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Non-Left-Orderable 3-Manifold Groups

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Abstract. We show that several torsion free 3-manifold groups are not left-orderable. Our examples are groups of cyclic branched coverings of S^3 branched along links. The figure eight knot provides simple nontrivial examples. The groups arising in these examples are known as Fibonacci groups which we show not to be left-orderable. Many other examples of non-orderable groups are obtained by taking 3-fold branched covers of S^3 branched along various hyperbolic 2-bridge knots. The manifold obtained in such a way from the 5_2 knot is of special interest as it is conjectured to be the hyperbolic 3-manifold with the smallest volume.

We investigate the orderability properties of fundamental groups of 3-dimensional manifolds. We show that several torsion free 3-manifold groups are not left-orderable. Many of our manifolds are obtained by taking n-fold branched covers along various hyperbolic 2-bridge knots. The paper is organized in the following way: after defining left-orderability we state our main theorem listing branched set links and multiplicity of coverings from which we obtain manifolds with non-left-orderable groups. Then we describe presentations of these groups in a way which allows the proof of non-left-orderability in a uniform way. The Main Lemma (Lemma 3) is the algebraic underpinning of our method and the non-left-orderability follows easily from it in almost all cases. Moreover we prove the non-left-orderability of a family of 3-manifold groups to which the Main Lemma does not apply. These groups, known as generalized Fibonacci groups $F(n-1,n)$, arise as groups of double covers of S^3 branched along pretzel links of type $(2,2,\dots,2,-1)$. We end the paper with some questions.

Definition 1 A group is *left-orderable* if there is a strict total ordering $\prec$ of its elements which is left-invariant: $x \prec y$ iff $zx \prec zy$ for all x, y and z.

Straight from the definition, it follows that a group with a torsion element is not left-orderable.

It is known that groups of compact, P^2-irreducible 3-manifolds with non-trivial first Betti number are left-orderable [BRW, H-S]. However, our main theorem below lists various classes of 3-manifolds with non-left-orderable groups. Non-left-orderability of 3-manifold groups has interesting consequences for the geometry of the corresponding manifolds [C-D, RSS].

Theorem 1 *Let $M_L^{(n)}$ denote the n-fold branched cover of S^3 branched along the link L, where $n > 1$. Then the fundamental group, $\pi_1(M_L^{(n)})$, is not left-orderable in the following cases:*

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(a) $L = T_{(2',2k)}$ *is the torus link of the type* $(2, 2k)$ *with the anti-parallel orientation of strings, and* n *is arbitrary (Figure 1).*
(b) $L = P(n_1, n_2, \dots, n_k)$ *is the pretzel link of the type* $(n_1, n_2, \dots, n_k)$, $k > 2$, *where either* (i) $n_1, n_2, \dots, n_k > 0$, *or* (ii) $n_1 = n_2 = \cdots = n_{k-1} = 2$ *and* $n_k = -1$ *(Figure 2). The multiplicity of the covering is* $n = 2$.
(c) $L = L_{[2k,2m]}$ *is a 2-bridge knot of the type* $\frac{p}{q} = 2m + \frac{1}{2k} = [2k, 2m]$, *where* $k, m > 0$, *and* n *is arbitrary (Figure 4).*
(d) $L = L_{[n_1,1,n_3]}$ *is the 2-bridge knot of the type* $\frac{p}{q} = n_3 + \frac{1}{1+1/n_1} = [n_1, 1, n_3]$, *where* n_1 *and* n_3 *are odd positive numbers. The multiplicity of the covering is* $n \leq 3$.

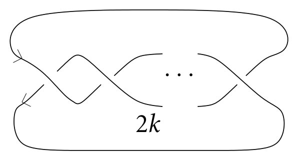

Figure 1

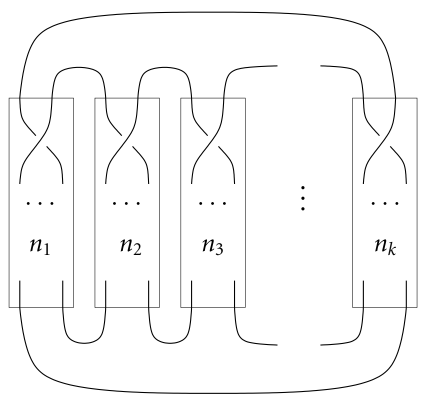

Figure 2

The manifolds described in parts (a), (b), and also for $n \leq 3$ and the figure-eight knot, $L = L_{[2,2]} = 4_1$, in part (c) are Seifert fibered manifolds. The non-left-orderability of their groups follows from the general characterization of Seifert fibered manifolds with a left-ordering [BRW]. Part (c) for the figure-eight knot when $n = 3$ is of historical interest because it was the first known example of a non-left-orderable torsion free 3-manifold group [Rol][1]. Part (c) for the figure-eight knot when $n > 3$, gives rise to hyperbolic manifolds that are related to examples discussed in [RSS], as they are Dehn fillings of punctured-torus bundles over S^1. The manifolds obtained in parts (c) and (d), when $n > 2$ (except $M^{(3)}_{4_1}$), are all hyperbolic manifolds as well[2].

[1]This Euclidean manifold was first considered by Hantzsche and Wendt [H-W]. J. Conway has proposed to call this manifold *didicosm*. It can be also described as the 2-fold branched cover over S^3 branched along the Borromean rings.

[2]It follows from the Orbifold Theorem that branched n-fold covers ($n > 2$) of S^3 branched along

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The case $\frac{p}{q} = \frac{7}{4} = 1 + \frac{1}{1+1/3} = [3, 1, 1]$, that is, the branching set being the 5_2 knot, is of special interest since $M^{(3)}_{5_2}$ is conjectured to be the hyperbolic 3-manifold with the smallest volume [Ki]. The fact that $\pi_1(M^{(3)}_{5_2})$ is not left-orderable was observed in [C-D, RSS]. The non-left-orderability in other cases is proved here for the first time.

The special form of the presentations of the groups listed in Theorem 1, allows us to conclude the theorem in most cases, using the Main Lemma formulated below (Lemma 3).

Proposition 2 *The groups listed in Theorem 1 have the following presentations:*

(a) $\pi_1(M^{(n)}_{T_{(2',2k)}}) =$
$\{x_1, x_2, \dots, x_n \mid x_1^k x_2^{-k} = e,\ x_2^k x_3^{-k} = e, \dots, x_n^k x_1^{-k} = e,\ x_1 x_2 \cdots x_n = e\}$

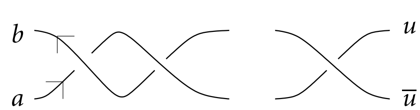
(b) (i) $\pi_1(M^{(2)}_{P_{(n_1,n_2,\dots,n_k)}}) =$
$\{x_1, x_2, \dots, x_k \mid x_1^{n_1} x_2^{-n_2} = e, x_2^{n_2} x_3^{-n_3} = e, \dots, x_k^{n_k} x_1^{-n_1} = e,\ x_1 x_2 \cdots x_k = e\}$

(ii) $\pi_1(M^{(2)}_{P_{(2,2,\dots,2,-1)}}) = \{x_1, x_2, \dots, x_k \mid x_1^2 = x_2^2 = \cdots = x_k^2 = x_1 x_2 \cdots x_k\}$

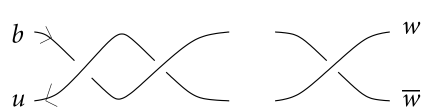
(c) $\pi_1(M^{(n)}_{L_{[2k,2m]}}) =$
$\{z_1, z_2, \dots, z_{2n} \mid z_{2i+1} = z_{2i}^{-k} z_{2i+2}^{k},\ z_{2i} = z_{2i-1}^{-m} z_{2i+1}^{m},\ z_2 z_4 \dots z_{2n} = e\}$ *where* $i = 1, 2 \dots n$ *and subscripts are taken modulo* $2n$.

(d) $\pi_1(M^{(n)}_{L_{[2k+1,1,2l+1]}}) = \{x_1, \dots, x_n \mid r_1 = e, \dots, r_n = e,\ x_1 x_2 \cdots x_n = e\}$, *where* $k \geq 0$, $l \geq 0$, $r_i = x_i^{-1}\big(x_i^{-k} x_{i+1}^{k+1} x_i^{-1}\big)^{l} x_i^{-k} x_{i+1}^{k+1}\big((x_{i+1}^{-k} x_{i+2}^{k+1} x_{i+1}^{-1})^{l} x_{i+1}^{-k} x_{i+2}^{k+1}\big)^{-1}$, *and subscripts are taken modulo* n.

Proof Since the presentations for all manifolds from Theorem 1 are obtained by similar calculations, therefore we shall only provide full details for the case (c) (compare [M-V]). Let T_1 denote the 2-tangle in Figure 3(a), $-[2k]$ in Conway's notation and let T_2 denote the 2-tangle in Figure 3(b), $[2m]$ in Conway's notation. Let us assume that the arcs of T_1 and T_2 are oriented in the way shown in Figure 3.



(a) $T_1 = -[2k]$ $2k$ right-handed half-twists



(b) $T_2 = -[2m]$ $2m$ left-handed half-twists

Figure 3

hyperbolic 2-bridge knots and links or along the Borromean rings are hyperbolic, except for $M^{(3)}_{4_1}$ which is a Euclidean manifold, didicosm [Bo, HLM, Ho, Th].

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Let $F_2 = \{a, b \mid \}$ be a free group generated by a and b. Assign to the initial arcs of T_1 the generators a and b. Then by successive use of Wirtinger relations, progressing from left to right in the diagram, we finally decorate the terminal arcs by $\bar{u} = (ba^{-1})^k a(ab^{-1})^k$ and $u = (ba^{-1})^k b(ab^{-1})^k$, respectively (see Figure 3(a)). Analogously, assigning to initial arcs of the tangle $T_2 = [2m]$ (Figure 3(b)) the elements b and u of F_2 and using Wirtinger relations successively one obtains terminal arcs decorated by $w = (u^{-1}b)^m b(b^{-1}u)^m$ and $\bar{w} = (u^{-1}b)^m u(b^{-1}u)^m$, respectively. Combining these calculations in the fashion illustrated in Figure 4, we obtain the relation $((ba^{-1})^k b^{-1}(ab^{-1})^k b)^m b = a((ba^{-1})^k b^{-1}(ab^{-1})^k b)^m$ and the presentation

$$\pi_1(S^3 - L_{[2k,2m]}) = \left\{ a, b \;\middle|\; r = \big((ba^{-1})^k b^{-1}(ab^{-1})^k b\big)^m b\big((ba^{-1})^k b^{-1}(ab^{-1})^k b\big)^{-m} a^{-1} \right\}.$$

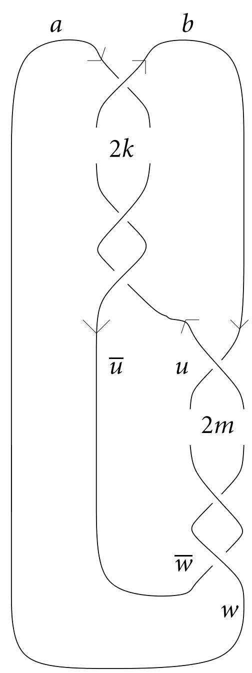

Figure 4: The 2-bridge knot $[2k, 2m]$

Using Fox non-commutative calculus [Cr], as explained in [Pr, P-R], we compute a presentation of $\pi_1(M^{(n)}_{L_{[2k,2m]}})$ by "lifting" the generators a and b as well as the defining relation r of $\pi_1(S^3 - L_{[2k,2m]})$.

We illustrate this by first computing a presentation of the fundamental group of the n-fold cyclic *unbranched* covering of $S^3 - L_{[2k,2m]}$. Since $\pi_1(S^3 - L_{[2k,2m]})$ has 2 generators, a and b, the covering space will have $n + 1$ generators, that is, $y =$

$ab^{-1}, \tau(y), \tau^2(y), \ldots, \tau^{n-1}(y)$ and b^n, where τ is the inner automorphism of F_2, given by $\tau(w) = bwb^{-1}$ (see Figure 5).

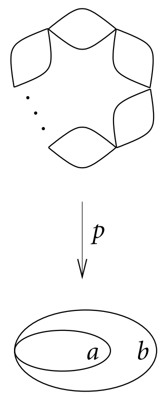

Figure 5

The relation r will also be lifted to n relations $\tilde{r}, \tau(\tilde{r}), \tau^2(\tilde{r}), \ldots, \tau^{n-1}(\tilde{r})$, in the group of the n-fold cyclic covering, where

$$\tilde{r} = \left((y^{-1})^k(\tau^{-1}(y))^k\right)^m \left((\tau(y^{-1}))^k(y)^k\right)^{-m} y^{-1}.$$

When dealing with the branched case, however, the relations $a^n = e$ and $b^n = e$ should also be added[3]. We then write the word a^n in terms of new generators as $y\tau(y)\ldots\tau^{n-1}(y)$. In order to simplify the presentation of $\pi_1(M^{(n)}_{L_{[2k,2m]}})$ we put $x_1 = \tau^{-1}(y)$, $x_2 = y$, $x_3 = \tau(y), \ldots, x_n = \tau^{n-2}(y)$. Thus

$$\pi_1\left(M^{(n)}_{L_{[2k,2m]}}\right) = \left\{x_1, x_2, \ldots, x_n \;\middle|\; x_i^{-1}\left(x_i^{-k}x_{i-1}^k\right)^m \left(x_{i+1}^{-k}x_i^k\right)^{-m} = e,\ x_1x_2\cdots x_n = e\right\},$$

where $i = 1, 2, \ldots, n$ and subscripts are taken modulo n.

To change this presentation to the one described in Proposition 2(c) we "deform" variables by putting $z_{2i} = x_i$ and $z_{2i+1} = x_i^{-k}x_{i+1}^k$. In new variables the presentation has the desired form

$$\pi_1\left(M^{(n)}_{L_{[2k,2m]}}\right) = \left\{z_1, z_2, \ldots, z_{2n} \;\middle|\; z_{2i+1} = z_{2i}^{-k}z_{2i+2}^k, z_{2i} = z_{2i-1}^{-m}z_{2i+1}^m, z_2z_4\cdots z_{2n} = e\right\},$$

where $i = 1, 2, \ldots, n$ and subscripts are taken modulo $2n$.[4] ■

It is worth mentioning that the case (c) that we singled out for illustrating the proof of Proposition 2 involves a step that the proofs for other cases do not require.

[3]Since $L_{[2k,2m]}$ is a knot, the relation $a^n = e$ follows from the relation $b^n = e$ and the relations $\tau^i(\tilde{r})$.

[4]In the special case of $k = m = 1$ we obtain the classical Fibonacci group $F(2, 2n)$ already known to be the fundamental group of $M^{(n)}_{4_1}$. We suggest that the presentation for any k and m to be called the (k, m)-*deformation*, $F((k, m), 2n)$, of the classical Fibonacci group.

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More specifically, all of the presentations given in the statement of Proposition 2, except for the case (c), are results of straightforward calculations and we do not need to deform the variables in any way in those cases in order to obtain the desired presentation.

The following definition and Main Lemma capture the algebraic properties of listed groups.

Definition 2

(i) Given a finite sequence $\epsilon_1, \epsilon_2, \dots, \epsilon_n$, $\epsilon_i \in \{-1, 1\}$, for all $i = 1, 2, \dots, n$ and a nonempty reduced word $w = x_{a_1}^{b_1} x_{a_2}^{b_2} \dots x_{a_m}^{b_m}$ of the free group $F_n = \{x_1, x_2, \dots, x_n \mid \}$, we say w *blocks* the sequence $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ if either $\epsilon_{a_j} b_j > 0$ for all j or $\epsilon_{a_j} b_j < 0$ for all $j = 1, 2, \dots, m$.

(ii) A set W of reduced words of F_n is *complete* if for any given sequence

$$\epsilon_1, \epsilon_2, \dots, \epsilon_n, \epsilon_i \in \{-1, 1\},$$

for $i = 1, 2, \dots, n$, there is a word $w \in W$ that blocks $\epsilon_1, \epsilon_2, \dots, \epsilon_n$.

(iii) The presentation $\{x_1, x_2, \dots, x_n \mid W\}$ of a group G is called *complete* if the set W of relations is complete.

Lemma 3 (Main Lemma) *Any nontrivial group G that admits a complete presentation is not left-orderable.*

Proof Suppose, on the contrary, that $\prec$ is a left-ordering on G. Let

$$G = \{x_1, x_2, \dots, x_n \mid W\}$$

be a complete presentation of G. Let

$$E = \left\{ (\epsilon_1, \epsilon_2, \dots, \epsilon_n) \mid x_i^{\epsilon_i} \preceq e \text{ in the group } G, \epsilon_i \in \{-1, 1\}, i = 1, 2, \dots, n \right\}.$$

Since W is complete, each sequence

$$(\epsilon_1, \epsilon_2, \dots, \epsilon_n) \in E$$

is blocked by a word $w \in W$. Since w is a relator, this is impossible, because the product of a number of "positive" elements in a left-orderable group will be "positive", not the identity. This contradiction completes the proof. ■

Theorem 1 follows easily from the Main Lemma and Proposition 2 in all cases except for part (b)(ii) which we deal with separately in the following lemma.

Lemma 4 *Let*

$$F(n-1, n) = \{x_1, \cdots, x_n \mid x_1 x_2 \cdots x_{n-1} = x_n, x_2 x_3 \cdots x_n = x_1, \cdots, x_n x_1 \cdots x_{n-2} = x_{n-1}\}.$$

If $n > 2$, then $F(n-1, n)$ is not left-orderable.

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Proof $F(2,3)$ is finite (it is the quaternion group Q_8), hence it is not left-orderable. Let us assume, then, that $n > 3$. First of all, note that the mapping

$$x_i \mapsto g\colon F(n-1,n) \to \{g \mid g^{n-2} = e\} = Z_{n-2}$$

defines an epimorphism, and since $n-2 > 1$ our group is not the trivial group.

It is not hard to see that in $F(n-1,n)$ we have $x_1^2 = x_2^2 = \cdots = x_n^2 = x_1x_2\cdots x_n$. Let $t = x_i^2 = x_1x_2\cdots x_n$ for any i. Suppose that $\prec$ is a left-ordering on $F(n-1,n)$. Since $F(n-1,n)$ is not the trivial group, hence $t \neq e$ unless our group has a torsion, which is not the case. Consider the case $t \prec e$. The case $e \prec t$ can be dealt with similarly.

Since $t = x_i^2$, we must have $x_i \prec e$ for all i. In particular, $x_i \neq e$ for all i. This makes $x_1 \preceq x_2 \leq \cdots \preceq x_n \preceq x_1$ impossible, because if $x_1 = x_2 = \cdots = x_n \neq e$, then $x_1^2 = t = x_1x_2\cdots x_n = x_1^n$ implies $x_1^{n-2} = e$, which in turn makes $F(n-1,n)$ a torsion group and thus non-left-orderable.

Therefore, $x_{i+1} \prec x_i$ for some i modulo n. Assume, without loss of generality, that $x_n \prec x_{n-1}$. Multiplying from the left by $x_1x_2\cdots x_{n-1}$ one obtains

$$t = x_1x_2\cdots x_{n-1}x_n \prec x_1x_2\cdots x_{n-2}x_{n-1}x_{n-1} = x_1x_2\cdots x_{n-2}t = tx_1x_2\cdots x_{n-2}.$$

The last equality holds because $t = x_i^2$ commutes with all x_i. Multiplying both sides from the left by t^{-1} gives $e \prec x_1x_2\cdots x_{n-2}$, contradicting the fact that $x_i \prec e$ for all i. ■

Left-orderability of a countable group G is equivalent to G being isomorphic to a subgroup of $Homeo_+(\mathbf{R})$ (compare [BRW]). Calegari and Dunfield related left-orderability of the group of a 3-manifold M with foliations on M. Therefore we have

Corollary 5

(i) *The groups of manifolds described in Theorem 1 do not admit a faithful representation to* $Homeo_+(\mathbf{R})$.
(ii) *Manifolds described in Theorem 1 do not admit a co-orientable* $\mathbf{R}$*-covered foliation* [C-D].

Thurston proved that if an atoroidal 3-manifold M has a taut foliation then there exists a faithful action of $\pi_1(M)$ on S^1 [C-D]. Exploring the fact that the group of the manifold of the smallest known volume, $M^{(3)}_{5_2}$, (together with some of its subgroups) is not left-orderable, Calegari and Dunfield showed that $\pi_1(M^{(3)}_{5_2})$ does not admit a faithful action of $\pi_1(M)$ on S^1 and therefore $M^{(3)}_{5_2}$ does not admit a taut foliation [C-D]. The connection between faithful actions of $\pi_1(M)$ on S^1 and on $\mathbf{R}$ is to be explored further.

We would like to contrast our non-left-orderability results with some examples of left-orderable 3-manifold groups.

It is known that if $M^{(n)}_K$ is irreducible (as is always the case for a hyperbolic knot K) and the group $H_1(M^{(n)}_K)$ is infinite, then the group $\pi_1(M^{(n)}_K)$ is left-orderable [BRW,

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H-S]. There are several examples of 2-bridge knots with infinite homology groups of cyclic branched coverings along them. For the trefoil knot 3_1 we have $H_1(M_{3_1}^{(6k)}) = Z \oplus Z$. For hyperbolic 2-bridge knots $9_6 = K_{[2,2,5]}$ and $10_{21} = K_{[3,4,1,2]}$ the groups $H_1(M_{9_6}^{(6)})$ and $H_1(M_{10_{21}}^{(10)})$ are also infinite.[5]

We end the paper with some questions about possible generalizations of our results.

Problem 1

(i) Are the groups $\pi_1(M_{5_2}^{(n)})$ non-left-orderable for $n > 3$?

(ii) Are the groups $\pi_1(M_K^{(n)})$ of hyperbolic 2-bridge knots K with finite $H_1(M_K^{(n)})$ non-left-orderable?

(iii) Are the groups $\pi_1(M_K^{(n)})$ of hyperbolic knots K with finite $H_1(M_K^{(n)})$ non-left-orderable?

(iv) In general, for which links L and multiplicities of covering n, is the group $\pi_1(M_L^{(n)})$ non-left-orderable?

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[5]To see quickly that $H_1(M_K^{(n)})$ is infinite one can use the Fox theorem which says that $H_1(M_K^{(n)})$ is infinite if and only if the Alexander polynomial, $\Delta_K(t)$, is equal to zero for some nth root of unity. To test the last condition for small knots one can use tables of knots with $\Delta_K(t)$ decomposed into irreducible factors [B-Z]. We check, for example, that $\Delta_K(e^{\pi i/3}) = 0$ for hyperbolic 2-bridge knots $K = 8_{11}, 9_6, 9_{23}, 10_5, 10_9, 10_{32}$ and 10_{40}. Note also that Casson and Gordon proved that p^k-fold cyclic branched coverings along a knot, where p is prime, are rational homology spheres.

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