On a Certain Type of Space-Tableau

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(*Received 6th August*, 1951. *Read 2nd November*, 1951.)

1. *Tableaux and Standard Tableaux.*

An integer n may be *partitioned* into the set of integers $(\alpha_1, \alpha_2, \ldots, \alpha_t)$, where $\Sigma\alpha_i = n$ and $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_t$. Whenever $\alpha_{i-1} > \alpha_i$ the substitution of $\alpha_i + 1$ for α_i, where α_i may be $\alpha_{t+1} \equiv 0$, would give a partition of $n+1$ whilst the substitution of $\alpha_{i-1} - 1$ for α_{i-1} would give a partition of $n-1$; partitions of $n+1$ and of $n-1$ so arising will be said to be *associated* with the given partition of n.

We shall consider, corresponding to each partition, a set of n elements arranged in the top left-hand corner of a matrix, occupying the first α_i places in the i-th row, $i = 1, 2, \ldots, t$. This array of elements will be called a tableau on the given partition. Tableaux were introduced and studied by Alfred Young, references to whose work are given by Rutherford in an earlier paper in these Proceedings [1]. If the n elements have a prescribed order, in which case a convenient realisation of the set of elements is given by the first n integers in ascending or descending order of magnitude or the first n letters of an alphabet in direct or reversed dictionary order, we may demand that the order shall be preserved in both rows and columns; the tableau is then said to be *standard.*

There is a stock process for analysing standard tableaux on a partition by way of the associated partitions of $n-1$. It can be used to determine the *number* of standard tableaux [2] but we shall be concerned only with the process itself. We take as a realisation of the set of elements the first n integers in descending order of magnitude and consider the standard tableaux *on a given partition of n.* The possible positions of the element 1 are exactly those the removal of which would give the associated partitions of $n-1$. If from any of our set of standard tableaux we remove the element 1, we are left with a standard tableau of the integers $n, \ldots, 3, 2$; every standard tableau on an associated partition of $n-1$ will arise thus

[1] Rutherford, *Proc. Edinburgh Math. Soc.*, 7 (1942), 51–4.

[2] Rutherford, *loc. cit.*, §2.

https://doi.org/10.1017/S0013091500021337 Published online by Cambridge University Press

from exactly one of the standard tableaux of our set. We thus have the iterative rule:

(1.1) *The number of standard tableaux on a given partition of n is equal to the sum of the numbers of standard tableaux on the associated partitions of $n-1$.*

As an example, we illustrate diagrammatically the analysis of the standard tableaux on the partition (4, 2); in the right-hand column are enumerated the nine standard tableaux arising, given by the integers 6, ..., 2, 1; these integers, in natural order, prescribe the order in which the corresponding positions are omitted as the analysis proceeds (Fig. 1).

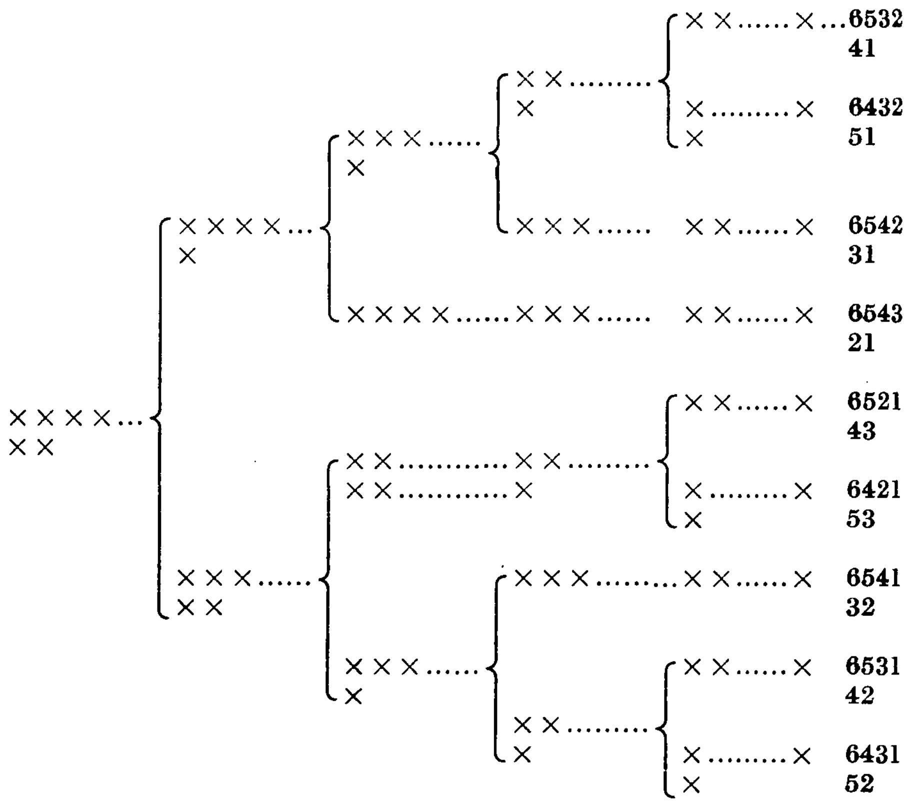

Fig. I.

The iterative relation for standard tableaux when we consider the associated partitions of $n+1$ is less immediate and we quote without proof

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the result[1] that:

(1 . 2) *The number of standard tableaux on a given partition of n, multiplied by* $n+1$, *is equal to the sum of the numbers of standard tableaux on the associated partitions of* $n+1$.

Tableaux may also be considered in which the *repetition* of elements is allowed in either rows or columns or both; here again we may standardise by requiring that the prescribed order shall not be violated in either rows or columns. Of such tableaux we remark only that when we allow repetition in either ordering we can require that each element shall be followed in this ordering either by a replica or by its natural successor; the ordering will then be said to be *conjunct.*

2. *Space-Tableaux.*

The above tableaux are *plane tableaux*: they are ordered in two dimensions. We now consider the tableaux built up as before but with three dimensions instead of two. It will be convenient to describe the third ordering as *vertical* and to refer to the *layers* formed by the other two.

We shall here be concerned only with a certain type of three-dimensional tableau, for which the name *space-tableau* will be reserved. It is defined by the conditions that the layers, enumerated from the lowest or *basic layer*, shall be standard tableaux (without repetition) formed respectively from elements such as

$$\begin{array}{l} n, \; \ldots, \; b, \; a; \\ \cdots \;\; \cdots \;\; \cdots \;\; \cdots \\ \cdots \;\; \cdots \;\; \cdots \\ b, \; a; \\ a \end{array}$$

and that order, reverse-alphabetical in this notation, shall also be preserved vertically.

There are two immediate consequences of the definition:

(2 . 1) *The numbers of elements in vertical columns form a standard tableau of* $n, \ldots, 2, 1$.

(2 . 2) *The vertical ordering is conjunct.*

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[1] Rutherford, *loc. cit.*, §3.

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3. *Construction of Space-Tableaux from above.*

It follows from the definition that the top block of r layers of a space-tableau is itself a space-tableau. Space-tableaux may therefore be constructed from the top by finding at each stage layers of $r+1$ elements that may be placed beneath a space-tableau with basic layer of r elements to form a space-tableau with basic layer of $r+1$ elements.

The relation between two adjacent layers is, however, a simple one. One element of the lower layer is exposed; if, for example, this were f, the letters k, ..., g would be covered by their alphabetical predecessors and the letters, e, ..., b, a by replicas. It follows that, if the upper layer be a standard tableau on a given partition of r, the lower layer must be a standard tableau on an associated partition of $r+1$ and that each such standard tableau of $r+1$ elements is a possible lower layer for exactly one standard tableau on the given partition of r.

We now have, by 1.2, that the number of space-tableaux with a basic layer of $r+1$ elements is obtained from the number of space-tableaux with a basic layer of r elements by multiplication by $r+1$. Hence:

(3.1) *The number of space-tableaux with a basic layer of n elements is n!*

4. *The Correspondence Theorem.*

Now consider the method of building up a space-tableau from the basic layer. Suppose that at any stage in the process it be decided to leave exposed the element f; it is of course necessary for the position of f to be such that its removal leaves an associated partition. Then it is clear, from our previous consideration of the relation between adjacent layers, that this decision completely determines the next layer. We thus have:

(4.1) *A Correspondence Theorem: The process of determining the number of space-tableaux with a given basic layer is isomorphic with that of analysing the standard tableaux on the corresponding partition as explained in* §1.

We again illustrate, choosing as basic layer for our space-tableau one of the standard tableaux on the partition (4, 2) used in Fig. I. The

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standard tableaux in the right-hand column (*cf.* Fig. I) give the numbers of elements in the vertical columns of the space-tableaux (Fig. II).

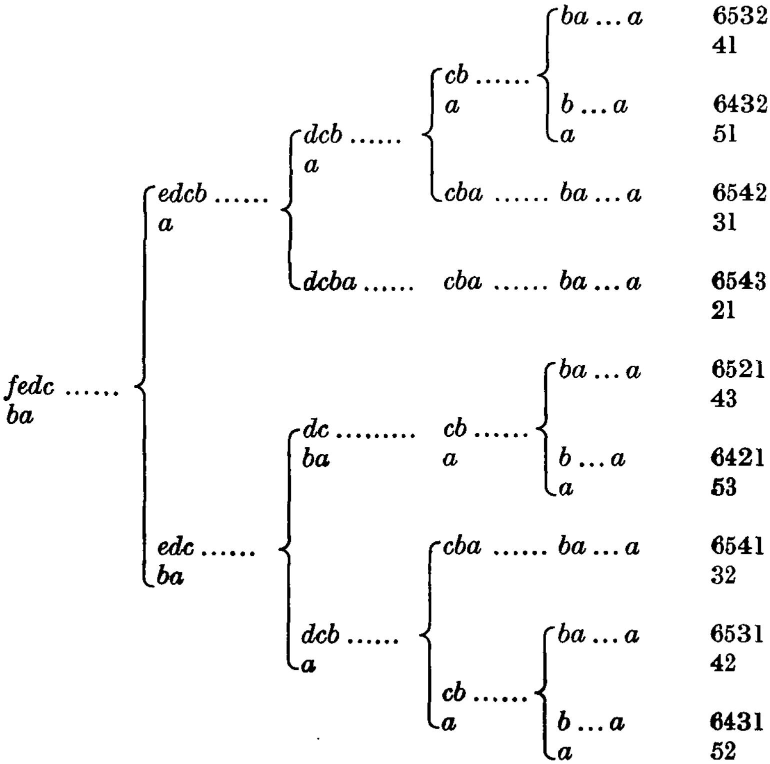

Fig. II.

We now have the following results, of which (4.3) is known:

(4.1) *The number of space-tableaux with a given basic layer is equal to the number of standard tableaux on the partition of this basic layer.*

(4.2) *The number of space-tableaux with basic layer of given partition is equal to the square of the number of standard tableaux on this partition.*

(4.3) *The sum of the squares of the numbers of standard tableaux on the partition of n is n*! (3.1 and 4.2).

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