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Characters for summary functions associated with cartesian products

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In deriving macroscopic descriptions of microscopic phenomena one often uses a vector valued summary function which is defined on a cartesian product in terms of component summary functions. We show that any character of such a summary function is the product of characters of its component summary functions.

1. Introduction

We adopt without further comment the notation and terminology of Finch [1]. In many practical situations one is considering two non-empty sets X' and X'' together with a universal M'-valued summary function $\xi' : X'_* \to M'$ and a universal M''-valued summary function $\xi'' : X''_* \to M''$. One's immediate interest is in the cartesian product $X = X' \times X''$ and the universal $M' \times M''$-valued summary function $\xi : X_* \to M' \times M''$ defined by

(1.1) $$\forall n \geq 1 : \xi_n(x'_1, x''_1)(x'_2, x''_2) \ldots (x'_n, x''_n) = (\xi'_n x'_1 x'_2 \ldots x'_n, \xi''_n x''_1 x''_2 \ldots x''_n) .$$

If χ' is a character for ξ' and χ'' is a character for ξ'' then $\chi : M' \times M'' \to \mathbb{C}$ with

$$\mathrm{dom}\chi = \mathrm{dom}\chi' \times \mathrm{dom}\chi''$$

and

(1.2) $$\chi(m', m'') = \chi'(m')\chi''(m'') ,$$

for all m' in $\mathrm{dom}\chi'$ and all m'' in $\mathrm{dom}\chi''$, is a character of ξ.

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However it is not immediately obvious that a general character of ξ must be of the form (1.2) where χ' and χ'' are characters of ξ' and ξ'' respectively, and it is our purpose in this paper to show that this is so. However we require a preliminary result about identity elements for summary functions and this is established in the next section.

2. Adjunction of an identity element

Let X be a non-empty set and let $\xi : X_* \to M$ be a universal M-valued summary function. We say that ξ has an identity element e when there is an element e in X such that, for any positive integer $n > 1$ and any elements $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$ in X,

$$\xi_n x_1 \ldots x_{k-1} e x_{k+1} \ldots x_n = \xi_{n-1} x_1 \ldots x_{k-1} x_{k+1} \ldots x_n .$$

In particular

$$\xi_n ee \ldots e = \xi_1 e .$$

When ξ does have an identity e one has $\chi(\xi_1 e) = 1$ for any non-trivial character χ of ξ.

If ξ does not have an identity element we can adjoin one to X in the following way. Let e be any symbol not representing an element of X_* and write $Y = X \cup e$. Similarly let g be any symbol not representing an element of M and write $L = M \cup g$. Define $\eta : Y_* \to L$ by

(i) $\eta | X_* = \xi$,

(ii) $\eta_n ee \ldots e = \eta_1 e = g$, $n \geq 1$,

(iii) for $n > 1$ and any $(y_1, y_2, \ldots, y_n)$ in Y_* with exactly $k < n$ occurrences of e,

$$\eta_n y_1 y_2 \ldots y_n = \xi_{n-k} x_1 x_2 \ldots x_{n-k}$$

where $(x_1, x_2, \ldots, x_{n-k})$ is the sequence obtained from $(y_1, y_2, \ldots, y_n)$ by deleting the k occurrences of e.

It is easily verified that η is a universal L-valued summary function with identity e. Moreover if $\alpha : L \to \mathbb{C}$ is a non-trivial character of η then $\alpha(g) = 1$ and $\chi = \alpha|\text{codom}\xi$ is a character of ξ. Conversely if χ is a character of ξ and we define $\alpha : \text{codom}\eta \to \mathbf{C}$ by extension from $\alpha|\text{codom}\xi = \chi$ and $\alpha(g) = 1$ then α is a character of η. It follows that there is no loss of generality in supposing that a summary function does have an identity element. It should be noted, however, that an identity element for a summary function ξ is not necessarily unique; if e and f are identities then we have $\xi_1 e = \xi_1 f$.

3. The theorem

We return now to the situation of Section 1 involving two summary functions ξ' and ξ'' and the derived summary function ξ defined by equation (1.1). We will, however, suppose now that ξ' has an identity e' and that ξ'' has an identity e''. We start with the

LEMMA. *Let* $\chi : \text{codom}\xi \to \mathbf{C}$ *be a character of* ξ *and define* $\chi' : \text{codom}\xi' \to \mathbf{C}$ *by*

$$\forall m' \in \text{codom}\xi' : \chi'(m') = \chi(m', \xi_1'' e'') .$$

Then χ' *is a character of* ξ'. *In like manner* $\chi'' : \text{codom}\xi \to \mathbf{C}$ *defined by*

$$\forall m'' \in \text{codom}\xi'' : \chi''(m'') = \chi(\xi_1' e', m'')$$

is a character of ξ''.

Proof. For any $n \geq 1$ and any $x_1', x_2', \ldots, x_n'$ in X' one has

$$\begin{aligned}
\chi'(\xi_n' x_1' x_2' \ldots x_n') &= \chi(\xi_n' x_1' x_2' \ldots x_n', \xi_1'' e'') \\
&= \chi(\xi_n' x_1' x_2' \ldots x_n', \xi_n'' e'' e'' \ldots e'') \\
&= \chi(\xi_n (x_1', e'')(x_2', e'') \ldots (x_n', e'')) \\
&= \chi(\xi_1 (x_1', e''))\chi(\xi_1 (x_2', e'')) \ldots \chi(\xi_1 (x_n', e'')) \\
&= \chi(\xi_1' x_1', \xi_1'' e'')\chi(\xi_1' x_2', \xi_1'' e'') \ldots \chi(\xi_1' x_n', \xi_1'' e'') \\
&= \chi'(\xi_1' x_1')\chi'(\xi_1' x_2') \ldots \chi'(\xi_1' x_n') .
\end{aligned}$$

This establishes that χ' is a character of ξ' and the result for χ'' is proved in the same way.

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We are now in a position to prove our main result, namely the

THEOREM. *Let* χ *be a character of* ξ . *Then for any* (m', m'') *in the codomain of* ξ ,

(3.1) $$\chi(m', m'') = \chi'(m')\chi''(m'') ,$$

where χ', χ'' *are the characters of* ξ', ξ'' *respectively which are defined in the lemma.*

Proof. We start by establishing that

(3.2) $$\forall x' \in X' \ \& \ \forall x'' \in X'' : \chi\left(\xi_1(x', x'')\right) = \chi'\left(\xi_1'x'\right)\chi''\left(\xi_1''x''\right) .$$

To prove (3.2) we observe that

$$\begin{aligned}\chi\left(\xi_1(x', x'')\right) &= \chi\left(\xi_1'x', \xi_1''x''\right)\\ &= \chi\left(\xi_2'x'e', \xi_2''e''x''\right)\\ &= \chi\left(\xi_2(x', e'')(e', x'')\right)\\ &= \chi\left(\xi_1(x', e'')\right)\chi\left(\xi_1(e', x'')\right)\\ &= \chi\left(\xi_1'x', \xi_1''e''\right)\chi\left(\xi_1'e', \xi_1''x''\right)\\ &= \chi'\left(\xi_1'x'\right)\chi''\left(\xi_1''x''\right) .\end{aligned}$$

Suppose now that (m', m'') is in the codomain of ξ ; then there is $n \geq 1$ and there are elements $x_1', x_2', \ldots, x_n'$ in X' and elements $x_1'', x_2'', \ldots, x_n''$ in X'' such that

$$m' = \xi_n'x_1'x_2' \ldots x_n' \ \& \ m'' = \xi_n''x_1''x_2'' \ldots x_n'' .$$

To establish (3.1) we make use of (3.2) in observing that

$$\begin{aligned}\chi(m', m'') &= \chi\left(\xi_n'x_1'x_2' \ldots x_n', \xi_n''x_1''x_2'' \ldots x_n''\right)\\ &= \chi\left(\xi_n\left(x_1', x_1''\right)\left(x_2', x_2''\right) \ldots \left(x_n', x_n''\right)\right)\\ &= \prod_{k=1}^{n} \chi\left(\xi_1\left(x_k', x_k''\right)\right)\\ &= \left[\prod_{j=1}^{n} \chi'\left(\xi_1'x_j'\right)\right]\left[\prod_{k=1}^{n} \chi''\left(\xi_1''x_k''\right)\right]\\ &= \chi'\left(\xi_n'x_1'x_2' \ldots x_n'\right)\chi''\left(\xi_n''x_1''x_2'' \ldots x_n''\right)\\ &= \chi'(m')\chi''(m'') .\end{aligned}$$

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This is the desired result.

Note that the result just established can be easily extended to the case of an arbitrary but finite number of component summary functions.

Reference

[1] Peter D. Finch, "Macroscopic descriptions of microscopic phenomena", *Bull. Austral. Math. Soc.* 9 (1973), 391-410.

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