ON EXPLICIT BOUNDS IN SCHOTTKY'S THEOREM

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1. Introduction. To Schottky is due the theorem which states that a function $F(Z)$, regular and not taking the values 0 and 1 in $|Z| < 1$ and for which $F(0) = a_0$, is bounded in absolute value in $|Z| \leqslant r$, $0 \leqslant r < 1$, by a number depending only on a_0 and r. Let $K(a_0, r)$ denote the best possible bound in this result. Various authors have dealt with the problem of giving an explicit estimate for this bound. Qualitative estimates were given by Landau **(2)** and Valiron **(7)** and numerically evaluable estimates were given by Ostrowski **(4)**, Pfluger **(5)**, Ahlfors **(1)**, Robinson **(6)**, and Hayman **(3)**. In this paper we shall develop a simple method of obtaining such bounds. The bounds obtained are, in various situations, superior to any previously given. Our approach will be seen to be closest in spirit to that of Robinson.

2. Preliminary remarks. Our method depends initially on a simple application of the theory of subordination, a feature which it shares with most earlier methods (a notable exception being that of Ahlfors). Indeed it is clear that the maximum is attained for the function $F_0(Z)$ mapping $|Z| < 1$ onto the universal covering surface of the finite W-plane punctured at 0 and 1 and taking the value a_0 at $Z = 0$.

The function $F_0(Z)$ maps a domain D in $|Z| < 1$ bounded by three arcs of circles orthogonal to $|Z| = 1$ and with $Z = 0$ in its closure in a (1, 1) manner onto a half plane $\mathfrak{J}W > 0$ or $\mathfrak{J}W < 0$ with a_0 in its closure. Then a suitable branch of $w = \log F_0(Z) \mp \pi i$ maps $D(1, 1)$ onto the strip $-\pi < \mathfrak{J}w < 0$ or $0 < \mathfrak{J}w < \pi$ in these respective cases. Let $|Z| < 1$ be mapped onto $\mathfrak{R}z > 0$ in the following manner. In the first case let the sides of D corresponding to

$$\mathfrak{J}w = 0;\ \mathfrak{J}w = -\pi,\ \mathfrak{R}w > 0;\ \mathfrak{J}w = -\pi,\ \mathfrak{R}w < 0$$

go into the sets

$$\mathfrak{J}z = 0,\ \mathfrak{R}z > 0;\ \mathfrak{J}z = -\pi,\ \mathfrak{R}z > 0;\ |z - \tfrac{1}{2}\pi i| = \tfrac{1}{2}\pi,\ \mathfrak{R}z > 0.$$

In the second case let the sides of D corresponding to

$$\mathfrak{J}w = 0;\ \mathfrak{J}w = \pi,\ \mathfrak{R}w > 0;\ \mathfrak{J}w = \pi,\ \mathfrak{R}w < 0$$

go into the sets

$$\mathfrak{J}z = 0;\ \mathfrak{R}z > 0;\ \mathfrak{J}z = \pi,\ \mathfrak{R}z > 0;\ |z - \tfrac{1}{2}\pi i| = \tfrac{1}{2}\pi,\ \mathfrak{R}z > 0.$$

Denote the image of D by Δ. Let the mapping function be $Z = Z(z)$ and let us denote the function $F_0(Z(z))$ by $f(z)$. Evidently $\log f(z) \mp \pi i$ (with the choice of a suitable branch) maps the union $\mathfrak{D}$ of Δ, its reflection in the real

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axis and the positive real axis onto the strip $-\pi < \Im w < \pi$ and carries the semi-circular arc $|z| = \pi$, $\Re z > 0$ into the segment $\Re w = 0$, $-\pi < \Im w < \pi$.

Further let $\zeta = e^{-z}$, $\omega = e^{-w}$. It is verified at once that the composed mapping $\omega = \psi(\zeta)$ carries a subdomain E of $|\zeta| < 1$ having the origin as a centre of symmetry into the ω-plane slit along the negative real axis to the left of -1. We observe that $\psi(\zeta) = -(f(z))^{-1}$ where $\zeta = e^{-z}$. Let $\zeta = \phi(\omega)$ be the function inverse to ψ.

Let the point $z = b$ correspond to $Z = 0$. We will make use of the important remark due to Robinson that the maximum of $|F_0(Z)|$ on $|Z| = r$ is attained on the intersection of this circle with $\bar{D}$. The image of $|Z| \leqslant r$ lies in the half plane $\Re z \leqslant (1 + r)(1 - r)^{-1}\Re b$ so that $K(a_0, r)$ is not greater than the least upper bound of $|f(z)|$ when z lies in the intersection S of $\bar{\mathfrak{D}}$ and $\Re z \leqslant (1 + r)(1 - r)^{-1}\Re b$.

With this formulation Hayman's Theorems I and III **(3)** reduce almost to trivialities. Indeed $|\psi(\zeta)/\zeta| \geqslant 1$ while the set $|\omega| < 1$ corresponds to a domain containing $|\zeta| < e^{-\pi}$, thus either $|\omega| \geqslant 1$ or $|\psi(\zeta)/\zeta| \leqslant e^{\pi}$. Thus for z in S

$$\log |f(z)| \leqslant \frac{1+r}{1-r}\Re b, \tag{1}$$

while if $|f(b)| \leqslant 1$, $\Re b \leqslant \pi$ and if $|f(b)| > 1$,

$$\Re b \leqslant \log |f(b)| + \pi.$$

Denoting $\mu = \max[1, |f(b)|]$ we have

$$\Re b \leqslant \log \mu + \pi; \tag{2}$$

combining this with (1) we get

$$\log |f(z)| \leqslant \frac{1+r}{1-r}(\log \mu + \pi),$$

that is,

$$K(a_0, r) \leqslant (\mu e^{\pi})^{(1+r)/(1-r)},$$

which is Hayman's Theorem I.

Let us impose the additional restriction $|f(b)| = e^{-\delta}$, $\delta > 0$. We verify at once that $(f(\pi^2/z))^{-1}$ has the same mapping properties as $f(z)$. Thus $\Re \pi^2 b^{-1} \geqslant \log |f(b)|^{-1} = \delta$. Evidently then $\Re b \leqslant \pi^2/\delta$ and by equation (1) for z in S

$$\log |f(z)| \leqslant \frac{\pi^2}{\delta}\frac{1+r}{1-r},$$

that is,

$$K(a_0, r) \leqslant \exp\left\{\frac{\pi^2}{\delta}\frac{1+r}{1-r}\right\},$$

which is Hayman's Theorem III.

The examples constituting Hayman's Theorems II and IV can be given just as simply in an explicit geometric manner. It should be remarked that at one point Hayman actually constructed mappings corresponding to ϕ and ψ above

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but his proofs are greatly complicated by the unnecessary introduction of hyperbolic measure.

We can also obtain at once the asymptotic behaviour of $K(a_0, r)$ for fixed r as $|a_0|$ approaches ∞. Indeed, setting up the chain of mappings $U = V^2$, $\omega = 4V(1 - V)^{-2}$, $\zeta = \phi(\omega)$, $\xi = \zeta^2$, we see that the mapping from $|U| < 1$ to the ξ-plane is just that given by $\zeta = \phi(\omega)$ from $|\omega| < 1$ to the ζ-plane, i.e., $\xi = \phi(U)$ and

$$\phi(U) = \left\{\phi\left(\frac{4U^{\frac{1}{2}}}{(1-U^{\frac{1}{2}})^2}\right)\right\}^2.$$

Inserting the power series

$$\phi(U) = c_1 U + c_2 U^2 + \dots$$

in this functional equation, we can calculate all its coefficients. In particular $c_1 = 16c_1^2$, thus $c_1 = 1/16$. From this we derive readily the asymptotic behaviour due to Ostrowski **(4)**

$$K(a_0, r) \sim (16)^{2r/(1-r)} |a_0|^{(1+r)/(1-r)},$$

which, in fact, holds uniformly in r.

Now Robinson obtained not only the asymptotic behaviour but also universal bounds which imply the asymptotic result and which, indeed, for certain values of $|a_0|$ are better than Hayman's. In the next section we turn to the consideration of this question.

3. Bounds in Schottky's Theorem. We decompose the mapping $\omega = \psi(\zeta)$ into two stages, first mapping E by a function $\eta = \chi(\zeta)$ onto $|\eta| < 1$ so that $\chi(0) = 0$, $\chi'(0) > 0$ and $\chi(e^{-b}) = \eta_0$, say. Let the inverse function be $\zeta = \theta(\eta)$. Second we map $|\eta| < 1$ by the function $\omega = \lambda(\eta) = 4\eta(1 - \eta)^{-2}$ onto the ω-plane slit along the negative real axis to the left of -1. Clearly $\psi(\zeta) = \lambda(\chi(\zeta))$ and $\lambda(\eta_0) = -(f(b))^{-1}$. Note that $\lambda'(0) = 4$, $\chi'(0) = 4$.

Now, letting $|\eta| = r_1$,

$$|\lambda(\eta)| \geqslant \frac{4r_1}{(1 + r_1)^2},$$

thus

$$|\lambda(\eta)|^{-1} \leqslant \tfrac{1}{4}(r_1 + 2 + r_1^{-1}) \leqslant \tfrac{1}{4} r_1^{-1} + \tfrac{3}{4}. \tag{3}$$

Combining this with the evident estimate $|\chi(\zeta)| \geqslant |\zeta|$ we have

$$|\psi(\zeta)|^{-1} \leqslant \tfrac{1}{4}|\zeta|^{-1} + \tfrac{3}{4},$$

i.e., for z in S

$$|f(z)| \leqslant \tfrac{1}{4}\{\exp(\Re b)\}^{(1+r)/(1-r)} + \tfrac{3}{4};$$

using the estimate (2) we obtain

$$K(a_0, r) \leqslant \tfrac{1}{4}(\mu e^{\pi})^{(1+r)/(1-r)} + \tfrac{3}{4},$$

a bound better than Hayman's in all cases.

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This bound is not asymptotically best possible. To obtain such a bound we note that $\theta'(0) = \frac{1}{4}$ and that $\theta(\eta)$ is an odd function. Thus

$$|\theta(\eta)| \leqslant \tfrac{1}{4}\frac{r_1}{1 - r_1^2}$$

and

$$|\theta(\eta)|^{-1} \geqslant 4(r_1^{-1} - r_1).$$

Combining this with (3),

$$\begin{aligned} |\lambda(\eta)|^{-1} &\leqslant \tfrac{1}{4}(\tfrac{1}{4}|\theta(\eta)|^{-1} + 2r_1) + \tfrac{1}{2} \\ &\leqslant 16^{-1}|\theta(\eta)|^{-1} + 1; \end{aligned}$$

that is, for z in S,

(4) $$|f(z)| \leqslant 16^{-1}\{\exp(\Re b)\}^{(1+r)/(1-r)} + 1.$$

We want now a corresponding bound for $\exp(\Re b)$. For this set $|\eta_0| = r_0$ and note

$$|\lambda(\eta_0)| \leqslant \frac{4r_0}{(1 - r_0)^2},$$

that is,

$$\frac{1 + r_0^2}{r_0} \leqslant 4|\lambda(\eta_0)|^{-1} + 2.$$

On the other hand

$$|\theta(\eta_0)| \geqslant \tfrac{1}{4}\frac{r_0}{1 + r_0^2},$$

so that

$$|\theta(\eta_0)|^{-1} \leqslant 4\,\frac{1 + r_0^2}{r_0}.$$

Hence

$$\exp(\Re b) \leqslant 16|f(b)| + 8.$$

Combining this with (4) we obtain

$$K(a_0, r) \leqslant 16^{-1}(16|a_0| + 8)^{(1+r)/(1-r)} + 1.$$

This bound is not only asymptotically best possible but in a universal bound of the form

$$K(a_0, r) \leqslant A(B|a_0| + C)^{(1+r)/(1-r)} + L,$$

the quantity B cannot be smaller than 16. If $B = 16$, A cannot be smaller than 1/16. If $B = 16$, $A = 1/16$, C cannot be smaller than $e^{\pi} - 16$ (between 7 and 8). In the estimate (4) the additive term 1 could not be replaced by anything smaller than 15/16.

We collect up these results and state:

THEOREM 1. *If $K(a_0, r)$ denotes the best possible bound in Schottky's Theorem for functions $F(Z)$ with $F(0) = a_0$ and if $\mu = \max\,[1, |a_0|]$ then*

$$\begin{aligned} K(a_0, r) &\leqslant \tfrac{1}{4}\,(\mu e^{\pi})^{(1+r)/(1-r)} + \tfrac{3}{4}, \\ K(a_0, r) &\leqslant \tfrac{1}{16}\,(16|a_0| + 8)^{(1+r)/(1-r)} + 1. \end{aligned}$$

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The second bound is much the better except for $|a_0|$ near 1 and r fairly large. It is clear that numerous other bounds could be obtained by combinations and modifications of the above estimates. Further, a small improvement of the constant 1 in (4) can be obtained by using the fact that $\theta(\eta)$ is a bounded function.

4. A bound in Landau's Theorem. Suppose that $F(Z)$ is regular for $|Z| < 1$ and does not take the values 0 and 1 and that

$$F(Z) = a_0 + a_1 Z + \ldots$$

is its Taylor expansion about $Z = 0$. Landau's Theorem states that $|a_1|$ has a bound depending only on a_0. Hayman **(3)** gave the explicit bound

$$|a_1| \leqslant 2|a_0|\{|\log|a_0|| + 5\pi\}.$$

We shall show that the constant 5π can be notably reduced using the approach presented above.

We observe first that obtaining a bound of the form

$$|a_1| \leqslant 2|a_0|\{|\log|a_0|| + K\}$$

it is enough to confine ourselves to the case $|a_0| \geqslant 1$, $|a_0 - 1| \geqslant 1$. Indeed, if it is proved in this case and we have $|F(0)| \geqslant 1$, $|F(0) - 1| \leqslant 1$, we consider the function $\Phi(Z) = F(Z)/(F(Z) - 1)$. For it

$$|\Phi(0)| = |F(0)/(F(0) - 1)| \geqslant 1,$$
$$|\Phi(0) - 1| = |F(0) - 1|^{-1} \geqslant 1;$$

thus

$$|\Phi'(0)| = |a_1/(a_0 - 1)^2| \leqslant 2|a_0/(a_0 - 1)|\{|\log|a_0/(a_0 - 1)|| + K\},$$

that is,

$$\begin{aligned}|a_1| &\leqslant 2|a_0||a_0 - 1|\{|\log|a_0/(a_0 - 1)|| + K\}\\ &\leqslant 2|a_0|\{|\log|a_0|| + K - |a_0 - 1|\log|a_0 - 1| - K(1 - |a_0 - 1|)\}.\end{aligned}$$

Since, for $|a_0 - 1| \leqslant 1$,

$$|a_0 - 1|\log|a_0 - 1| + K(1 - |a_0 - 1|) \geqslant 0$$

provided $K \geqslant 1$, which is necessarily so in the present situation, the result is true for all $|a_0| \geqslant 1$. Applying a similar argument to the function $\Psi(Z) = (F(Z))^{-1}$ in the case $|a_0| < 1$ we obtain the same bound for all $|a_0|$.

Now for the function $\phi(\omega)$ we verify by an elementary calculation

$$\phi'(-a_0^{-1}) = -\frac{2a_0^{\,2}\Re b}{a_1 e^b}.$$

Further we have for $|\omega| < 1$ the well-known estimate

$$\left|\frac{1}{\zeta}\frac{d\zeta}{d\omega}\right| \geqslant \frac{1}{|\omega|}\frac{1 - |\omega|}{1 + |\omega|}.$$

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Applying this at the value $\omega = -a_0^{-1}$ we derive

$$|a_1| \leqslant \frac{1 + |a_0|^{-1}}{1 - |a_0|^{-1}} 2|a_0|\Re b.$$

It is clear that this will lead to an advantageous bound for $|a_0|$ large. In particular, if $|a_0|^{-1} \leqslant \epsilon(1 + \epsilon)^{-1}$,

$$\begin{aligned}|a_1| &\leqslant (1 + 2(1 + \epsilon)|a_0|^{-1})2|a_0|\Re b \\ &\leqslant 2|a_0|\Re b + 4(1 + \epsilon)\Re b.\end{aligned}$$

We recall

$$\Re b \leqslant \log(16\,|a_0| + 8).$$

Thus for $|a_0| \geqslant t$ we have

$$\begin{aligned}|a_1| \leqslant 2|a_0|\Big\{&\log |a_0| + \log 16 + \log\left(1 + \frac{1}{2t}\right) \\ &+ \frac{2t}{t-1}\left(e^{-1} + \frac{1}{t}\left[\log 16 + \log\left(1 + \frac{1}{2t}\right)\right]\right)\Big\},\end{aligned}$$

using the facts that $x^{-1}\log x \leqslant e^{-1}$ and $1 \leqslant |a_0|/t$ for $|a_0| \geqslant t$. In particular for $|a_0| \geqslant 2.63$

$$|a_1| \leqslant 2|a_0|\{\log|a_0| + 7.77\}. \tag{5}$$

It remains to find a bound advantageous when $|a_0|$ is near 1. Originally we used for this an improved version of Hayman's technique. However we now use an improved version of a suggestion by the referee which gives a somewhat simpler argument and also a slightly better value of the final constant.

We first make the trivial observation that if a domain D_1 in the z-plane is mapped conformally into a domain D_2 in the w-plane by a function $w = w(z)$ with the point z_0 going into the point w_0 and if the inner radius of D_1, D_2 with respect to z_0, w_0 is r_1, r_2 then $|w'(z_0)| \leqslant r_2/r_1$. Now the mapping constructed above from the z-plane to the w-plane corresponds to the maximal value of $|a_1|$ for given a_0, by the majoration principle. Under it the half-plane $\Re z > \frac{1}{2}\pi$ is mapped into the w-plane slit along the half-infinite segments $\Im w = (2n+1)\pi$, $\Re w < 0$, n running through all integers. Under the assumptions $|a_0| \geqslant 1$, $|a_0 - 1| \geqslant 1$ we have $\Re b \geqslant 3^{\frac{1}{2}}\pi/2$. The inner radius of $\Re z > \pi/2$ with respect to the point b is $2(\Re b - \pi/2)$. The inner radius of the w-plane slit as above with respect to the image of b (namely $\log a_0 \mp \pi i$) is $2|1 - a_0^{-1}|^{\frac{1}{2}} \log |2a_0 - 1 + 2(a_0(a_0 - 1))^{\frac{1}{2}}|$, the radicals being properly determined. The derivative of the mapping function at the point $z = b$ is $a_1/2a_0\Re b$. Thus

$$|a_1| \leqslant 2(|a_0||a_0 - 1|)^{\frac{1}{2}} \log |2a_0 - 1 + 2\{a_0(a_0 - 1)\}^{\frac{1}{2}}|\Re b/(\Re b - \pi/2).$$

Since $\Re b \geqslant 3^{\frac{1}{2}}\pi/2$ we have for $|a_0| = t$, $1 \leqslant t$,

$$\begin{aligned}|a_1| &\leqslant (3 + 3^{\frac{1}{2}})|a_0|(1 + t^{-1})^{\frac{1}{2}}\{\log|a_0| + \log[2 + t^{-1} + 2(1 + t^{-1})^{\frac{1}{2}}]\} \\ &\leqslant 2|a_0|\{\log|a_0| + L(t)\},\end{aligned}$$

where

$$L(t) = \tfrac{1}{2}(3 + 3^{\frac{1}{2}})(1 + t^{-1})^{\frac{1}{2}} \log[2t^2 + t + 2t(t^2 + t)^{\frac{1}{2}}] - 2\log t.$$

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A direct calculation verifies that $L(t)$ is an increasing function of t for $t \geqslant 1$. Thus for $1 \leqslant |a_0| \leqslant t$ we have

$$|a_1| \leqslant 2|a_0|\{\log|a_0| + L(t)\}.$$

In particular, for $1 \leqslant |a_0| \leqslant 2.63$,

$$|a_1| \leqslant 2|a_0|\{\log|a_0| + 7.77\}.$$

Combining this with (5) and our earlier remarks we have

THEOREM 2. *If $F(Z)$ is regular for $|Z| < 1$, does not take the values* 0 *and* 1 *and has Taylor expansion about $Z = 0$,*

$$F(Z) = a_0 + a_1 Z + \ldots,$$

then

$$|a_1| \leqslant 2|a_0|\{|\log|a_0|| + 7.77\}.$$

It should be observed that in an expression of this form the number 2 could not be replaced by any smaller number. We have replaced Hayman's constant 5π by 7.77 which is less than half as big. Our earlier method gave the constant 8.58.

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