ON INGHAM'S SUMMATION METHOD

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Ingham **(2)** has defined the following summation method. A series $\sum a_n$ will be called summable (I) to s if

$$\lim_{x\to\infty} \sum_{n\leqslant x} \frac{n}{x}\left[\frac{x}{n}\right] a_n = s,$$

where as usual $[x]$ is the greatest integer $\leqslant x$. (An equivalent method was described somewhat earlier by Wintner **(7)** ,who called it "an Eratosthenian method"; however, the notation (I) and the name "Ingham summability" introduced by Hardy **(1)** seem to have become usual.)

Ingham has proved that although the method (I) is not comparable with convergence (and so, in particular, not regular), for every δ, $0 < \delta < 1$, summability $(C, -\delta)$ implies summability (I), and for every $\delta > 0$, summability (I) implies summability (C, δ), where (C, k) denotes the Cesàro mean of order k. Pennington **(4)** and Rajagopal **(5)** have in fact given explicit constructions of convergent series whose Ingham sum is unbounded as $x \to \infty$.

In the present paper we are mainly concerned with Tauberian converses of the above results and the possibility or impossibility of inferring (I)-summability from Abel or (C, k)-summability for some $k \geqslant 0$ under certain auxiliary restrictions on $\{a_n\}$. Other Tauberian theorems stated by Rajagopal **(5)** with brief indications of proof connect generalized Lambert summability and Pennington's generalization of Ingham summability **(4)**. A special case of one of these results is that Lambert summability (and so *a fortiori* (C, k) summability) together with a Schmidt condition on $I(t)$ (see below) implies the convergence of $I(t)$ as $t \to \infty$.

The related question of "limitation" theorems for (I)-summability is also discussed.

Throughout this paper all sequences are of real numbers. $\mu(n)$ will denote the Möbius function

$$M(x) = \sum_{n\leqslant x} \mu(n), \qquad N(x) = \sum_{n\leqslant x} \frac{\mu(n)}{n},$$

$\phi(n)$ is Euler's ϕ-function, $[x]$ is the integral part of x, and $\{x\} = x - [x]$. $\sum_{d\mid n}$ will denote a sum over the positive divisors of n. Given a series $\sum a_n$, it will be convenient to denote the sum

$$\sum_{n\leqslant t} \frac{n}{t}\left[\frac{t}{n}\right] a_n$$

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by $I(t)$, so that the (I)-sum of $\sum a_n$ is equal to $\lim_{t\to\infty} I(t)$ (if it exists). The series $\sum a_n$ will be called (I)-bounded if

$$-\infty < \liminf_{t\to\infty} I(t) \leqslant \limsup_{t\to\infty} I(t) < \infty.$$

Since no more than one series will be under consideration at any one time, the notation $I(t)$ (which depends implicitly on $\sum a_n$) should not lead to any confusion. All other unexplained notation or terminology is as in **(1)**.

THEOREM 1. *Let $\sum a_n$ be a series of real numbers which is Abel-summable and such that*

$$\textstyle\sum_{d\mid n} da_d = O(1) \qquad \text{as } n \to \infty.$$

Then $\sum a_n$ is summable (I).

Proof. We show first that it is sufficient to prove the theorem if Abel-summability is replaced by summability $(C, 1)$. Let $\sum_{d\mid n} da_d = b_n$. Then

$$a_n = \frac{1}{n} \sum_{d\mid n} \mu\left(\frac{n}{d}\right) b_d \tag{1}$$

by Möbius inversion and

$$\sum_{n\leqslant r} a_n = \sum_{n\leqslant r} \sum_{d\mid n} \frac{\mu(r/d)}{r/d} \frac{b_d}{d} = \sum_{d\leqslant r} \frac{b_d}{d} N(r/d) = \sum_{d=1}^{\infty} \frac{b_d}{d} N(r/d)$$

(for $d > r$, the sum defining $N(r/d)$ is void and $N(r/d) = 0$). And so

$$\left|\sum_{n\leqslant r} a_n\right| \leqslant \sum_{d=1}^{\infty} \frac{|b_d|}{d} |N(r/d)| \leqslant \operatorname*{Sup}_{d} |b_d| \operatorname*{Sup}_{r} \sum_{d=1}^{\infty} \frac{1}{d} |N(r/d)|.$$

But the second factor is known to be bounded **(6**, correction**)** and b_d is bounded by hypothesis. Hence $\sum a_n$ is bounded, and so by a well-known Tauberian theorem **(1**, Theorem 92**)**, $\sum a_n$ is $(C, 1)$ summable.

The condition $\sum_{d\mid n} da_d = O(1)$ also implies that

$$I(t) = \sum_{n\leqslant t} a_n \frac{n}{t}\left[\frac{t}{n}\right] = \frac{1}{t} \sum_{n\leqslant t} \sum_{d\mid n} da_d = O(1), \tag{2}$$

and that for $y > x$,

$$\begin{aligned} I(y) - I(x) &= \frac{1}{y} \sum_{n\leqslant y} \sum_{d\mid n} da_d - \frac{1}{x} \sum_{n\leqslant x} \sum_{d\mid n} da_d \\ &= \left(\frac{1}{y} - \frac{1}{x}\right) \sum_{n\leqslant x} \sum_{d\mid n} da_d + \frac{1}{y} \sum_{x<n\leqslant y} \sum_{d\mid n} da_d \\ &= -\left(\frac{y-x}{y}\right) I(x) + O\left(\frac{y-x}{y}\right) = O\left(\frac{y-x}{y}\right), \end{aligned}$$

since $I(x) = O(1)$ by (2). Hence $I(y) - I(x) \to 0$ whenever x and y are such that $y > x$ and as $x \to \infty$, $y/x \to 1$. It follows that $I(t)$ is a slowly oscillating function defined on $(0, \infty)$ in the sense of Schmidt. (There are two definitions of slowly oscillating functions according as the function in question is defined over $(0, \infty)$ or $(-\infty, \infty)$. Here we use the definition appropriate to $(0, \infty)$; see (**1**, pp. 124, 288).)

We could now appeal to Rajagopal's result quoted above. However, a direct proof based on the known result

$$\int_0^\infty \frac{I(t)}{t} N(x/t)\, dt = \frac{1}{x} \sum_{n \leqslant x} (x - n) a_n \tag{3}$$

(e.g. **1**, p. 377) and using the Wiener–Pitt Tauberian theory is also possible and is given below. For completeness, we give first a proof of (3) slightly different from that in (**1**).

Since for $t < 1$, $N(t) = 0$ and $I(t) = 0$, we have

$$\begin{aligned}
\int_0^\infty \frac{I(t)N(x/t)}{t}\, dt &= \int_1^x \frac{I(t)}{t} N(x/t)\, dt = \int_1^x \frac{N(x/t)}{t^2} \sum_{d \leqslant t} da_d \left[\frac{t}{d}\right] dt \\
&= \sum_{d \leqslant x} da_d \int_d^x \frac{N(x/t)}{t^2} \left[\frac{t}{d}\right] dt = \frac{1}{x} \sum_{d \leqslant x} da_d \int_1^{x/d} N(u) \left[\frac{x}{ud}\right] du \\
&= \frac{1}{x} \sum_{d \leqslant x} da_d \sum_{k \leqslant x/d} \int_1^{x/kd} N(u)\, du \\
&= \frac{1}{x} \sum_{d \leqslant x} da_d \sum_{k \leqslant x/d} \left\{\frac{x}{kd} N(x/kd) - M(x/kd)\right\} \\
&= \frac{1}{x} \sum_{d \leqslant x} (x - d) a_d,
\end{aligned}$$

on letting $x/t = u$ and using the fact that by partial summation

$$xN(x) - M(x) = x \sum_{d \leqslant x} \frac{\mu(d)}{d} - \sum_{d \leqslant x} \mu(d) = \int_1^x N(t) dt,$$

and that, as is easily verified,

$$\sum_{k \leqslant x} \frac{1}{k} N(x/k) = \sum_{k \leqslant x} M(x/k) = 1.$$

But as noted above, $\sum a_n$ is $(C, 1)$-summable to A, say, under the conditions of the theorem. Hence writing $G(t) = (1/t)N(1/t)$, we have, by (3),

$$\begin{aligned}
\lim_{x \to \infty} \frac{1}{x} \int_0^\infty I(t) G(t/x)\, dt &= \lim_{x \to \infty} \int_0^\infty \frac{I(t)}{t} N(x/t)\, dt \\
&= \lim_{x \to \infty} \frac{1}{x} \sum_{d \leqslant x} (x - d) a_d = A.
\end{aligned} \tag{4}$$

Also, by a classical result of Landau, $|N(u)| = O(e^{-\alpha \sqrt{\log u}})$ for some $\alpha > 0$, and $N(u) = 0$ for $0 \leqslant u < 1$; hence $|N(u)|/u$ is integrable in $(0, \infty)$ and

$$K = \int_0^\infty \frac{|N(u)|}{u}\, du = \int_0^\infty \frac{1}{t} |N(1/t)| dt = \int_0^\infty |G(t)|\, dt,$$

on setting $t = 1/u$. Furthermore, by partial summation, we have, by classical results of Landau,

$$(5)\qquad \int_0^\infty G(t)dt = \int_0^1 \frac{1}{t} N(1/t)\,dt = \int_1^\infty \frac{N(u)}{u}\,du$$
$$= \lim_{k\to\infty}\left(N(k)\log k - \sum_{d\leqslant k} \frac{\mu(d)\log d}{d}\right) = 1.$$

Hence by (5), (4) may be written as

$$(6)\qquad \lim_{x\to\infty} \frac{1}{x}\int_0^\infty I(t)G(t/x)\,dt = A\int_0^\infty G(t)\,dt.$$

If

$$\int_0^\infty G(t)t^{-iy}\,dt \neq 0$$

for any real y, it will then follow by a theorem of Pitt (**1**, Theorem 233) that

$$\lim_{t\to\infty} I(t) = A,$$

which will prove the theorem. But this last condition is readily verified, since for $y = 0$ it is contained in (5), while for real $y \neq 0$ we have

$$\int_0^\infty G(t)t^{-iy}\,dt = \int_0^1 N(1/t)t^{-1-iy}\,dt = \int_1^\infty N(u)u^{-1+iy}\,du$$
$$= \lim_{k\to\infty}\left(\frac{k^{iy}N(k)}{iy} - \frac{1}{iy}\sum_{d\leqslant k}\frac{\mu(d)}{d^{1-iy}}\right) = -\frac{1}{iy\zeta(1-iy)} \neq 0$$

(where $\zeta(s)$ is the Riemann zeta-function). Hence the theorem follows.

Remarks. If the assumption that $\sum a_n$ is Abel-summable is replaced by the stronger hypothesis of $(C, 1)$-summability, then the hypothesis $\sum_{d|n} da_d = 0(1)$ can be altered to the weaker pair of conditions

(i) $\sum_{d|n} da_d > -K$ for some constant K,

(ii) $\sum a_n$ is (I)-bounded,

and again we may deduce that $\sum a_n$ is (I)-summable since $\mathrm{I}(x)$ can then be shown exactly as above to be slowly decreasing in the sense of Schmidt, the remainder of the proof going through as before. In fact, using (**1**, Theorem 94) instead of (**1**, Theorem 92) we may assume Abel summability. However, $(C, 1)$-summability and condition (ii) alone are not sufficient to imply (I)-summability as the explicit example of Theorem 2 below shows. (It will be noted that the examples of Pennington and Rajagopal of convergent series that are not (I)-summable are not (I)-bounded. On the other hand, the example below, although it is (I)-bounded and $(C, 1)$-summable, is not convergent.)

Also, under the assumption of $(C, 1)$-summability of $\sum a_n$ and condition (i), if we replace (ii) by the weaker condition that

$$\frac{1}{x}\int_1^x I(t)\,dt = O(1),$$

we may deduce the weaker conclusion:

$$\lim_{x\to\infty} \frac{1}{x} \int_1^x I(t)\,dt$$

exists. For it is easily verified that under these conditions,

$$S(x) = \frac{1}{x} \int_1^x I(t)\,dt$$

is slowly decreasing in the sense of Schmidt, and it then follows by an argument of Hardy (**1**, p. 304) that

$$\int_1^x \frac{S(t)}{t} N(x/t)\,dt = A + o(1) \qquad \text{as } x \to \infty,$$

the remainder of the proof being the same as in Theorem 1.

THEOREM 2. *There exists a series $\sum a_n$ that is (I)-bounded and $(C, 1)$-summable but not (I)-summable.*

Proof. Let

$$a_n = \sum_{d|n} \frac{\mu(d)}{d} (-1)^{n/d}.$$

That $\sum a_n$ is (I)-bounded is almost trivial since by Möbius inversion

$$\textstyle\sum_{d|n} da_d = n(-1)^n$$

and hence

$$I(t) = \sum_{d\leqslant t} \frac{d}{t}\left[\frac{t}{d}\right]a_d = \frac{1}{t}\sum_{n\leqslant t}\sum_{d|n} da_d = \frac{1}{t}\sum_{n\leqslant t} n(-1)^n$$
$$= \frac{1}{t}\left[\frac{t+1}{2}\right](-1)^{[t]} = O(1);$$

but does not converge as $t \to \infty$. Hence $\sum a_n$ is (I)-bounded but not (I)-summable.

To show that $\sum a_n$ is $(C, 1)$ summable, consider first the function

$$F(n) = 4\sum_{\substack{d|n \\ d \text{ even}}} \phi(\tfrac{1}{2}d).$$

Then

$$F(n) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 4\sum_{k|\frac{1}{2}n} \phi(k) = 2n, & \text{if } n \text{ is even.} \end{cases}$$

Hence

$$n(-1)^n = F(n) - n = 4\sum_{\substack{d|n \\ d \text{ even}}} \phi(\tfrac{1}{2}d) - \sum_{d|n} \phi(d).$$

Hence by Möbius inversion,

$$na_n = \sum_{d|n} \mu(d) \frac{n}{d} (-1)^{n/d} = \begin{cases} -\phi(n), & \text{if } n \text{ is odd,} \\ 4\phi(\tfrac{1}{2}n) - \phi(n), & \text{if } n \text{ is even.} \end{cases}$$

Now by a well-known elementary theorem of Mertens,

$$\sum_{n \leqslant x} \phi(n) = \frac{3}{\pi^2} x^2 + R(x),$$

where $R(x) = O(x \log x)$. Hence,

$$(7) \qquad \sum_{n \leqslant x} na_n = -\sum_{n \leqslant x} \phi(n) + 4 \sum_{\substack{n \leqslant x \\ n \text{ even}}} \phi(\tfrac{1}{2}n) = -\frac{3}{\pi^2} x^2 - R(x) + 4 \sum_{n \leqslant \frac{1}{2}x} \phi(n)$$
$$= 4R(\tfrac{1}{2}x) - R(x),$$

and also by partial summation,

$$(8) \qquad \sum_{n \leqslant x} a_n = \frac{4R(\frac{1}{2}x) - R(x)}{x} + \int_1^x \frac{4R(\frac{1}{2}t) - R(t)}{t^2}\, dt$$

(where $R(t) = -3t^2/\pi^2$ for $0 \leqslant t < 1$). Equations (7) and (8) give for the $(C, 1)$-mean of $\sum a_n$

$$(9) \qquad \frac{1}{x} \sum_{n \leqslant x} (x - n) a_n = \int_1^x \frac{4R(\frac{1}{2}t)}{t^2}\, dt - \int_1^x \frac{R(t)}{t^2}\, dt$$
$$= \int_1^{\frac{1}{2}x} \frac{R(t)}{t^2}\, dt - \int_{\frac{1}{2}x}^x \frac{R(t)}{t^2}\, dt - 3/\pi^2.$$

It remains to show that

$$\int_1^x \frac{R(t)}{t^2}\, dt$$

converges as $x \to \infty$. We have on the one hand, by partial summation,

$$(10) \qquad \sum_{n \leqslant x} \frac{\phi(n)}{n} = \frac{6}{\pi^2} x + \frac{R(x)}{x} - \frac{3}{\pi^2} + \int_1^x \frac{R(t)}{t^2}\, dt$$

and, on the other,

$$(11) \quad \sum_{n \leqslant x} \frac{\phi(n)}{n} = \sum_{n \leqslant x} \sum_{d|n} \frac{\mu(d)}{d} = \sum_{d \leqslant x} \frac{\mu(d)}{d} \left[\frac{x}{d}\right] = \frac{1}{x} \sum_{d \leqslant x} \mu(d) \frac{x}{d} \left[\frac{x}{d}\right]$$
$$= \frac{x}{2} \sum_{d \leqslant x} \frac{\mu(d)}{d^2} + \frac{1}{2x} \sum_{d \leqslant x} \mu(d) \left[\frac{x}{d}\right]^2 - \frac{1}{2x} \sum_{d \leqslant x} \mu(d) \left\{\frac{x}{d}\right\}^2$$
$$= \frac{3}{\pi^2} x - \frac{x}{2} \sum_{d > x} \frac{\mu(d)}{d^2} + \frac{1}{x} \sum_{d \leqslant x} \mu(d) \sum_{m \leqslant x/d} m - \frac{1}{2x} \sum_{d \leqslant x} \mu(d) \left[\frac{x}{d}\right]$$
$$- \frac{1}{2x} \sum_{d \leqslant x} \mu(d) \left\{\frac{x}{d}\right\}^2$$
$$= \frac{3}{\pi^2} x + \frac{1}{x} \sum_{n \leqslant x} \phi(n) - \frac{1}{2x} \sum_{d \leqslant x} \mu(d) \left\{\frac{x}{d}\right\}^2 + o(1)$$
$$= \frac{6}{\pi^2} x + \frac{R(x)}{x} + o(1),$$

since

$$\sum_{d\leqslant x} \mu(d)\left[\frac{x}{d}\right] = 1, \qquad \sum_{d>x} \frac{\mu(d)}{d^2} = o(1/x), \qquad \sum_{d\leqslant x} \mu(d)\left\{\frac{x}{d}\right\}^2 = o(x).$$

(This last can be deduced from "Axer's Theorem" (**1**, pp. 378, 386).)

Comparing (10) and (11), we have that

$$\lim_{x\to\infty} \int_1^x \frac{R(t)}{t^2}\,dt = \frac{3}{\pi^2}$$

and hence by (9) that $\sum a_n$ is $(C, 1)$-summable to 0, which proves the result.

Remarks. It is obvious that, as mentioned above, $\sum a_n$ is not convergent, since if p is any odd prime, $a_p = (1/p) - 1$ and hence $a_n \neq o(1)$ as $n \to \infty$.

Pennington (**4**, p. 79) and Hardy (**1**, Theorem 266) independently noted the following "limitation" theorem for (I)-summability:

If $\sum a_n$ *is (I)-summable, then* $a_n = o(\log\log n)$.

A limitation theorem of a somewhat different sort is:

If $\sum a_n$ *is (I)-summable, then*

$$\sum_{n\leqslant x} a_n = o(x^\delta)$$

for every $\delta > 0$.

This latter is a corollary of Ingham's result that (I)-summability implies (C, δ)-summability for every $\delta > 0$ and a well-known theorem on Cesàro means (**1**, Theorem 46). This latter result can be improved to:

THEOREM 3. *If* $\sum a_n$ *is (I)-summable, then*

$$\sum_{n\leqslant x} a_n = o(\log x).$$

Proof. By altering a_1, if necessary, we need only consider the case where $\sum a_n$ is summable (I) to 0. For m a non-negative integer, let

$$K(m) = \begin{cases} mI(m) = \sum_{d\leqslant m} da_d\left[\frac{m}{d}\right] = \sum_{n\leqslant m}\sum_{d\mid n} da_d & \text{if } m \geqslant 1, \\ 0 \quad \text{if } m = 0. \end{cases}$$

Then $K(m) = o(m)$ as $m \to \infty$.

Subtracting and using Möbius inversion gives

$$a_m = \frac{1}{m}\sum_{d\mid m} \mu(m/d)(K(d) - K(d-1)) = \sum_{dr=m} \frac{\mu(r)}{r}\frac{K(d) - K(d-1)}{d}$$

and hence

$$(12)\qquad \sum_{m\leqslant x} a_m = \sum_{dr\leqslant x} \frac{\mu(r)}{r}\frac{K(d)-K(d-1)}{d} = \sum_{d\leqslant x}\frac{K(d)-K(d-1)}{d} N(x/d)$$
$$= \sum_{d\leqslant x}\frac{K(d)}{d}\, d\left(\frac{N(x/d)}{d} - \frac{N(x/(d+1))}{d+1}\right)$$

since $K(0) = 0$ and $N(x/[x]+1) = 0$.

Since $K(m) = o(m)$ as $m \to \infty$, the required result will follow by a simple Abelian argument, provided we can prove that

$$(13)\qquad \sum_{d\leqslant x} d\left|\frac{N(x/d)}{d} - \frac{N(x/(d+1))}{d+1}\right| = O(\log x) \qquad \text{as } x \to \infty$$

and, for any *fixed* $D \geqslant 1$,

$$(14)\qquad \sum_{d\leqslant D} d\left|\frac{N(x/d)}{d} - \frac{N(x/(d+1))}{d+1}\right| = O(1) \qquad \text{as } x \to \infty.$$

But the left-hand sides of (13) and (14) are maximized by

$$(15)\qquad \sum_{d\leqslant v} |N(x/d) - N(x/(d+1))| + \sum_{d\leqslant v}\frac{|N(x/(d+1))|}{d+1},$$

with $v = x$ and $v = D$ respectively.

But since $|\mu(m)| \leqslant 1$,

$$\sum_{d\leqslant v} |N(x/d) - N(x/(d+1))| = \left|\sum_{d\leqslant v}\ \sum_{x/(d+1)<m\leqslant x/d}\frac{\mu(m)}{m}\right|$$
$$\leqslant \sum_{d\leqslant v}\ \sum_{x/(d+1)<m\leqslant x/d}\frac{1}{m} = \sum_{x/([v]+1)<m\leqslant x}\frac{1}{m} = O(1+\log v);$$

and since $|N(x)| \leqslant 1$,

$$\sum_{d\leqslant v}\frac{|N(x/(d+1))|}{d+1} \leqslant \sum_{d\leqslant v}\frac{1}{d+1} = O(1+\log v).$$

Taking $v = x$ and $v = D \geqslant 1$ (D fixed) we obtain (13) and (14) respectively, and so the theorem.

If $\sum a_n$ is (I)-bounded, then $K(m) = O(m)$ and the same proof gives

$$\sum_{n\leqslant x} a_n = O(\log x),$$

immediately.

Remark. By Rubel's result quoted above, the second sum in (15) is in fact $O(1)$ for $v = x$ also.

References

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