradiction. Hence P cannot be ${}_R\mathcal{F}$.

We note that the above example can be modified to form a triangular product T of three isomorphic groups A_i so that T is not ${}_R\mathcal{F}$. The interesting thing here is that the unpleasantness mentioned in Section 1 is absent – namely the triangular amalgam underlying the product is at least isomorphically embeddable in some group – albeit not a residually finite one. Example 3.1 can also be modified to give an example of a polygonal product of four free groups with disjoint cyclic amalgamations which is not ${}_R\mathcal{F}$. We omit the details.

4. **Polygonal products of p-groups.** Polygonal products of finite p-groups could be quite unpleasant. Unlike g.f.p. of finite p-groups amalgamating a cyclic subgroup, which is a residually finite p-group [5], whence residually nilpotent, polygonal products of finite p-groups amalgamating disjoint cyclic subgroups may not even be residually nilpotent. The following is an example to illustrate this.

EXAMPLE 4.1. Let $A_i = \langle a_ib_i; a_i^4 = b_i^2 = (a_ib_i)^2 = 1\rangle$, $i = 1, 2, 3, 4$. Form the polygonal product P by identifying $[a_i, b_i]$ with b_{i+1} for $i = 1, 2, 3, 4$ with $b_5 = b_1$. Now,

$$[b_1, a_1, a_2, a_3, a_4] = [b_2, a_2, a_3, a_4] = [b_3, a_3, a_4] = [b_4, a_4] = b_1.$$

This implies $b_1 \in \Gamma_i(P)$ for each i, whence P is not residually nilpotent. We note that P is ${}_R\mathcal{F}$ by Theorem 4.4 below. We also note that the A_i's need not be 2-groups. Indeed nilpotent groups generated by a_i, b_i such that $\operatorname{ord}[a_i, b_i] = \operatorname{ord} b_i$ and $\langle [a_i, b_i] \rangle \cap \langle b_i\rangle = 1$ can be used. The resulting P will not be residually nilpotent.

We now show that the square product of finite p-groups in which one pair of the opposite edges consists of cyclic p-groups while the other pair consists of groups of order p, is ${}_R\mathcal{F}$. We first prove the following lemma.

LEMMA 4.2. *Let $G = A *_U B$ where A, B are finite p-groups and U is cyclic. Let H, K be subgroups of order p in A and B respectively such that $H \cap U = K \cap U = 1$. Let $W \lhd_f G$ such that G/W embeds the amalgam $A = (A, B; U)$.*

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If N is any normal subgroup of finite index in $H * K$ such that $N \subseteq W \cap (H * K)$, then there exists $N_G \lhd_f G$ such that $N_G \cap (H * K) = N$.

PROOF. Let $H = \langle h \rangle$, $K = \langle k \rangle$ and $U = \langle u \rangle$. Let $L = \langle h, u \rangle$ and $M = \langle k, u \rangle$. Let Φ(L) be the Frattini subgroup of L. Let θ be the canonical map L to $\bar{L} = L/\Phi(L)$. Since L is a finite p-group, by the Burnside basis theorem, $\bar{L}$ is an elementary abelian p-group. Indeed $\bar{L} = \langle \bar{h} \rangle \times \langle \bar{u} \rangle$ where $\bar{h} = h\theta$ and $\bar{u} = u\theta$. Let L_0 be the preimage of $\langle \bar{u} \rangle$ in L under θ. Since H is of order p, it follows that $L_0 \lhd L$ is of index p. Moreover $L_0 \cap H = 1$. In the same way there exists $M_0 \lhd M$ of index p such that $M_0 \cap K = 1$. Let $F = \langle L, M \rangle$. Then $F = L *_U M$. Let φ: $F \to \hat{F} = F/T$ where T is the normal closure of $\langle L_0, M_0 \rangle$ in F. Clearly $\hat{F} = \hat{H} * \hat{K}$ where $\hat{H} = H\varphi$ and $\hat{K} = K\varphi$ are isomorphic copies of H and K respectively. Consider $\hat{N} = N\varphi$ clearly $\hat{N} \lhd \hat{H} * \hat{K}$. Let N_1 be the preimage of $\hat{N}$ in F under φ. Then $N_1 \lhd_f F$. Moreover $N_1 \cap (H * K) = N$. Now let $N_2 = W \cap N_1$. Since $W \lhd_f G$, it follows that $W \cap F \lhd_f F$ whence $N_2 \lhd_f F$. It is easy to check that $N_2 \cap (H * K) = N$. Now G/W embeds A. This implies $W \cap A = W \cap B = 1$. Thus $N_2 \cap L = N_2 \cap M = 1$. Let $X = A *_U M$. Then

$$X = A *_L (L *_U M) = A *_L F.$$

Since $N_2 \cap L = 1$ and $N_2 \lhd_f F$, this implies there is a natural homomorphism ψ of X to $\bar{X} = A *_L F/N_2$. Now A and F/N_2 are both finite. Thus there exists π: $\bar{x} \to \tilde{X}$ where $\tilde{X}$ is finite and $\tilde{X}$ embeds the amalgam $(A, F/N_2; L)$. Let $N_3 = \ker \psi\pi$. Then $N_3 \lhd_f X$. Moreover $N_3 \cap F = N_2$ whence $N_3 \cap (H * K) = N$. Furthermore $N_3 \cap M = 1$.

We now consider the group $Y = X *_M B$. Since $N_3 \lhd_f X$, as in the case of N_2 in F, we can find $N_4 \lhd_f Y$ such that $N_4 \cap (H * K) = N$. But,

$$\begin{aligned} Y &= X *_M B = (A *_L F) *_M B = (A *_L (L *_U M)) *_M B \\ &= (A *_U M) *_M B = A *_U B. \end{aligned}$$

Hence let $N_G = N_4$ then $N_G \cap (H * K) = N$ as required.

LEMMA 4.3. *Let $G = A *_U B$ as in Lemma 4.2. Then G is $H * K$-separable.*

PROOF. Immediate from Theorem 2.1.

We are now ready to prove our main result, verifying, in particular, the residual finiteness of Example 4.1.

THEOREM 4.4 *Let P be the polygonal product of the finite p-groups A, B, C, D amalgamating the cyclic subgroups H, U, K, V, where $A \cap B = U$, $B \cap C = K$, $C \cap D = V$, $D \cap A = H$ with $|H| = |K| = p$ and $U \cap K = K \cap V = V \cap H = H \cap U = 1$. Then P is ${}_R\mathcal{F}$.*

PROOF. Let $E = A *_U B$ and $F = C *_V D$. Then $P = E *_Q F$ where $Q = H * K$. Thus a typical element x of P of length $\geqq 1$ is of the form $x =$

$e_1 f_1 \ldots e_n f_n$ where $e_i \in E\backslash Q$ and $f_i \in F\backslash Q$ except possibly $e_1 = 1$ or $f_n = 1$. By Lemma 4.3, E and F are Q-separable. Therefore, for each e_i and f_i, there exist $N_i \lhd_f E$ and $M_i \lhd_f F$ such that $e_i \notin \langle Q, N_i \rangle$ and $f_i \notin \langle Q, M_i \rangle$. Let $N = \cap_{i=1}^n N_i$ and $M = \cap_{i=1}^n M_i$. Then $M \lhd_f E$ and $N \lhd_f F$ such that $e_i \notin \langle Q, M \rangle$ and $f_i \notin \langle Q, N \rangle$ for each i. Since A, B are finite, there exists $W_1 \lhd_f E$ such that E/W_1 embeds the amalgam $(A, B; U)$. In the same way, there exists $W_2 \lhd_f F$ such that F/W_2 embeds $(C, D; V)$. Let $J = W_1 \cap W_2 \cap M \cap N$. Then $J \lhd_f Q$. Thus, by Lemma 4.2, there exists $N_E \lhd_f E$, $N_F \lhd_f F$ such that $N_E \cap Q = J = N_F \cap Q$. Now let $R = N_E \cap W_1 \cap N$ and $S = N_F \cap W_2 \cap M$. Then clearly $R \lhd_f E$ and $S \lhd_f F$, and $R \cap Q = J = S \cap Q$. Furthermore, $e_i \notin \langle Q, R \rangle$ and $f_i \notin \langle Q, S \rangle$ for each i. Let $\bar{E} = E/R$, $\bar{F} = F/S$ and $\bar{Q} = Q/J$. Then we can form $\bar{P} = \bar{E} *_{\bar{Q}} \bar{F}$. Since $\bar{E}$ and $\bar{F}$ are finite, it follows that $\bar{P} \in {}_R\mathscr{F}$. Let θ be the natural homomorphism of P onto $\bar{P}$. Then, because of the choice of R and S, we have $e_i\theta \notin \bar{Q}$ and $f_i\theta \notin \bar{Q}$ for each i, whence $x\theta \neq 1$ in $\bar{P}$. This implies that there exists $\bar{T} \lhd_f \bar{P}$ such that $x\theta \notin \bar{T}$. Let T be the preimage of $\bar{T}$ in P under θ. Then $T \lhd_f P$ and $x \notin T$.

If $x \in Q$ then there exists $J \lhd_f Q$ such that $x \notin J$. Applying Lemma 4.2, we can again find $N_E \lhd_f E$ and $N_F \lhd_f F$ such that $N_E \cap Q = J = N_F \cap Q$. By a similar argument as above we can then show that there exists $T \lhd_f P$ such that $x \notin T$. This completes our proof that P is ${}_R\mathscr{F}$.

References

1. G. Baumslag, *On the residual finiteness of generalized free products of nilpotent groups*, Trans. Amer. Math. Soc. **106** (1963), pp. 193-209.

2. A. M. Brunner, M. L. Frame, Y. W. Lee and N. J. Wielenberg, *Classifying the torsion-free subgroups of the Picard group*, Trans. Amer. Math. Soc. **282** (1984), pp. 205-235.

3. R. G. Burns, *A note on free groups*, Proc. Amer. Math. Soc. **23** (1969), pp. 14-17.

4. M. Hall, Jr., *Coset representations in free groups*, Trans. Amer. Math. Soc., **67** (1949), pp. 421-432.

5. G. Higman, *Amalgam of p-groups*, J. of Algebra **1** (1964), pp. 301-305.

6. A. Karrass, A. Pietrowski and D. Solitar, *The subgroups of polygonal products of groups*, unpublished manuscript.

7. B. H. Neumann, *An essay on free products of groups with amalgamations*, Philos. Trans. Roy. Soc. London, Ser. A **246** (1954), pp. 503-554.

8. B. H. Neumann and H. Neumann, *A contribution to the embedding theory of group amalgams*, Proc. London Math. Soc. (3) **3** (1953), pp. 243-256.

9. J. Wiegold, *Nilpotent products of groups with amalgamations*, Publ. Math. Debrecen **6** (1959), pp. 131-168.

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