

ON AN ELLIPTIC EQUATION OF
 p -KIRCHHOFF TYPE VIA VARIATIONAL METHODS

FRANCISCO JÚLIO S.A. CORRÊA AND GIOVANY M. FIGUEIREDO

This paper is concerned with the existence of positive solutions to the class of nonlocal boundary value problems of the p -Kirchhoff type

$$-\left[M\left(\int_{\Omega} |\nabla u|^p dx\right)\right]^{p-1} \Delta_p u = f(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

and

$$-\left[M\left(\int_{\Omega} |\nabla u|^p dx\right)\right]^{p-1} \Delta_p u = f(x, u) + \lambda|u|^{s-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded smooth domain of \mathbb{R}^N , $1 < p < N$, $s \geq p^* = (pN)/(N - p)$ and M and f are continuous functions.

1. INTRODUCTION

The purpose of this article is to investigate the existence of positive solutions to the class of nonlocal boundary value problems of the p -Kirchhoff type

$$(P) \quad \begin{cases} -\left[M(\|u\|^p)\right]^{p-1} \Delta_p u = f(x, u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

and

$$(P)_\lambda \quad \begin{cases} -\left[M(\|u\|^p)\right]^{p-1} \Delta_p u = f(x, u) + \lambda|u|^{s-2}u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where, through this work, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $f : \bar{\Omega} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ and $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are continuous functions that satisfy conditions which will be stated later, $\Delta_p u$ is the p -Laplacian operator, that is,

$$\Delta_p u = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right), \quad 1 < p < N,$$

Received 2nd May, 2006

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9727/06 \$A2.00+0.00.

and $\|\cdot\|$ is the usual norm in $W_0^{1,p}(\Omega)$ given by

$$\|u\|^p = \int_{\Omega} |\nabla u|^p.$$

The main goal of this paper is establishing conditions on M and f under which problem (P) and $(P)_\lambda$ possess positive solutions.

Problem (P) and $(P)_\lambda$ are called nonlocal because of the presence of the term $M(\|u\|^p)$ which implies that the equations in (P) and $(P)_\lambda$ are no longer pointwise identities. This provokes some mathematical difficulties which makes the study of such a problem particularly interesting. This problem has a physical motivation. The operator $\left[M(\|u\|^p)\right]^{p-1} \Delta_p u$, with $p = 2$, appears in the Kirchhoff equation, which arises in nonlinear vibrations, namely

$$(1.1) \quad \begin{cases} u_{tt} - M(\|u\|^2) \Delta u = f(x, u) & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x) \quad , \quad u_t(x, 0) = u_1(x). \end{cases}$$

Hence, problem (P) and $(P)_\lambda$, in case $p = 2$, are the stationary counterpart of the above evolution equation.

Such a hyperbolic equation is a general version of the Kirchhoff equation

$$(1.2) \quad \rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0$$

presented by Kirchhoff [8]. This equation extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. The parameters in equation (1.2) have the following meanings: L is the length of the string, h is the area of cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension.

Problem (1.1) began to attract the attention of several researchers mainly after the work of Lions [9], where a functional analysis approach was proposed to attack it.

The reader may consult [1, 2, 4, 10, 13] and the references therein, for more informations on (P) and $(P)_\lambda$, in case $p = 2$.

Motivated by papers [2, 5] and by some ideas developed in [3, 6], we prove the existence of positive solutions to (P) and $(P)_\lambda$. However, in this work, we use a different approach to those explored in [2, 5, 3], because here we are working with the p -Laplacian operator. Some estimates for this type of operator can not be obtained using the same kind of ideas explored for the case $p=2$. For example, results involving uniform a priori estimate of the Gidas and Spruck type [7] does not hold for the p -Laplacian. To overcome these difficulties, we use comparison between minimax levels of energy.

This paper is organised as follows: in Section 2, we show the existence of positive solutions for the equation (P) . In Section 3, we show the existence of positive solutions for the equation $(P)_\lambda$.

2. THE SUBCRITICAL CASE

In this section we assume that $f : \bar{\Omega} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous function and satisfies the subcritical growth conditions

$$(f_1) \quad |f(x, t)| \leq C|t|^{q-1},$$

for all $x \in \Omega$ and for all $t \in \mathbb{R}$, where $p < q < p^* = (pN)/(N - p)$.

We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of the problem (P) if it satisfies

$$\left[M(\|u\|^p) \right]^{p-1} \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi - \int_{\Omega} f(x, u) \phi = 0$$

for all $\phi \in W_0^{1,p}(\Omega)$.

We shall look for solutions of (P) by finding critical points of the C^1 -functional $I : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ given by

$$I(u) = \frac{1}{p} \widehat{M}(\|u\|^p) - \int_{\Omega} F(x, u) dx$$

where $\widehat{M}(t) = \int_0^t [M(s)]^{p-1} ds$ and $F(x, t) = \int_0^t f(x, s) ds$.

Note that

$$I'(u)\phi = \left[M(\|u\|^p) \right]^{p-1} \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi - \int_{\Omega} f(x, u) \phi,$$

for all $\phi \in W^{1,p}(\Omega)$.

In order to use critical point theory we first derive results related to the Palais–Smale compactness condition.

We say that a sequence (u_n) is a Palais–Smale sequence for the functional I if

$$I(u_n) \rightarrow c \text{ and } \|I'(u_n)\| \rightarrow 0 \text{ in } (W_0^{1,p}(\Omega))'$$

If every Palais–Smale sequence of I has a strongly convergent subsequence, then one says that I satisfies the Palais–Smale condition ((PS) for short).

Through this paper, we assume that M is a continuous function and satisfies:

$$(M_1) \quad M(t) \geq m_0 > 0 \text{ for all } t \in \mathbb{R}^+.$$

We have the following lemma:

LEMMA 2.1. *Assume that conditions (f_1) and (M_1) hold. Then, any bounded Palais–Smale sequence of I has a strongly convergent subsequence.*

PROOF: Let (u_n) be a bounded Palais–Smale sequence for I . Thus, passing to a subsequence if necessary, we have

$$(2.1) \quad \|u_n\|^p \rightarrow t_0$$

and there exists $u \in W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$. From (f_1) , the Lebesgue Dominated Convergence Theorem and the Sobolev Imbedding, we see that

$$\int_{\Omega} f(x, u_n)u \rightarrow \int_{\Omega} f(x, u)u \text{ and } \int_{\Omega} f(x, u_n)u_n \rightarrow \int_{\Omega} f(x, u)u.$$

Let us now consider the sequence

$$P_n = I'(u_n)u_n + \int_{\Omega} f(x, u_n)u_n - I'(u_n)u - \int_{\Omega} f(x, u_n)u.$$

We have that $P_n \rightarrow 0$ and

$$P_n = [M(\|u_n\|^p)]^{p-1} \|u_n\|^p - [M(\|u_n\|^p)]^{p-1} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla u.$$

Moreover, from (2.1), we get $M(\|u_n\|^p) \rightarrow M(t_0)$ and from the weak convergence, we have

$$-[M(\|u_n\|^p)]^{p-1} \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla u_n + [M(\|u_n\|)]^{p-1} \|u\|^p = o_n(1).$$

Hence,

$$\begin{aligned} o_n(1) + P_n &= [M(\|u_n\|^p)]^{p-1} \|u_n\|^p - [M(\|u_n\|^p)]^{p-1} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla u \\ &\quad - [M(\|u_n\|^p)]^{p-1} \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla u_n + [M(\|u_n\|^p)]^{p-1} \|u\|^p. \end{aligned}$$

Consequently,

$$o_n(1) + P_n = [M(\|u_n\|^p)]^{p-1} \int_{\Omega} \langle |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u, \nabla u_n - \nabla u \rangle.$$

Using the standard inequality in \mathbb{R}^N given by

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle \geq C_p|x - y|^p \text{ if } p \geq 2$$

or

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle \geq \frac{C_p|x - y|^2}{(|x| + |y|)^{2-p}} \text{ if } 2 > p > 1,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidian inner product in \mathbb{R}^N (see appendix or [12]) and from (M_1) , we obtain

$$o_n(1) + P_n \geq m_0^{p-1}C_p \int_{\Omega} |\nabla u_n - \nabla u|^p.$$

Thus, we conclude that $\|u_n - u\| \rightarrow 0$ in $W_0^{1,p}(\Omega)$. □

Now, let us show a basic existence result as a motivation to our main theorem. Here, we use the version due to Willem for the Mountain Pass Theorem (see [15, p. 12]).

LEMMA 2.2. Let X be a Banach space, $I \in C^1(X, \mathbb{R})$ with $I(0) = 0$. Suppose that:

(H_1) There exists $\alpha, r > 0$ such that $I(u) \geq \alpha > 0$ for all $u \in X$ with $\|u\| = r$

(H_2) There exists $e \in X$ such that $\|e\| > r$ and $I(e) < 0$.

Then there exists a sequence $(u_n) \subset X$ such that

$$I(u_n) \rightarrow c \text{ and } I'(u_n) \rightarrow 0 \text{ in } X'$$

where

$$0 < c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t))$$

and

$$\Gamma = \left\{ \gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e \right\}.$$

THEOREM 2.3. Assume that (f_1) and (M_1) hold. Furthermore, let us suppose that

$$(f_2) \quad 0 < \mu F(x, t) \leq f(x, t)t \text{ for all } t > 0.$$

for some $\mu \in \mathbb{R}$ with $p < \mu < q$. Then, if

$$(2.2) \quad \widehat{M}(t) \geq [M(t)]^{p-1}t \text{ for all } t \geq 0,$$

problem (P) has a positive solution.

PROOF: Note that $I(0) = 0$ and using the condition (2.2), we obtain

$$\begin{aligned} I(u) &= \frac{1}{p} \widehat{M}(\|u\|^p) - \int_{\Omega} F(x, u) \\ &\geq \frac{1}{p} [M(\|u\|^p)]^{p-1} \|u\|^p - \int_{\Omega} F(x, u). \end{aligned}$$

From (f_1) and (M_1), there exists $r, \alpha > 0$ such that $I(u) \geq \alpha > 0$, for all $u \in W_0^{1,p}(\Omega)$ with $\|u\| = r$. From (f_2), there exists $u \in W_0^{1,p}(\Omega)$ such that $I(u) < 0$. By Lemma 2.2, we find a Palais-Smale sequence $(u_n) \subset W_0^{1,p}(\Omega)$, for the functional I . We claim that such a sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$. Indeed, using (f_2) and (2.2) again, we have

$$\begin{aligned} C + \|u_n\| &\geq I(u_n) - \frac{1}{\mu} I'(u_n)u_n \\ &= \frac{1}{p} \widehat{M}(\|u_n\|^p) - \frac{1}{\mu} [M(\|u_n\|^p)]^{p-1} \|u_n\|^p + \int_{\Omega} \left[\frac{1}{\mu} f(x, u_n)u_n - F(x, u_n) \right] \\ &\geq \left(\frac{1}{p} - \frac{1}{\mu} \right) [M(\|u_n\|^p)]^{p-1} \|u_n\|^p \geq C \|u_n\|^p \end{aligned}$$

with $C > 0$. Hence, (u_n) is bounded in $W_0^{1,p}(\Omega)$ and from lemma 2.1, there exists $u \in W_0^{1,p}(\Omega)$ such that $I(u) = c_M > 0$, where

$$c_M = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \text{ and } \Gamma = \left\{ \gamma \in C([0, 1]) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0 \right\}.$$

Using u^- as a test function, the condition (f_1) and the maximum principle, we obtain $u > 0$. Thus, we conclude the proof of Theorem 2.3. \square

REMARK 2.4. As we can see in [2], to the original meaning of M , in the Kirchhoff equation

$$u_{tt} - M(\|u\|^2)\Delta u = f(x, u),$$

it should be an increasing function. Then

$$\widehat{M}(u) < \int_0^u M(s) ds = M(u)u, \text{ for all } u > 0$$

and therefore, condition (2.2) cannot be satisfied.

In what follows, we consider the existence of positive solutions of (P) where M may be increasing. To this end, we first suppose that M is bounded. More precisely, we assume that there exists $m_1 \geq m_0$ and $t_0 > 0$ such that

$$(2.3) \quad M(t) = m_1 \text{ for all } t \geq t_0.$$

THEOREM 2.5. *Suppose that f satisfies (f_1) and (f_2) . Assume, in addition, that M is a function satisfying (M_1) and (2.3) with*

$$(2.4) \quad \left(\frac{m_0^{p-1}}{p} - \frac{m_1^{p-1}}{\mu} \right) > 0.$$

Then, problem (P) has a positive solution.

PROOF: We argue as in Theorem 2.3 to show the functional I has a nonzero critical point. From (M_1) and (2.3), we see that

$$(2.5) \quad \widehat{M}(t) \geq m_0^{p-1}t \text{ for all } t \geq 0$$

and

$$(2.6) \quad \widehat{M}(t) \leq m_1^{p-1}t + m_2 \text{ for all } t \geq t_0$$

where

$$m_2 = \left| \int_0^{t_0} [M(s)]^{p-1} ds - m_1^{p-1}t_0 \right|.$$

Using standard arguments, we infer that I satisfies

$$I(u) \geq C\|u\|^p - C\|u\|^q$$

for all $u \in W_0^{1,p}(\Omega)$, where here and elsewhere we may use the same letter C to indicate (possibly different) positive constants. If $\phi \geq 0$ is a nonzero function, we get from (f_2) and (2.6) that

$$I(t\phi) \leq \frac{t^p m_1^{p-1}}{p} \|\phi\|^p - t^\mu C \|\phi\|^\mu + C < 0 \text{ (} t > 0 \text{ large)}.$$

Thus, from Lemma 2.2, there exists a $(PS)_c$ sequence $(u_n) \subset W_0^{1,p}(\Omega)$ for I . We claim that (u_n) is bounded in $W_0^{1,p}(\Omega)$. Indeed, assume, by contradiction, that, up to a subsequence, $\|u_n\| \rightarrow +\infty$. Thus $M(\|u_n\|) = m_1$, if n is large enough, and by (2.5) and (f_2) , we have

$$C + \|u_n\| \geq I(u_n) - \frac{1}{\mu} I'(u_n)u_n \geq \left(\frac{m_0^{p-1}}{p} - \frac{m_1^{p-1}}{\mu} \right) \|u_n\|^p.$$

Using (2.4), we conclude that (u_n) is bounded in $W_0^{1,p}(\Omega)$, which contradicts $\|u_n\| \rightarrow +\infty$. □

Our goal is to extend Theorem 2.5 to a large class of M , including the increasing linear functions. This is done by using truncation arguments and a priori estimates obtained via relations between minimax levels c_M and c_0 , related to functional I_0 associated to the problem

$$(P_0) \quad \begin{cases} -\Delta_p u = \frac{1}{m_1^{p-1}} f(x, u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

that is,

$$I_0(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p - \frac{1}{m_1^{p-1}} \int_{\Omega} F(x, u).$$

Next we prove a lemma that establishes a relation between the $W_0^{1,p}(\Omega)$ norm of the solutions of problem (P) and $M(\|u\|^p)$.

LEMMA 2.6. *Let u be the solution of (P) obtained in Theorem 2.5. Then, there exist $\tilde{C} > 0$ and $\theta > 0$ independent on M , such that*

$$\|u\| \leq \tilde{C} \text{ and } \|u\|^p \leq [M(\|u\|^p)]^{1-p} \theta.$$

PROOF: If $\|u\| < t_0$, we choose $\tilde{C} = t_0$. If $\|u\|^p \geq t_0$, we have $M(\|u\|^p) = m_1$ and

$$c_M = I(u) - \frac{1}{\mu} I'(u)u \geq \left(\frac{m_0^{p-1}}{p} - \frac{m_1^{p-1}}{\mu} \right) \|u\|^p + \int_{\Omega} \left[\frac{1}{\mu} f(x, u)u - F(x, u) \right].$$

By (f_2) again

$$(2.7) \quad c_M \geq I(u) - \frac{1}{\mu} I'(u)u \geq \left(\frac{m_0^{p-1}}{p} - \frac{m_1^{p-1}}{\mu} \right) \|u\|^p.$$

Moreover, by (2.3) and (2.6), for all $u \in W^{1,p}(\Omega)$, we obtain

$$\begin{aligned} I(u) &= \frac{1}{p} \widehat{M}(\|u\|^p) - \int_{\Omega} F(x, u) \leq \frac{1}{p} m_1^{p-1} \|u\|^p - \int_{\Omega} F(x, u) + m_2 \\ &= m_1^{p-1} \left[I_0(u) + \frac{m_2}{m_1^{p-1}} \right]. \end{aligned}$$

Thus, we conclude that

$$(2.8) \quad c_M \leq m_1^{p-1} c_0 + \frac{m_2}{m_1^{p-1}}.$$

By (2.7) and (2.8), we get

$$\|u\|^p \leq \left(\frac{p\mu}{m_0^{p-1}\mu - m_1^{p-1}p} \right) \left(m_1^{p-1} c_0 + \frac{m_2}{m_1^{p-1}} \right) = \tilde{C}^p.$$

By (f₁)

$$\left[M(\|u\|^p) \right]^{p-1} \|u\|^p = \int_{\Omega} f(x, u)u \leq C\tilde{C}^q.$$

Hence

$$\|u\|^p \leq \left[M(\|u\|^p) \right]^{1-p} \theta,$$

where $\theta = C\tilde{C}^q$. □

THEOREM 2.7. *Suppose that f satisfies (f₁) and (f₂). Assume, in addition, that M satisfies (M₁) and there exists $k > 0$ such that*

$$(M_2) \quad [M(k)]^{p-1} < \mu \frac{m_0^{p-1}}{p}$$

and

$$(M_3) \quad [M(k)]^{1-p} \leq \frac{k}{\theta},$$

where θ was given in Lemma 2.5. Then, problem (P) has a positive solution.

PROOF: Let us define the truncated function

$$M_k(t) = \begin{cases} M(t) & \text{if } t \leq k \\ M(k) & \text{if } t > k. \end{cases}$$

Then, the assumption (M₂) imply that M_k satisfies (2.4) with $m_1 = M(k)$. We can apply Theorem 2.5 to obtain a solution $u_k > 0$ of the truncated problem

$$\begin{cases} -[M_k(\|u\|^p)]^{p-1} \Delta_p u = f(x, u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

From Lemma 2.6, we know that

$$\|u_k\|^p \leq [M_k(\|u\|^p)]^{1-p} \theta.$$

This implies that if $\|u_k\|^p > k$, so

$$k < [M(k)]^{1-p} \theta,$$

which contradicts (M₃). Therefore, $\|u_k\|^p \leq k$, which shows that u_k is, in fact, a positive solution of the (nontruncated) problem (P). □

3. CASE CRITICAL/SUPERCritical

First of all, we have to note that because $f(x, t) + \lambda|t|^{s-2}t$ has a supercritical growth we can not use directly the variational techniques, by virtue of the lack of compactness of the Sobolev immersions. So, we construct a suitable truncation of $f(x, t) + \lambda|t|^{s-2}t$ in order to use variational methods or, more precisely, the Mountain Pass Theorem. This truncation was used by Rabinowitz [14] (see [3, 6]).

Let $K > 0$ be a real number, whose precise value will be fixed later, and consider the function $g_K : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$g_K(x, t) = \begin{cases} 0 & \text{if } t < 0 \\ f(x, t) + \lambda t^{s-1} & \text{if } 0 \leq t \leq K \\ f(x, t) + \lambda K^{s-q} t^{q-1} & \text{if } t \geq K. \end{cases}$$

We study the truncated problem associated to g_K

$$(T)_\lambda \quad \begin{cases} -[M(\|u\|^p)]^{p-1} \Delta_p u = g_K(x, u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Such a function enjoys the following conditions:

$$(g_{K,1}) \quad |g_K(x, t)| \leq (C + \lambda K^{s-q}) t^{q-1}$$

for all $x \in \Omega$, for all $t \in \mathbb{R}$, where $C > 0$ and $p < q < p^* = (pN)/(N - p)$ and

$$(g_{K,2}) \quad 0 < \mu G_K(x, t) \leq g_K(x, t)t,$$

for all $x \in \Omega$, for all $t > 0$ and where $G_K(x, t) = \int_0^t g_K(x, \xi) d\xi$. Assuming $(M_1) - (M_2) - (M_3)$, by Theorem 2.7, we have a positive solution u_λ of $(T)_\lambda$, such that $I_\lambda(u_\lambda) = c_\lambda$, where c_λ is the Mountain Pass level associated to the functional

$$I_\lambda(u_\lambda) = \frac{1}{p} \widehat{M}(\|u_\lambda\|^2) - \int_\Omega G_K(x, u_\lambda)$$

which is related to the problem $(T)_\lambda$, where $\widehat{M}(t) = \int_0^t M(s) ds$. Furthermore,

$$(3.1) \quad I_\lambda(u_\lambda) - \frac{1}{\mu} I'_\lambda(u_\lambda) u_\lambda \geq \left(\frac{m_0^{p-1}}{p} - \frac{M(k)^{p-1}}{\mu} \right) \|u_\lambda\|^p + \int_\Omega \left[\frac{1}{\mu} g_K(x, u_\lambda) u_\lambda - G_K(x, u_\lambda) \right].$$

To prove the main result of this section, we need the following estimate.

LEMMA 3.1. *If u_λ is a solution (positive) of problem $(T)_\lambda$, then $\|u_\lambda\| \leq \bar{C}$ for all $\lambda \geq 0$, where $\bar{C} > 0$ is a constant does not depend on λ .*

PROOF: Since $G_K(x, t) \geq F(x, t)$ for all $x \in \Omega$ and for all $t \geq 0$, one has $c_\lambda \leq c_M$, where c_M is the Mountain Pass level related to the functional I . Furthermore

$$c_M \geq c_\lambda = I_\lambda(u_\lambda) = I_\lambda(u_\lambda) - \frac{1}{\mu} I'_\lambda(u_\lambda)u_\lambda$$

and from (3.1)

$$c_M \geq \left(\frac{m_0^{p-1}}{p} - \frac{M(k)^{p-1}}{\mu} \right) \|u_\lambda\|^p + \int_\Omega \left[\frac{1}{\mu} g_K(x, u_\lambda)u_\lambda - G_K(x, u_\lambda) \right].$$

From $(g_{K,2})$, we get

$$\|u_\lambda\| \leq \left(\frac{p\mu^{p-1}c_M}{\mu m_0^{p-1} - pM(k)^{p-1}} \right)^{1/p} := \bar{C},$$

for all $\lambda \geq 0$. □

Next, we are going to use the Moser iteration method [11](see [3, 6]).

THEOREM 3.2. *Let us suppose that the function M satisfies $(M_1) - (M_2) - (M_3)$ and f satisfies $(f_1) - (f_2)$. Then there exists $\lambda_0 > 0$ such that problem $(P)_\lambda$ possesses a positive solution for each $\lambda \in [0, \lambda_0]$.*

PROOF: Let u_λ be a solution of problem $(T)_\lambda$. We shall show that there is K_0 such that for all $K > K_0$ there exists a corresponding λ_0 for which

$$|u_\lambda|_{L^\infty(\Omega)} \leq K \text{ for all } \lambda \in [0, \lambda_0].$$

This is the case one has $g_K(x, u_\lambda) = f(x, u_\lambda) + \lambda|u_\lambda|^{s-1}$ and so u_λ is a solution of problem $(P)_\lambda$, for all $\lambda \in [0, \lambda_0]$.

For the sake of simplicity, we shall use the following notation:

$$u_\lambda := u$$

For $L > 0$, let us define the following functions

$$u_L = \begin{cases} u & \text{if } u \leq L \\ L & \text{if } u > L, \end{cases}$$

$$z_L = u_L^{p(\beta-1)}u \quad \text{and} \quad w_L = uu_L^{\beta-1}$$

where $\beta > 1$ will be fixed later. Let us use z_L as a test function, that is,

$$\left[M(\|u\|^p) \right]^{p-1} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla z_L = \int_\Omega g_K(x, u)z_L$$

which implies

$$M(\|u\|^p)^{p-1} \int_\Omega u_L^{p(\beta-1)} |\nabla u|^p = -p(\beta-1)M(\|u\|^p)^{p-1} \int_\Omega u_L^{p\beta-3} u |\nabla u|^{p-2} \nabla u \nabla u_L + \int_\Omega g_K(x, u)uu_L^{p(\beta-1)}.$$

From the definition of u_L , we have

$$p(\beta - 1)M(\|u\|^p)^{p-1} \int_{\Omega} u_L^{p\beta-3} u |\nabla u|^{p-2} \nabla u \nabla u_L \geq 0$$

and using $(g_{K,1})$ and (M_1) , we have

$$\int_{\Omega} u_L^{p(\beta-1)} |\nabla u|^p \leq (C + \lambda K^{s-q}) \frac{1}{m_0^{p-1}} \int_{\Omega} u^q u_L^{p(\beta-1)},$$

that is,

$$(3.2) \quad \int_{\Omega} u_L^{p(\beta-1)} |\nabla u|^p \leq C_{\lambda,K} \int_{\Omega} u^q u_L^{p(\beta-1)},$$

where $C_{\lambda,K} = (C + \lambda K^{s-q}) 1/m_0^{p-1}$.

On the other hand, from the continuous Sobolev immersions, one gets

$$|w_L|_{p^*}^p \leq C_1 \int_{\Omega} |\nabla w_L|^p = C_1 \int_{\Omega} |\nabla (u u_L^{\beta-1})|^p.$$

Consequently

$$|w_L|_{p^*}^p \leq C_1 \int_{\Omega} u_L^{p(\beta-1)} |\nabla u|^p + C_1(\beta - 1)^p \int_{\Omega} u_L^{p(\beta-2)} u^p |\nabla u_L|^p$$

which gives

$$(3.3) \quad |w_L|_{p^*}^p \leq C_2 \beta^p \int_{\Omega} u_L^{p(\beta-1)} |\nabla u|^p.$$

From (3.2) and (3.3), we get

$$|w_L|_{p^*}^p \leq C_2 \beta^p C_{\lambda,K} \int_{\Omega} u^q u_L^{p(\beta-1)}$$

and hence,

$$|w_L|_{p^*}^p \leq C_2 \beta^p C_{\lambda,K} \int_{\Omega} u^{q-p} (u u_L^{\beta-1})^p = C_2 \beta^p C_{\lambda,K} \int_{\Omega} u^{q-p} w_L^p.$$

We now use Hölder inequality, with exponents $p^*/[q-p]$ and $p^*/[p^* - (q-p)]$, to obtain

$$(3.4) \quad |w_L|_{p^*}^p \leq C_2 \beta^p C_{\lambda,K} \left(\int_{\Omega} u^{p^*} \right)^{(q-p)/p^*} \left(\int_{\Omega} w_L^{pp^*/[p^* - (q-p)]} \right)^{[p^* - (q-p)]/p^*},$$

where $p < (pp^*)/(p^* - (q-p)) < p^*$. Considering the continuous Sobolev immersion $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $p-1 \leq q \leq p^*$, we obtain

$$|w_L|_{p^*}^p \leq C_2' \beta^p C_{\lambda,K} \|u\|^{q-p} |w_L|_{\Omega}^p.$$

where $\alpha^* = pp^*/(p^* - (q - p))$.

Using Lemma 3.1

$$(3.5) \quad |w_L|_{p^*}^p \leq C_3 \beta^p C_{\lambda,K} \overline{C}^{q-p} |w_L|_{\alpha^*}^p.$$

Since $w_L = uu_L^{\beta-1} \leq u^\beta$ and supposing that $u^\beta \in L^{\alpha^*}(\Omega)$, we have from (3.5) that

$$\left(\int_{\Omega} |uu_L^{\beta-1}|^{p^*} \right)^{p/p^*} \leq C_4 \beta^p C_{\lambda,K} \left(\int_{\Omega} u^{\beta\alpha^*} \right)^{p/\alpha^*} < +\infty.$$

We now apply Fatou's lemma to the variable L to obtain

$$|u|_{\beta p^*}^{p\beta} \leq C_4 C_{\lambda,K} \beta^p |u|_{\beta\alpha^*}^{p\beta}.$$

and so

$$(3.6) \quad |u|_{\beta p^*} \leq (C_4 C_{\lambda,K})^{1/\beta p} \beta^{1/\beta} |u|_{\beta\alpha^*}.$$

Furthermore, by considering $\chi = p^*/\alpha^*$, we have $p^* = \chi\alpha^*$ and $\beta\chi\alpha^* = \beta p^*$, for all $\beta > 1$ so that $u^\beta \in L^{\alpha^*}(\Omega)$. Let us consider two cases:

FIRST CASE. First we consider $\beta = p^*/\alpha^*$ and note that

$$u^\beta \in L^{\alpha^*}(\Omega).$$

Hence, from the Sobolev immersions, Lemma 3.1 and relation (3.6), we get

$$|u|_{(p^*)^2/\alpha^*} \leq (C_4 C_{\lambda,K})^{1/p\beta} \beta^{1/\beta} \overline{C} C_5.$$

and so

$$(3.7) \quad |u|_{\chi^2\alpha^*} \leq C_6 (C_4 C_{\lambda,K})^{1/\chi p} \chi^{1/\chi}$$

SECOND CASE. We now consider $\beta = (p^*/\alpha^*)^2$, and note again that

$$u^\beta \in L^{\alpha^*}(\Omega).$$

From inequality in (3.6) we obtain,

$$|u|_{(p^*)^2/(\alpha^*)^2} \leq C_6 (C_4 C_{\lambda,K})^{1/\beta p} \beta^{1/\beta} |u|_{(p^*)^2/\alpha^*},$$

which implies

$$|u|_{\chi^3\alpha^*} \leq C_6 (C_4 C_{\lambda,K})^{1/\chi^2 p} (\chi^2)^{1/\chi^2} |u|_{\chi^2\alpha^*}$$

or,

$$|u|_{\chi^3\alpha^*} \leq C_7 (C_4 C_{\lambda,K})^{1/\chi^2 p + 1/\chi p} (\chi^2)^{p/\chi^2 + 1/\chi} \overline{C}.$$

An iterative process leads to

$$|u|_{\chi^{(m+1)\alpha^*}} \leq C_8(C_4C_{\lambda,K})^{\sum_{i=1}^m \frac{\lambda^{-i}}{p}} \chi^{\sum_{i=1}^m i\chi^{-i}} \bar{C}.$$

Taking limit as $m \rightarrow \infty$, we obtain

$$|u|_{L^\infty(\Omega)} \leq C_8(C_4C_{\lambda,K})^{\sigma_1} \chi^{\sigma_2} \bar{C}.$$

where $\sigma_1 = \sum_{i=1}^{\infty} (\chi^{-i})/p$ and $\sigma_2 = \sum_{i=1}^{\infty} i\chi^{-i}$. In order to choose λ_0 , we consider the inequality

$$C_8(C_4C_{\lambda,K})^{\sigma_1} \chi^{\sigma_2} \bar{C} = C_8 \left[C_4(C + \lambda K^{s-q}) \frac{1}{m_0^{p-1}} \right]^{\sigma_1} \chi^{\sigma_2} \bar{C} \leq K,$$

from which

$$(C + \lambda K^{s-q})^{\sigma_1} \leq \frac{K m_0^{(p-1)\sigma_1}}{C_4^{\sigma_1} \chi^{\sigma_2} \bar{C} C_8}.$$

Choosing λ_0 to satisfy the inequality

$$\lambda_0 \leq \left[\frac{K^{1/\sigma_1} m_0^{p-1}}{C_4 \chi^{\sigma_2/\sigma_1} C_8^{1/\sigma_1} \bar{C}^{1/\sigma_1}} - C \right] \frac{1}{K^{s-q}}$$

and fixing K such that

$$\left[\frac{K^{1/\sigma_1} m_0^{p-1}}{C_8^{1/\sigma_1} C_4 \chi^{\sigma_2/\sigma_1} \bar{C}^{1/\sigma_1}} - C \right] > 0,$$

we obtain

$$|u_\lambda|_{L^\infty(\Omega)} \leq K \quad \forall \lambda \in [0, \lambda_0].$$

and the proof of the theorem is over. □

APPENDIX

LEMMA 3.3. *Let $x, y \in \mathbb{R}^N$ and let $\langle \cdot, \cdot \rangle$ be the standard inner product in \mathbb{R}^N . Then*

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle \geq C_p |x - y|^p \text{ if } p \geq 2$$

or

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle \geq \frac{C_p |x - y|^2}{(|x| + |y|)^{2-p}} \text{ if } 2 > p > 1.$$

PROOF: By homogeneity we can assume that $|x| = 1$ and $|y| \leq 1$. Moreover by choosing a convenient basis in \mathbb{R}^N we can assume

$$x = (1, 0, 0, \dots, 0) \text{ , } y = (y_1, y_2, 0, \dots, 0) \text{ and } \sqrt{y_1^2 + y_2^2} \leq 1.$$

(i) Case $2 > p > 1$. It is clear that the inequality is equivalent to the next one

$$\left\{ \left(1 - \frac{y_1}{(y_1^2 + y_2^2)^{(2-p)/2}} \right) (1 - y_1) + \frac{y_2^2}{(y_1^2 + y_2^2)^{(2-p)/2}} \right\} \frac{(1 - \sqrt{y_1^2 + y_2^2})^{2-p}}{(1 - y_1)^2 + y_2^2} \geq C$$

But

$$1 - \frac{y_1}{(\sqrt{y_1^2 + y_2^2})^{2-p}} \geq 1 - \frac{y_1}{|y_1|^{2-p}} \geq (p - 1)(1 - y_1) \text{ if } 0 \leq y_1 \leq 1$$

or

$$1 - \frac{y_1}{(\sqrt{y_1^2 + y_2^2})^{2-p}} \geq 1 - y_1 \geq (p - 1)(1 - y_1) \text{ if } y_1 \leq 0,$$

then

$$(p - 1) \{ (1 - y_1)^2 + y_2^2 \} \frac{(1 + y_1 + y_2)^{(2-p)/2}}{(1 - y_1)^2 + y_2} \geq p - 1.$$

(ii) Case $p \geq 2$. The inequality is equivalent to prove

$$\frac{[1 - y_1(y_1^2 + y_2^2)^{(p-2)/2}](1 - y_1) + y_2^2(y_1^2 + y_2^2)^{(p-2)/2}}{((1 - y_1)^2 + y_2^2)^{p/2}} \geq C.$$

Denoting $t = |y|/|x|$ and $s = \langle x, y \rangle / (|x||y|)$ then, we must show that the function

$$f(t, s) = \frac{1 - (t^{p-1} + t)s + t^p}{(1 - 2ts + t^2)^{p/2}}$$

is bounded from below. Direct calculation shows that fixed t , $\frac{\partial f}{\partial s} = 0$ if

$$1 - (t^{p-1} + t)s + t^p = \frac{t^{p-2} + 1}{p}(1 - 2ts + t^2),$$

we have

$$f(t, s) = \frac{t^{p-2} + 1}{p(1 - 2ts + t^2)^{(p-2)/2}} \geq \frac{1}{p} \min_{0 \leq t \leq 1} \frac{t^{p-2} + 1}{(t + 1)^{p-2}} \geq \frac{1}{2p},$$

which concludes the proof of lemma. □

REFERENCES

- [1] C.O. Alves and F.J.S.A. Corrêa, 'On existence of solutions for a class of problem involving a nonlinear operator', *Comm. Appl. Nonlinear Anal.* **8** (2001), 43-56.
- [2] C.O. Alves, F.J.S.A. Corrêa and T.F. Ma, 'Positive solutions for a quasilinear elliptic equation of Kirchoff type', *Comput. Math. Appl.* **49** (2005), 85-93.

- [3] J. Chabrowski and J. Yang, 'Existence theorems for elliptic equations involving supercritical Sobolev exponents', *Adv. Differential Equations* **2** (1997), 231–256.
- [4] F.J.S.A. Corrêa and S.D.B. Menezes, 'Existence of solutions to nonlocal and singular elliptic problems via Galerkin method', *Electron. J. Differential Equations* (2004), 1–10.
- [5] F.J.S.A. Corrêa and G. M. Figueiredo, 'On the existence of positive solution for an elliptic equation of Kirchhoff type via Moser iteration method', *Bound. Value Probl.* (to appear).
- [6] G. M. Figueiredo, *Multiplicidade de soluções positivas para uma classe de problemas quasilineares*, Doct. dissertation (Unicamp, 2004).
- [7] G. Gidas and J. Spruck, 'A priori bound for positive solutions of nonlinear elliptic equation', *Comm. Partial Differential Equations* **6** (1981), 883–901.
- [8] G. Kirchhoff, *Mechanik* (Teubner, Leipzig, 1883).
- [9] J.L. Lions, 'On some questions in boundary value problems of mathematical physics', *International Symposium on Continuum, Mechanics and Partial Differential Equations, Rio de Janeiro(1977)* (1978), 284–346.
- [10] T.F. Ma, 'Remarks on an elliptic equation of Kirchhoff type', *Nonlinear Anal.* (to appear).
- [11] J. Moser, 'A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations', *Comm. Pure Appl. Math.* **13** (1960), 457–468.
- [12] I. Peral, *Multiplicity of solutions for the p -Laplacian*, Second School of Nonlinear Functional Analysis and Applications to Differential Equations (ICTP, Trieste, 1997).
- [13] K. Perera and Z. Zhang, 'Nontrivial solutions of Kirchhoff-type problems via the Yang index', *J. Differential Equations* **221** (2006), 246–255.
- [14] P.H. Rabinowitz, 'Variational methods for nonlinear elliptic eigenvalue problems', *Indiana Univ. Math. J.* **23** (1974), 729–754.
- [15] M. Willem, *Minimax theorems* (Birkhäuser, Boston, MA, 1996).

Departamento de Matemática-CCEN
Universidade Federal do Pará
66.075-110-Belém-Pará
Brazil
e-mail: fjulio@ufpa.br
e-mail: giovany@ufpa.br