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# EQUIVALENCE OF TOPOLOGICALLY-ALGEBRAIC AND SEMI-TOPOLOGICAL FUNCTORS

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**1. Preliminaries.** Throughout let $U:\mathscr{A} \rightarrow \mathscr{X}$ be faithful.

1.1. A *U-morphism* with domain $X$ is a pair $(e, A)$, where $e \in \text{Hom}\,(X, UA)$. A $U$-morphism $(e, A)$ is called *U-epi* (=*generating*) provided that $r, s \in \text{Hom}\,(A, A')$ and $(Ur)e = (Us)e$ imply that $r = s$.

1.2. A *U-source* is a pair $(X, (f_i, A_i)_I)$, (written more simply $(X, f_i, A_i)_I$), where $(f_i, A_i)_I$ is a family of $U$-morphisms each with domain $X$.

1.3. A *factorization* of a $U$-source $(X, f_i, A_i)_I$ is a triple $(e, A, g_i)_I$ such that $(e, A)$ is a $U$-morphism with domain $X$ and for each $i \in I$ $g_i \in \text{Hom}\,(A, A_i)$ and $(Ug_i)e = f_i$.

1.4. A factorization $(e, A, g_i)_I$ is called *semi-initial* provided that for each $\mathscr{A}$-object $A'$ and each family of morphisms $g_i' \in \text{Hom}\,(A', A_i)$, $(Ug_i)ee' = Ug_i'$ for each $i$ in $I$ implies that there exists some $h \in \text{Hom}\,(A', A)$ with $Uh = ee'$.

1.5. $U$ is called *topologically-algebraic* if every $U$-source $(X, f_i, A_i)_I$ has a factorization $(e, A, g_i)_I$ with $(e, A)$ $U$-epi, and $(A, g_i)_I$ initial.

1.6. $U$ is called *semi-topological* if every $U$-source $(X, f_i, A_i)_I$ has a factorization $(e, A, g_i)_I$ with $(e, A)$ $U$-epi and $(e, A, g_i)_I$ semi-initial.

1.7. Let $\mathscr{E}$ be a family of $U$-morphisms and $\mathscr{M}$ a family of $\mathscr{A}$-sources. $U$ is called $(\mathscr{E}, \mathscr{M})$*-factorizable* provided:

(1) every $U$-source has an $(\mathscr{E}, \mathscr{M})$*-factorization*; i.e., a factorization $(e, A, m_i)_I$ with $(e, A) \in \mathscr{E}$ and $(A, m_i)_I \in \mathscr{M}$.

$U$ is called an $(\mathscr{E}, \mathscr{M})$*-functor* if in addition:

(2) every commutative square:

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & UB \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle Uf_i} \\ UA & \xrightarrow[\;Um_i\;]{} & UA_i \end{array}$$

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with $(e, B) \in \mathscr{E}$ and $(A, m_i)_I \in \mathscr{M}$, has a *diagonal* i.e., there exists a unique $d \in \text{Hom}(B, A)$ such that:

$$(Ud)e = f \text{ and } m_i d = f_i \text{ for each } i \text{ in } I.$$

1.8. $\mathscr{A}$ is called an $(\mathscr{E}, \mathscr{M})$-*category* in case the identity functor on $\mathscr{A}$ is an $(\mathscr{E}, \mathscr{M})$-functor.

PROPOSITION 1.9. *Let $(e, A, g_i)_I$ be a factorization of a U-source $(X, f_i, A_i)_I$. Then each of the statements below implies those that follow it:*

(1) *$(e, A, g_i)_I$ is semi-initial, and e is a retraction.*

(2) *$(A, g_i)_I$ is initial.*

(3) *$(e, A, g_i)_I$ is semi-initial.*

PROPOSITION 1.10. *Each of the statements below implies those that follow it.*

(1) *U is a (U-epi, $\mathscr{M}$)-functor for some $\mathscr{M}$.*

(2) *U is topologically-algebraic.*

(3) *U is semi-topological.*

(4) *U has a left adjoint.*

*Proof.* (1) implies (2): To show that $U$ is topologically-algebraic we need only show that every $(A, m_i)_I \in \mathscr{M}$ is initial. But this follows immediately since every commutative square of the form:

$$\begin{array}{ccc} UB & \xrightarrow{1_{UB}} & UB \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Uf_i} \\ UA & \xrightarrow[Um_i]{} & UA_i \end{array}$$

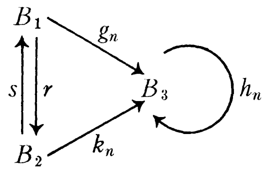
with $(A, m_i)_I \in \mathscr{M}$ has a diagonal.

That (2) implies (3) follows from Proposition 1.9. That (3) implies (4) is obvious.

**2. "Are not equivalent".** None of the implications in Proposition 1.10 is reversible. For the first consider the identity functor on the category $\cdot \rightrightarrows \cdot$ that is not (epi, $\mathscr{M}$) for any $\mathscr{M}$; for the third consider the functor from a non-complete partially ordered class with smallest element into the terminal category; for the second consider the following:

THEOREM 2.1. *There exists a subcategory $\mathscr{A}$ of a countable category $\mathscr{X}$ such that the inclusion functor from $\mathscr{A}$ into $\mathscr{X}$ is semi-topological but not topologically-algebraic.*

*Proof.* Consider the subcategory $\mathscr{X}$ of *Set*,




with

$$B_1 = \{1\},\ B_2 = \{2\},\ B_3 = \mathbf{N} = \{0, 1, 2, \ldots\},\ s(2) = 1,\ r(1) = 2,$$
$$g_n(1) = n,\quad k_n(2) = n,\quad h_n^{(m)} = n + m,\quad (\text{here } n, m \in \mathbf{N}),$$
$$1_{B_1}(1) = 1,\ 1_{B_2}(2) = 2, \text{ and } 1_{B_3} = h_0.$$

Let $\mathscr{A}$ be the subcategory of $\mathscr{X}$ obtained by removing $r$ and $g_0$, and let $U:\mathscr{A} \to \mathscr{X}$ be the inclusion functor. Then $U$ fails to be topologically-algebraic since the $U$-source $(B_3, \emptyset)$ has no ($U$-epi, initial)-factorization.

To see that $U$ is semi-topological consider the following cases:

| $U$-source $(X, f_i, A_i)_I$ | $U$-epi semi-initial factorization |
|---|---|
| $X = B_1$, some $f_i \in \{g_0, r\}$ | $(r, B_2, f_i^1)_I$* |
| $X = B_1$, no $f_i \in \{g_0, r\}$ | $(1_{B_1}, B_1, f_i)_I$ |
| $X = B_2$, some $f_i \in \{k_0, 1_{B_2}\}$ | $(1_{B_2}, B_2, f_i)_I$ |
| $X = B_2$, no $f_i \in \{k_0, 1_{B_2}\}$ | $(s, B_1, f_i^2)_I$** |
| $X = B_3$, some $f_i = h_0$ | $(1_{B_3}, B_3, f_i)_I$ |
| $X = B_3$, no $f_i = h_0$ | $(h_1, B_3, f_i^3)_I$*** |

*$f_i^1$ is the unique $\mathscr{A}$-morphism such that $f_i = f_i^1 r$.
*$f_i^2$ is the unique $\mathscr{A}$-morphism such that $f_i = f_i^2 s$.
***$f_i^3$ is the unique $\mathscr{A}$-morphism such that $f_i = f_i^3 h_1$.

COROLLARY 2.2. *Topologically-algebraic functors are not closed under compositions.*

*Proof.* Every semi-topological functor admits a factorization into a reflective full embedding followed by a topological functor, both of which are topologically-algebraic. (See [**9**], [**5**] and [**3**]).

COROLLARY 2.3. *There exists a category over a category $\mathscr{X}$ which has both a reflective Mac Neille completion and a universal initial completion that is not reflective.*

*Proof.* Every semi-topological functor has a reflective Mac Neille completion (see references in 2.2). Also every small faithful functor has a universal initial

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completion (see [**2**]). But any faithful functor having a reflective universal initial completion is topologically-algebraic (see [**3**]).

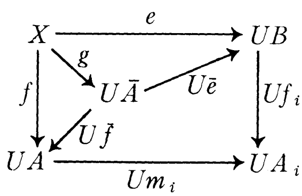
**3. "Are equivalent".** We now provide rather painless conditions under which the concepts topologically-algebraic and semi-topological coincide.

THEOREM 3.1. *Let $\mathscr{A}$ be an (epi, $\mathscr{M}$)-category for some family $\mathscr{M}$ of $\mathscr{A}$-sources. Then the following are equivalent:*
(1) *U is a (U-epi, $\mathscr{M}$)-functor.*
(2) *U is topologically-algebraic.*
(3) *U is semi-topological.*
(4) *U has a left adjoint.*
(5) *U is (U-epi, $\mathscr{M}$)-factorizable.*

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follows from Proposition 1.10.

(4) $\Rightarrow$ (5). Let $\eta$ be the unit of the adjunction. Universality of each $\eta_X$ together with (epi, $\mathscr{M}$)-factorizations in $\mathscr{A}$ give ($U$-epi, $\mathscr{M}$)-factorizations of $U$-sources.

(5) $\Rightarrow$ (1). Consider the commutative square of 1.7 with $(e, B)$ $U$-epi and $(A, m_i)_I$ in $\mathscr{M}$. The (U-epi, $\mathscr{M}$)-factorization of its upper left corner induces the diagram




in which $f_i\bar{e} = m_i\bar{f}$ since $(g, \bar{A})$ is a $U$-epi. Also $\bar{e}$ is epi since $(e, B)$ is a $U$-epi. Because $\bar{A}$ is an (epi, $\mathscr{M}$)-category, the square $f_i\bar{e} = m_i\bar{f}$ has a diagonal $d$, which is then a diagonal for the original square.

COROLLARY 3.2. *Let $\mathscr{A}$ be complete, cocomplete and co-well-powered. Then the following are equivalent:*
(1) *U is a (U-epi, $\mathscr{M}$)-functor for some family $\mathscr{M}$ of $\mathscr{A}$-sources.*
(2) *U is topologically-algebraic.*
(3) *U is semi-topological.*
(4) *U has a left adjoint.*
(5) *U is (U-epi, extremal monosource)-factorizable.*

*Proof.* This follows from 1.10, 3.1 and the following lemma.

LEMMA 3.3. *Let $\mathscr{A}$ be complete, cocomplete and co-well-powered. Then $\mathscr{A}$ is an (epi extremal monosource)-category.*

*Proof.* Cointersections and products give (epi, extremal-mono)-factorizations of $\mathscr{A}$-sources, and diagonalization follows from the existence of pushouts.

THEOREM 3.4. *Let $\mathscr{A}$ be complete, co-well-powered and well-powered. Then the following are equivalent:*

(1) *U is topologically-algebraic.*

(2) *U is semi-topological.*

(3) *U has a left adjoint.*

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) follows from Proposition 1.10.

(3) $\Rightarrow$ (1). This will follow if we provide (epi, initial-mono)-factorizations of $\mathscr{A}$-sources (see [**6**, Theorem 2.5]). By co-well-poweredness it suffices to consider a set-indexed $\mathscr{A}$-source $(A, f_i, A_i)_I$. This source has a factorization $(f, P, \pi_i)_I$ where $(P, \pi_i)$ is a product. Let $(h_j, B_j, g_j)_J$ be the class of all factorizations of $f$ in which $g_j$ is a $U$-initial mono and let $(D, d)$ be the intersection of the $(B_j, g_j)_J$. There exists $e: A \to D$ with $de = f$. Then $(e, D, \pi_i d)_I$ is the required factorization of $(A, f_i)_I$.

The example given in 2.1 is countable and object-finite. A modification of 2.1 gives an example which is countable and has finite hom sets. The following theorem shows that these two examples are in some sense minimal.

THEOREM 3.5. *Let $\mathscr{A}$ be finite. Then the following are equivalent:*

(1) *U is topologically-algebraic.*

(2) *U is semi-topological.*

*Proof.* (2) $\Rightarrow$ (1). Let $(X, f_i, A_i)_I$ be a $U$-source. Then $(X, f_i, A_i)$ has a semi-initial factorization $(e_1, B_1, f_{1,i})_I$ with $(e_1, B_1)$ $U$-epi. Now for $k = 2, 3 \ldots$ we have semi-initial factorizations $(e_k, B_k, f_{k,i})_I$ of $(UB_{k-1}, Uf_{k-1,i}, A_i)_I$, with $(e_k, B_k)$ $U$-epi. It follows, by semi-initiality, that for each $k \geqq 2$ there exists a unique $h_k \in \mathrm{Hom}(B_{k-1}, B_k)$ with $Uh_k = e_k$. Now define $g_{m,n}: B_m \to B_n$ by:

$$g_{m,n} = \begin{cases} 1_{B_n} \text{ if } m = n \\ h_n \circ \ldots \circ h_{m+2} \circ h_{m+1} \text{ if } n > m \geqq 1. \end{cases}$$

Since $\mathscr{A}$ is finite there exists $k < p$ such that $g_{1,k} = g_{1,p}$. Thus

$$h_p \circ g_{k,p-1} \circ g_{1,k} = g_{1,k}$$

or (since $g_{1,k}$ is an epi)

$$h_p \circ g_{k,p-1} = 1_{B_k}.$$

So $h_p$ (therefore $Uh_p$) is a retraction. It follows by Proposition 1.9 that $(B_p, f_{p,i})_I$ is initial and hence $((Ug_{1,p})e_1, B_p, f_{p,i})_I$ is a ($U$-epi, initial)-factorization of $(X, f_i, A_i)_I$.

The authors would like to thank the referee for many helpful suggestions. We wish to mention that after completion of this manuscript we received a letter from W. Tholen stating that:

(1) R. Börger has independently obtained an example of a functor which is semi-topological but not topologically-algebraic.

(2) W. Tholen has independently obtained essentially our results 3.2 and 3.4.

These results will appear in [**1**].

Finally, the authors wish to acknowledge their indebtedness to Y. T. Rhineghost for many useful discussions.

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