Fredholm Toeplitz operators with VMO symbols and the duality of generalized Fock spaces with small exponents

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We characterize Fredholmness of Toeplitz operators acting on generalized Fock spaces of the n-dimensional complex space for symbols in the space of vanishing mean oscillation VMO. Our results extend the recent characterizations for Toeplitz operators on standard weighted Fock spaces to the setting of generalized weight functions and also allow for unbounded symbols in VMO for the first time. Another novelty is the treatment of small exponents $0 < p < 1$, which to our knowledge has not been seen previously in the study of the Fredholm properties of Toeplitz operators on any function spaces. We accomplish this by describing the dual of the generalized Fock spaces with small exponents.

1. Introduction

The Fock space (also known as the Segal–Bargmann space) consists of all holomorphic functions in the n-dimensional complex Euclidean space $\mathbb{C}^n$ square-integrable with respect to the Gaussian measure $\exp(-|z|^2)\,\mathrm{d}v$, where $\mathrm{d}v$ is the Lebesgue measure on $\mathbb{C}^n$. It plays an important role in many areas of analysis and its applications, especially in quantum mechanics. Its study is genuinely different from other function spaces, such as the Hardy space of the unit circle or the Bergman space of the unit ball, and features unique phenomena that require a distinct set of tools and techniques. Some of these applications arise from the theory of operators, such as Toeplitz and Hankel operators, and there is currently considerable interest in the study of these operators on the Fock space and its generalizations, which illustrates

the fruitful interplay between geometry, function theory of several complex variables and functional analysis; see [**18**].

Indeed, relatively recently, Schuster and Varolin [**15**] began the study of Toeplitz operators T_f with symbol f on generalized Fock spaces F_φ^p of $\mathbb{C}^n$ with respect to the measure $\exp(-p\varphi)\,\mathrm{d}v$, where the weight function φ is of C^2 with $dd^c\varphi$ comparable to the Euclidean Kähler form. When $n = 1$, this simply means that $m \leqslant \Delta\varphi \leqslant M$ for some positive constants m and M. Their results characterize boundedness and compactness of T_μ in terms of the so-called Carleson and vanishing Carleson conditions for symbols that are positive measures. This was followed by the work [**6**] of Lv and one of the present authors, in which similar characterizations are given for bounded and compact Toeplitz operators between possibly different Fock spaces F_φ^p and F_φ^q, covering the full range of parameters $0 < p, g < \infty$ and also symbols of bounded mean oscillation when $p = q$. For further results and recent developments in the study of operators and operator algebras on these function spaces, we refer to the papers [**8**, **9**] and the references therein.

The goal of the present paper is to study the Fredholm properties of Toeplitz operators T_f on generalized Fock spaces in terms of the Berezin transform of f at infinity for symbols of vanishing mean oscillation. Fredholm theory of Toeplitz operators originated in the works of Coburn, Douglas, Gohberg and Krupnik in the late 1960s, and it is closely connected with (singular) integral operators, Hankel operators and the theory of function spaces, such as Hardy, Bergman and Fock spaces.

In the late 1980s, Berger and Coburn [**3**], who were the first to study Fredholm Toeplitz operators on the Fock space F^2, showed that for $f \in L^\infty \cap VMO$, the Toeplitz operator is Fredholm if and only if there are positive numbers R and ϵ such that $|f(z)| \geqslant \epsilon$ whenever $|z| > R$. Their approach was based on C^*-algebra techniques and Hilbert space methods, which they used to obtain isomorphic representations as quotients of function algebras for the image in the Calkin algebra generated by Toeplitz operators with symbols in $L^\infty \cap VMO$. A few years later, Stroethoff [**16**] obtained the same characterization using methods more suited to operators on Banach spaces. Indeed, very recently, the Fredholm theory was extended to the setting of standard weighted Fock spaces F_α^p for $1 < p < \infty$ in [**2**] (in the one-dimensional setting) using similar ideas and elementary methods and in [**4**] using newly developed limit operator techniques. It is worth noting that analogous characterizations are well known in Hardy spaces of the unit circle for symbols in the Douglas algebra $C + H^\infty$ and in Bergman spaces of the unit ball for symbols in the algebra $L^\infty \cap VMO$.

In the present paper, we prove that the Toeplitz operator T_f with $f \in VMO$ is Fredholm on the generalized Fock space F_φ^p with $0 < p < \infty$ if and only if

$$0 < \liminf_{|z|\to\infty} |\widetilde{f}(z)| \quad \text{and} \quad \limsup_{|z|\to\infty} |\widetilde{f}(z)| < \infty, \tag{1.1}$$

where $\tilde{f}$ is the Berezin transform of the symbol f. Our result covers Berger and Coburn's characterization and its recent generalizations to standard weighted Fock spaces F_α^p. Another important feature of our analysis is the discovery of the second inequality in (1.1), which allows us to treat Toeplitz operators with all symbols

in VMO, in contrast to the previous results, which are all restricted to bounded symbols in VMO. Finally, we emphasize that our results remain valid for any parameter $0 < p < \infty$, which to our knowledge is a novelty in the study of the Fredholm properties of Toeplitz operators on any function space. For this reason, we also provide an account of the theory of Fredholm operators on quasi-Banach spaces in §4 and show that the dual of F_φ^p with $0 < p < 1$ can be identified with F_φ^∞; the dual spaces of the other generalized Fock spaces F_φ^p with $1 \leqslant p \leqslant \infty$ were described in [**15**] while the treatment of the standard weighted Fock spaces can be found in [**10**, **17**]. This will likely stimulate new interest and research activity in the study of operators on these and other small exponent function spaces.

In what follows, we first introduce basic background material that is essential to the rest of the paper. In §3, we discuss some more advanced preliminary results, such as characterizations of boundedness and compactness of Toeplitz and Hankel operators in terms of functions of bounded and vanishing oscillation and the Berezin transform. In §4, we recall the basic Fredholm theory both in the Banach and quasi-Banach space settings, describe the dual of F_φ^p for $0 < p < 1$ and then proceed to prove our main result (theorem 4.5). We conclude the paper with a list of open problems related to our work and other important questions about Toeplitz and Hankel operators on Fock spaces.

2. Notation and definitions

In this section, we define the generalized Fock spaces, Toeplitz and Hankel operators, the spaces of mean oscillation, and their close relatives.

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidean space. For $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we write

$$(z, w) = z_1 \overline{w_1} + \cdots + z_n \overline{w_n}$$

and $|z| = \sqrt{(z, z)}$. We denote by $B(z, r)$ the Euclidean ball in $\mathbb{C}^n$, that is,

$$B(z, r) = \{ w \in \mathbb{C}^n : |w - z| < r \}.$$

We write $\omega_0 = dd^c |z|^2$ for the Euclidean Kähler form on $\mathbb{C}^n$, where $d^c = \frac{\sqrt{-1}}{4}(\overline{\partial} - \partial)$. Throughout the paper, we assume that $\varphi \in C^2(\mathbb{C}^n)$ is real-valued and

$$dd^c \varphi \simeq \omega_0, \tag{2.1}$$

that is, there are two positive numbers M_1, M_2 satisfying the two inequalities

$$M_1 \omega_0 \leqslant dd^c \varphi \leqslant M_2 \omega_0 \tag{2.2}$$

in the sense of currents; see [**15**] for more details. For $0 < p < \infty$, the space L_φ^p consists of all Lebesgue measurable functions f on $\mathbb{C}^n$ for which

$$\|f\|_{p,\varphi} = \left(\int_{\mathbb{C}^n} \left| f(z) e^{-\varphi(z)} \right|^p \, \mathrm{d}v(z) \right)^{1/p} < \infty,$$

where v is the Lebesgue volume measure on $\mathbb{C}^n$. With the norm $\|\cdot\|_{p,\varphi}$, it is easy to check that L_φ^p is a Banach space when $p \geqslant 1$ and a quasi-Banach space when

$0 < p < 1$ (see definition 4.1). We denote by L^∞ the usual space of all bounded Lebesgue measurable functions f on $\mathbb{C}^n$ with the norm given by

$$\|f\|_\infty = \sup_{z \in \mathbb{C}^n} |f(z)| < \infty.$$

Finally, we denote by $H(\mathbb{C}^n)$ the family of all holomorphic functions on $\mathbb{C}^n$. We can now define the weighted Fock spaces.

DEFINITION 2.1. For $0 < p < \infty$, the Fock space with weight φ is defined by

$$F_\varphi^p = L_\varphi^p \cap H(\mathbb{C}^n).$$

It is well known that each F_φ^p is a closed subspace of L_φ^p, and hence it is a Banach space when $p \geqslant 1$ and a quasi-Banach space when $0 < p < 1$.

Functions of bounded and vanishing mean oscillation, which we will define next, play an important role in our analysis, and can be used to characterize boundedness and compactness of large classes of Toeplitz and Hankel operators. The mean oscillation $MO_r^p(f)$ of a locally integrable function f on $\mathbb{C}^n$ for $r > 0$ and $1 \leqslant p < \infty$ is defined by

$$MO_r^p(f)(z) = \left(\frac{1}{v(B(z,r))} \int_{B(z,r)} |f(w) - \widehat{f}_r(z)|^p \, dv(w) \right)^{1/p}$$

where the averaging function $\widehat{f}_r$ is given by

$$\widehat{f}_r(z) = \frac{1}{v(B(z,r))} \int_{B(z,r)} f(w) \, dv(w).$$

DEFINITION 2.2. Let $1 \leqslant p < \infty$. The space of functions of bounded mean oscillation BMO^p consists of all f in L_{loc}^1 for which

$$\|f\|_{BMO^p} = \sup_{z \in \mathbb{C}^n} MO_r^p(f)(z) < \infty$$

for some $r > 0$. The space of functions of vanishing mean oscillation VMO^p is a subspace of all f in BMO^p for which

$$\lim_{z \to \infty} MO_r^p(f)(z) = 0$$

for some $r > 0$.

To describe the structure of both BMO and VMO, we define the following related function spaces.

DEFINITION 2.3. The space BO of functions of bounded oscillation consists of all continuous functions f on $\mathbb{C}^n$ for which

$$\omega_r(f)(\cdot) = \sup_{w \in B(\cdot, r)} |f(w) - f(\cdot)| \in L^\infty$$

for some $r > 0$. Set $\|f\|_{BO} = \sup_{z \in \mathbb{C}^n} \omega_r(f)(z)$. The space VO of functions of vanishing oscillation consists of those $f \in BO$ for which

$$\lim_{z \to \infty} \omega_r(f)(z) = 0$$

for some $r > 0$.

DEFINITION 2.4. Let $1 \leqslant p < \infty$. The space of functions of bounded average BA^p consists of $f \in L^1_{loc}$ for which $\|f\|_{BA^p} = \sup_{z \in \mathbb{C}^n} \widehat{|f|^p}_r(z) < \infty$. The space VA^p of functions of vanishing average consists of those $f \in BA^p$ satisfying $\lim_{z \to \infty} \widehat{|f|^p}_r(z) = 0$.

For $1 \leqslant p < \infty$, the spaces BMO^p, VMO^p, BO, VO, BA^p and VA^p are all independent of r, and different values of r correspond to equivalent semi-norms on BMO^p, BO and BA^p. For this reason, we write $\omega(\cdot)$ for $\omega_1(\cdot)$. We also note that BMO^p, VMO^p, BA^p and VA^p are properly contained in BMO^q, VMO^q, BA^q and VA^q, respectively, if $q < p$.

Other useful and well-known properties of these spaces are seen in the following two canonical decompositions

$$BMO^p = BO + BA^p \tag{2.3}$$

and

$$VMO^p = VA + VA^p \tag{2.4}$$

which hold for $1 \leqslant p < \infty$ and are obtained by writing

$$f = \hat{f}_r + (f - \hat{f}_r) \text{ or } f = \tilde{f} + (f - \tilde{f}).$$

Let $K(\cdot, \cdot)$ be the reproducing kernel (also known as the Bergman kernel) of F^2_φ. The orthogonal projection P of L^2_φ onto F^2_φ can be represented as the integral operator

$$Pf(z) = \int_{\mathbb{C}^n} f(w) K(z, w) \, e^{-2\varphi(w)} \, dv(w)$$

for $z \in \mathbb{C}^n$. Using this expression, P can be extended to a bounded linear operator from L^p_φ to F^p_φ for $1 \leqslant p < \infty$, and in particular, we have

$$Pf = f \tag{2.5}$$

for all $f \in F^p_\varphi$, which was verified for $1 \leqslant p \leqslant \infty$ in [15], and the other case $0 < p < 1$ follows from the fact that $F^p_\varphi \subset F^1_\varphi$. The normalized reproducing kernel of F^2_φ is

denoted by k_z, that is,

$$k_z(\cdot) = \frac{K(\cdot, z)}{\sqrt{K(z,z)}}.$$

We set

$$\text{span}\{k_z : z \in \mathbb{C}^n\} = \left\{ \sum_{j=1}^{m} a_j k_{z_j} : a_j \in \mathbb{C}, z_j \in \mathbb{C}^n, m \in \mathbb{N} \right\}.$$

As we have not found the following density result in the literature in the case $0 < p < 1$, we include its proof for completeness. Recall first that a sequence (z_k) in $\mathbb{C}^n$ is an r-lattice with $r > 0$ if the balls $B(z_k, r)$ cover $\mathbb{C}^n$ and $B(z_k, r/2)$ are pairwise disjoint. It is not difficult to see that for any $r > 0$, an r-lattice exists in $\mathbb{C}^n$ and there is an $N \in \mathbb{N}$ such that each $z \in \mathbb{C}^n$ can belong to at most N balls in $\{B(z_k, 2r)\}$; that is,

$$1 \leqslant \sum_{k=1}^{\infty} \chi_{B(z_k, 2r)}(z) \leqslant N.$$

PROPOSITION 2.5. *For $0 < p < \infty$, span$\{k_z : z \in \mathbb{C}^n\}$ is dense in F_φ^p.*

Proof. Write $K_z(w)$ for $K(w, z)$ and let $S = \{K_z : z \in \mathbb{C}^n\}$. Suppose that $1 \leqslant p < \infty$. If S is not dense in F_φ^p, then, using the duality $(F_\varphi^p)^* = F_\varphi^q$, we have a nonzero function $g \in F_\varphi^q$, where $1/p + q/q = 1$, such that

$$\langle K_z, g \rangle = \int_{\mathbb{C}^n} K(\xi, z) \overline{g(\xi)} \, e^{-2\varphi(\xi)} \, dv(\xi) = 0.$$

which is a contradiction.

Suppose next that $0 < p < 1$ and let $f \in F_\varphi^p$. By (2.5) and lemma 2.4 of [9], there exists a constant $C > 0$ such that

$$\|f - P(f\chi_{B(0,R)})\|_{p,\varphi}^p$$

$$= \left\| \int_{\mathbb{C}^n \setminus B(0,R)} f(\xi) K(z, \xi) \, e^{-2\varphi(\xi)} \, dv(\xi) \right\|_{p,\varphi}^p$$

$$\leqslant C \int_{\mathbb{C}^n} e^{-p\varphi(z)} \, dv(z) \int_{\mathbb{C}^n \setminus B(0,R-1)} |f(\xi) K(\xi, z)|^p \, e^{-2p\varphi(\xi)} \, dv(\xi)$$

$$= C \int_{\mathbb{C}^n \setminus B(0,R-1)} |f(\xi)|^p \, e^{-p\varphi(\xi)} \, dv(\xi) \int_{\mathbb{C}^n} |K(\xi, z)|^p \, e^{-p\varphi(z)} \, e^{-p\varphi(\xi)} \, dv(\xi)$$

$$\leqslant C \int_{\mathbb{C}^n \setminus B(0,R-1)} |f(\xi)|^p \, e^{-p\varphi(\xi)} \, dv(\xi) \longrightarrow 0$$

as $R \to \infty$. This means

$$\lim_{R \to \infty} \|f - P(f\chi_{B(0,R)})\|_{p,\varphi}^p = 0. \tag{2.6}$$

Now for an r-lattice $\{z_j^{(r)}\}$ of $\mathbb{C}^n$, set

$$B_{j,r} = B(z_j^{(r)}, r) \setminus \bigcup_{k=1}^{j-1} B(z_k^{(r)}, r)$$

and define

$$S_{r,R}f(w) = \sum_j K(w, z_j^{(r)}) \int_{B_{j,r} \cap B(0,R)} f(\xi)\, \mathrm{e}^{-2\varphi(\xi)}\, \mathrm{d}v(\xi) \qquad (2.7)$$

for $w \in \mathbb{C}^n$. Then, as in the proof of theorem 4.4 of [7], it follows that $S_{r,R}f \in S$ and

$$\|S_{r,R}f - P(f\chi_{B(0,R)})\|_{p,\varphi} \longrightarrow 0 \qquad (2.8)$$

as $r \to 0$. Thus, S is dense in F_φ^p. $\qquad\square$

DEFINITION 2.6. Let Γ be the family of all measurable functions f on $\mathbb{C}^n$ satisfying $fk_z \in \cup_{p \geqslant 1} L_\varphi^p$ for each $z \in \mathbb{C}^n$. Given some $f \in \Gamma$, we define the Toeplitz operator T_f and Hankel operator H_f on F_φ^p as

$$T_f(g) = P(fg)$$

and

$$H_g f = (I - P)(gf)$$

respectively, where I is the identity operator on L_φ^p.

Observe that, since BMO^1 is contained in Γ, it follows from proposition 2.5 that both T_f and H_f are well-defined on F_φ^p for $0 < p < \infty$ when $f \in BMO^1$.

We finish the section with the definition of the Berezin transform, which is another useful tool for studying the properties of Toeplitz and Hankel operators, and indeed, we will use it heavily in our present work. For a function f in Γ, the Berezin transform $\tilde{f}$ of f is defined by

$$\tilde{f}(z) = \langle fk_z, k_z \rangle = \int_{\mathbb{C}^n} |k_z(w)|^2 f(w)\, \mathrm{d}v(w)$$

for $z \in \mathbb{C}^n$.

3. Preliminaries

Because the Bergman kernel of the generalized Fock space F_φ^p has no explicit expression, we need to rely on the following estimates instead. The first inequality is due to Christ in the case $n = 1$ and to Delin when $n \geqslant 2$; for further details, references and an alternate proof, see [15].

LEMMA 3.1. *The Bergman kernel $K(\cdot, \cdot)$ for F_φ^2 satisfies the following properties.*

(i) *There exist positive numbers C and θ such that*

$$|K(z,w)|\,\mathrm{e}^{-\varphi(z)}\,\mathrm{e}^{-\varphi(w)} \leqslant C\,\mathrm{e}^{-\theta|z-w|} \quad for\ z,w \in \mathbb{C}^n. \tag{3.1}$$

(ii) *There exists some $r > 0$ such that*

$$|K(z,w)|\,\mathrm{e}^{-\varphi(z)}\,\mathrm{e}^{-\varphi(w)} \simeq 1\ \ whenever\ w \in B(z,r)\ \ and\ z \in \mathbb{C}^n. \tag{3.2}$$

(iii) *For $0 < p \leqslant \infty$,*

$$\|K(\cdot,z)\|_{p,\varphi} \simeq \mathrm{e}^{\varphi(z)} \simeq \sqrt{K(z,z)} \quad for\ z \in \mathbb{C}^n. \tag{3.3}$$

The meaning of the notation $\simeq$ above is explained in (2.2).

We can use the preceding lemma to obtain the following connection between functions of vanishing oscillation and their Berezin transform.

LEMMA 3.2. *Let $f \in VO$. Then*

$$\lim_{z \to \infty} \left(f - \widetilde{f}\right)(z) = 0. \tag{3.4}$$

Furthermore, we have $f - \widetilde{f} \in L^\infty \cap VO$ and

$$\lim_{z \to \infty} \widetilde{\left(f - \widetilde{f}\right)}(z) = 0. \tag{3.5}$$

Proof. Given $\varepsilon > 0$, by lemma 3.1, we have some $R > 0$ such that, for all $z \in \mathbb{C}^n$,

$$\int_{\mathbb{C}^n \setminus B(z,R)} |\xi - z||k_z(\xi)|^2\,\mathrm{e}^{-2\varphi(\xi)}\,\mathrm{d}v(\xi) < \varepsilon. \tag{3.6}$$

Since $f \in VO$, there is some $\rho > 0$ such that

$$\sup_{\xi \in B(z,R)} |f(\xi) - f(z)| < \varepsilon.$$

whenever $|z| > \rho$. Notice that $\int_{\mathbb{C}^n} |k_z(\xi)|^2\,\mathrm{e}^{-2\varphi(\xi)}\,\mathrm{d}v(\xi) = 1$. Thus, for $|z| > \rho$,

$$\begin{aligned}
\left|f - \widetilde{f}\right|(z) &\leqslant \int_{\mathbb{C}^n} |f(z) - f(\xi)|\,|k_z(\xi)|^2\,\mathrm{e}^{-2\varphi(\xi)}\,\mathrm{d}v(\xi) \\
&\leqslant \left\{\int_{B_R(z)} + \int_{\mathbb{C}^n \setminus B_R(z)}\right\} |f(z) - f(\xi)|\,|k_z(\xi)|^2\,\mathrm{e}^{-2\varphi(\xi)}\,\mathrm{d}v(\xi) \\
&\leqslant \varepsilon + \|f\|_{BO} \int_{\mathbb{C}^n \setminus B_R(z)} |z - \xi|\,|k_z(\xi)|^2\,\mathrm{e}^{-2\varphi(\xi)}\,\mathrm{d}v(\xi) \\
&\leqslant (1 + \|f\|_{BO})\,\varepsilon,
\end{aligned}$$

which gives (3.4).

Next we show that $\widetilde{f}$ is continuous on $\mathbb{C}^n$. Indeed, for $\varepsilon > 0$, we can choose R as in (3.6), and then by lemma 3.1 and Lebesgue's dominated theorem, it follows that

$$\lim_{w \to z} \int_{B_R(w)} |k_w(\xi)|^2 \, e^{-2\varphi(\xi)} \, dv(\xi) = \int_{B_R(z)} |k_z(\xi)|^2 \, e^{-2\varphi(\xi)} \, dv(\xi).$$

Therefore,

$$\begin{aligned} |\widetilde{f}(w) - \widetilde{f}(z)| &\leqslant \left| \int_{B_R(w)} |k_w(\xi)|^2 \, e^{-2\varphi(\xi)} \, dv(\xi) - \int_{B_R(z)} |k_z(\xi)|^2 \, e^{-2\varphi(\xi)} \, dv(\xi) \right| \\ &\quad + 2\varepsilon < 3\varepsilon \end{aligned}$$

provided that $|z - w|$ is sufficiently small. Thus, $f - \widetilde{f}$ is bounded on any compact subset of $\mathbb{C}^n$. This and (3.4) imply that $f - \widetilde{f} \in L^\infty \cap VO$. Using $f - \widetilde{f} \in L^\infty$ and (3.4), we have (3.5). The proof is now completed. $\qquad\square$

Recall that an operator $T : X \to Y$ between two normed spaces is said to be compact if for every bounded sequence (f_n) in X, the sequence (Tf_n) contains a convergent subsequence. When studying Toeplitz and Hankel operators, the equivalent formulation of compactness as the property that the image of the unit ball of X under T is relatively compact in Y is useful. Another useful simple observation is that T_f and H_f are compact if f has compact support.

As a consequence of the previous lemma and a general description of compact Toeplitz operators in [9], we obtain the following characterization for symbols of vanishing oscillation.

THEOREM 3.3. *Let* $0 < p < \infty$ *and* $f \in VO$. *Then*

(i) *The Toeplitz operator* T_f *is compact on* F_φ^p *if and only if* $\lim_{z \to \infty} \widetilde{f}(z) = 0$.

(ii) $T_{f - \widetilde{f}}$ *is compact on* F_φ^p.

Proof. We deduce the two assertions from theorem 3.2 of [9], which states that the Toeplitz operator T_f with $f \in BMO$ is compact on F_φ^p if and only if

$$\lim_{z \to \infty} \sup_{w \in B(z,r)} |\langle T_f k_w, k_z \rangle_{F_\varphi^2}| = 0 \tag{3.7}$$

for all $r > 0$.

Suppose that $f \in VO$. To prove the first assertion, we only need to verify (3.7) under the assumption that $\lim_{z \to \infty} \widetilde{f}(z) = 0$. To see this, let $\varepsilon > 0$ and use lemma 3.2 to obtain an $R > 0$ such that

$$|(f - \widetilde{f})(\xi)| < \varepsilon \quad \text{and} \quad |\widetilde{f}(\xi)| < \varepsilon$$

for $|\xi| > R$. It follows that $|f(\xi)| < 2\varepsilon$ whenever $|\xi| > R$, and therefore, for $w \in B(z, r)$, by lemma 3.1, we get

$$||\langle T_f k_w, k_z \rangle_{F_\varphi^2}|$$

$$= \left| \int_{\mathbb{C}^n} f(\xi) k_w(\xi) \overline{k_z(\xi)} \, \mathrm{e}^{-2\varphi(\xi)} \, \mathrm{d}v(\xi) \right|$$

$$\leqslant \left\{ \int_{\mathbb{C}^n \setminus B(0,R)} + \int_{B(0,R)} \right\} |f(\xi)| |k_w(\xi)| |k_z(\xi)| \, \mathrm{e}^{-2\varphi(\xi)} \, \mathrm{d}v(\xi)$$

$$\leqslant 2\varepsilon + \sup_{|\xi| \leqslant R} |f(\xi)| \, \mathrm{e}^{-2\theta(|z| - R - r)}$$

$$\leqslant 3\varepsilon$$

when $|z|$ is sufficiently large. From this the estimate (3.7) follows.

To prove the second assertion, we set $g = f - \widetilde{f}$. Then, by lemma 3.2, we have $g \in L^\infty$ and $\lim_{z \to \infty} \widetilde{g}(z) = 0$. Thus, the condition in (3.7) implies that T_g is compact on F_φ^p for all $0 < p < \infty$, which completes the proof. $\qquad\square$

We need one more preliminary result on Toeplitz operators.

LEMMA 3.4. *Let $0 < p < \infty$ and let $f \in VO$. If $z_j \in \mathbb{C}^n$,*

$$\lim_{j \to \infty} z_j = \infty, \quad \lim_{j \to \infty} f(z_j) = 0,$$

then

$$\lim_{k \to \infty} \|T_f(k_{z_j})\|_{p,\varphi} = 0. \tag{3.8}$$

Proof. For $0 < p < s < \infty$, there is a constant C such that $\|f\|_{s,\varphi} \leqslant \|f\|_{p,\varphi}$ for all $f \in H(\mathbb{C}^n)$; see [15]. We may therefore assume that $0 < p \leqslant 1$.

Let $\varepsilon > 0$. Since $f \in VO$, lemma 3.1 implies that there is an $R > 1$ such that

$$\int_{\mathbb{C}^n \setminus B(z,R)} |\xi - z|^p |k_z(\xi)|^p \, \mathrm{e}^{-p\varphi(\xi)} \, \mathrm{d}v(\xi) \leqslant \left(\frac{\varepsilon}{2\|f\|_{BO} + 1} \right)^p$$

for all $z \in \mathbb{C}^n$. Furthermore, for the fixed ε and R, we have some $\rho > 0$ so that

$$\sup_{\xi \in B(z,R)} |f(\xi) - f(z)| < \varepsilon$$

whenever $|z| > \rho$. Then for $|z_j| > \rho$, by lemma 2.4 of [9], we get

$$\int_{\mathbb{C}^n} \left\{ \int_{\mathbb{C}^n} |f(\xi) - f(z_j)| \, |k_{z_j}(\xi)| |K(\xi, z)| \, \mathrm{e}^{-2\varphi(\xi)} \, \mathrm{d}v(\xi) \right\}^p \mathrm{e}^{-p\varphi(z)} \, \mathrm{d}v(z)$$

$$\leqslant C \int_{\mathbb{C}^n} \mathrm{e}^{-p\varphi(z)} \, \mathrm{d}v(z) \int_{\mathbb{C}^n} |f(\xi) - f(z_j)|^p \, |k_{z_j}(\xi)|^p |K(\xi, z)|^p \, \mathrm{e}^{-2p\varphi(\xi)} \, \mathrm{d}v(\xi)$$

$$= C \int_{\mathbb{C}^n} |f(\xi) - f(z_j)|^p \, |k_{z_j}(\xi)|^p \, \mathrm{e}^{-2p\varphi(\xi)} \, \mathrm{d}v(\xi) \int_{\mathbb{C}^n} |K(\xi, z)|^p \, \mathrm{e}^{-p\varphi(z)} \, \mathrm{d}v(z)$$

$$= C \int_{\mathbb{C}^n} |f(\xi) - f(z_j)|^p \, |k_{z_j}(\xi)|^p \, \mathrm{e}^{-p\varphi(\xi)} \, \mathrm{d}v(\xi)$$

$$\leqslant C \left(\varepsilon^p \, \|k_{z_j}\|_{p,\varphi}^p + 2\|f\|_{BO}^p \int_{\mathbb{C}^n \setminus B(z_j, R)} |\xi - z_j|^p \, |k_{z_j}(\xi)|^p \, \mathrm{e}^{-p\varphi(\xi)} \, \mathrm{d}v(\xi) \right)$$

$$\leqslant C \varepsilon^p.$$

The constants C above are independent of ε. This, together with lemma 3.5 and the obvious inequality

$$\left| T_f(k_{z_j})(z) \right| \leqslant \int_{\mathbb{C}^n} \left(|f(\xi) - f(z_j)| + |f(z_j)| \right) |k_{z_j}(\xi)| K(\xi, z)| \, \mathrm{e}^{-2\varphi(\xi)} \, \mathrm{d}v(\xi),$$

gives

$$\limsup_{j \to \infty} \left\| T_f(k_{z_j}) \right\|_{p,\varphi}^p \leqslant C\varepsilon^p + \limsup_{j \to \infty} |f(z_j)|^p \left\| P_+(|k_{z_j}|) \right\|_{p,\varphi}^p = C\varepsilon^p,$$

which gives the limit in (3.8). $\qquad\square$

Relative to the Bergman projection, we define an integral operator P_+ by setting

$$P_+ f(z) = \int_{\mathbb{C}^n} f(\xi) |K(z, \xi)| \, \mathrm{e}^{-2\varphi(\xi)} \, \mathrm{d}v(\xi)$$

for $z \in \mathbb{C}^n$.

LEMMA 3.5. *The operator P_+ defined above has the following properties.*

(i) *For $1 \leqslant p \leqslant \infty$, P_+ is bounded on L_φ^p.*

(ii) *For $0 < p < 1$, P_+ is bounded from F_φ^p to L_φ^p.*

Proof. We start with the first assertion. For $f \in L_\varphi^1$, by lemma 3.1, we have

$$\|P_+(f)\|_{1,\varphi} = \int_{\mathbb{C}^n} \left| \int_{\mathbb{C}^n} f(\xi) |K(z, \xi)| \, \mathrm{e}^{-2\varphi(\xi)} \, \mathrm{d}v(\xi) \right| \mathrm{e}^{-\varphi(z)} \, \mathrm{d}v(z)$$

$$\leqslant \int_{\mathbb{C}^n} |f(\xi)| \, \mathrm{e}^{-2\varphi(\xi)} \, \mathrm{d}v(\xi) \int_{\mathbb{C}^n} |K(z, \xi)| \, \mathrm{e}^{-\varphi(z)} \, \mathrm{d}v(z)$$

$$\leqslant \int_{\mathbb{C}^n} |f(\xi)| \, \mathrm{e}^{-\varphi(\xi)} \, \mathrm{d}v(\xi) \int_{\mathbb{C}^n} \mathrm{e}^{-\theta|\xi - z|} \, \mathrm{d}v(z)$$

$$\leqslant C\|f\|_{1,\varphi}$$

and further

$$\|P_+(f)\|_{\infty,\varphi} = \sup_{z \in \mathbb{C}^n} \left| \int_{\mathbb{C}^n} f(\xi) |K(z,\xi)| \, \mathrm{e}^{-2\varphi(\xi)} \, \mathrm{d}v(\xi) \right| \mathrm{e}^{-\varphi(z)}$$

$$\leqslant \|f\|_{\infty,\varphi} \sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} |K(z,\xi)| \, \mathrm{e}^{-\varphi(\xi)} \, \mathrm{e}^{-\varphi(z)} \, \mathrm{d}v(\xi)$$

$$\leqslant C\|f\|_{\infty,\varphi}.$$

By interpolation, we obtain the desired conclusion.

For the second assertion, let $f \in H(\mathbb{C}^n)$. Applying lemma 2.4 of [**9**] with $\Omega = \mathbb{C}^n$ and $h = fK(\cdot, z)$, we have

$$|P_+(f)(z)|^p \leqslant \left(\int_{\mathbb{C}^n} |f(\xi)K(\xi,z)| \, \mathrm{e}^{-2\varphi(\xi)} \, \mathrm{d}v(\xi) \right)^p$$

$$\leqslant \int_{\mathbb{C}^n} |f(\xi)K(\xi,z)|^p \, \mathrm{e}^{-2p\varphi(\xi)} \, \mathrm{d}v(\xi).$$

Therefore,

$$\|P_+(f)\|_{p,\varphi}^p \leqslant \int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} |f(\xi)K(\xi,z)|^p \, \mathrm{e}^{-2p\varphi(\xi)} \, \mathrm{d}v(\xi) \right) \mathrm{e}^{-p\varphi(z)} \, \mathrm{d}v(z)$$

$$\leqslant \int_{\mathbb{C}^n} |f(\xi)|^p \, \mathrm{e}^{-2p\varphi(\xi)} \, \mathrm{d}v(\xi) \int_{\mathbb{C}^n} |K(\xi,z)|^p \, \mathrm{e}^{-p\varphi(z)} \, \mathrm{d}v(z)$$

$$\leqslant \int_{\mathbb{C}^n} |f(\xi)|^p \, \mathrm{e}^{-p\varphi(\xi)} \, \mathrm{d}v(\xi),$$

which completes the proof. □

As is well known, Hankel operators play an important role in the study Toeplitz operators and their spectral properties, and in particular we make use of compact Hankel operators to prove our main result on Fredholmness of Toeplitz operators. To show that Hankel operators with symbols of vanishing oscillation are compact (see theorem 3.8 below, which is also of independent interest), we need the following auxiliary result.

LEMMA 3.6. *Let $0 < p < \infty$. If f is continuous on $\mathbb{C}^n$ with compact support, then H_f is compact from F_φ^p to L_φ^p.*

Proof. Write $B(F_\varphi^p)$ for the unit ball of F_φ^p. Observe that $B(F_\varphi^p)$ is a normal family. Therefore, to show that $H_f(B(F_\varphi^p))$ is relatively compact in L_φ^p, it suffices to prove that

$$\lim_{j \to \infty} \|H_f(g_j)\|_{p,\varphi} = 0 \tag{3.9}$$

for any bounded sequence $\{g_j\}_{j=1}^{\infty}$ in F_φ^p converging to 0 uniformly on any compact subset of $\mathbb{C}^n$. Also, without loss of generality, we may assume that the support of

f is contained in $B(0, \sigma)$. For such f, we have

$$|H_f(g)(z)| \leqslant \int_{B(0,\sigma)} |f(\xi) - f(z)||g(\xi)||K(z,\xi)| \, \mathrm{e}^{-2\varphi(\xi)} \, \mathrm{d}v(\xi)$$

$$\leqslant 2\|f\|_\infty \int_{B(0,\sigma)} |g(\xi)||K(z,\xi)| \, \mathrm{e}^{-2\varphi(\xi)} \, \mathrm{d}v(\xi).$$

Hence, for $1 \leqslant p < \infty$,

$$|H_f(g_j)(z)| \leqslant 2\|f\|_\infty P_+(\chi_{B(0,\sigma)}|g_j|),$$

and so, by lemma 3.5, we have

$$\|H_f(g_j)\|_{p,\varphi} \leqslant C\|f\|_\infty \|\chi_{B(0,\sigma)}g_j\|_{p,\varphi}. \tag{3.10}$$

For $0 < p \leqslant 1$, using lemma 2.4 of [**9**] with $\Omega = B(0,\sigma)$ and $h(\xi) = g(\xi)K(\xi,z)$, we get

$$|H_f(g)(z)|^p \leqslant C\|f\|_\infty^p \int_{B(0,\sigma+1)} |g(\xi)|^p |K(z,\xi)|^p \, \mathrm{e}^{-2p\varphi(\xi)} \, \mathrm{d}v(\xi).$$

According to lemma 3.1,

$$\|H_f(g_j)\|_{p,\varphi} \leqslant C\|f\|_\infty \|\chi_{B(0,\sigma+1)}g_j\|_{p,\varphi}. \tag{3.11}$$

Since the constants C above in (3.10) and (3.11) are independent of $\{g_j\}$, the limit in (3.9) follows for those $\{g_j\}$ which converge to 0 uniformly on any compact subset of $\mathbb{C}^n$. The proof is complete. $\qquad\square$

The simultaneous compactness of the Hankel operators H_f and $H_{\bar{f}}$ from the setting of the standard weighted Fock spaces F_α^p was recently described in [**12**] as follows.

THEOREM 3.7. *Let $1 \leqslant p < \infty$ and $f \in \Gamma$.*

(i) *The Hankel operators H_f and $H_{\bar{f}}$ are both bounded from F_φ^p to L_φ^p if and only if $f \in BMO^p$ with*

$$\|H_f\| + \|H_{\bar{f}}\| \simeq \|f\|_{BMO^p}.$$

(ii) *The Hankel operators H_f and $H_{\bar{f}}$ are both compact from F_φ^p to L_φ^p if and only if $f \in VMO^p$.*

Proof. See theorem 1.2 of [**12**]. $\qquad\square$

We can now give sufficient conditions for boundedness and compactness of Hankel operators, which are needed for the study of the Fredholm properties of Toeplitz operators.

THEOREM 3.8. *Let $0 < p < \infty$.*

(i) *If $f \in BO$, then H_f is bounded from F_φ^p to L_φ^p and the following norm estimate holds*:

$$\|H_f\|_{F_\varphi^p \to L_\varphi^p} \leqslant C\|f\|_{BO}.$$

(ii) *If $f \in VO$, then H_f is compact from F_φ^p to L_φ^p.*

Proof. When $1 \leqslant p < \infty$, the two assertions follow from the previous theorem using the decompositions in (2.3) and (2.4).

Suppose next that $0 < p < 1$ and $f \in BO$. Write $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $z = (z_1, z_2, \ldots, z_n)$. Then $|\xi - z| \simeq \sum_{j=1}^n |\xi_j - z_j|$, and for $g \in F_\varphi^p$ and $z \in \mathbb{C}^n$, we have

$$|H_f(g)(z)| \leqslant \int_{\mathbb{C}^n} |f(\xi) - f(z)||g(\xi)||K(z,\xi)| \, e^{-2\varphi(\xi)} \, dv(\xi)$$

$$\leqslant C\|f\|_{BO} \int_{\mathbb{C}^n} (|\xi - z| + 1) \, |g(\xi)||K(z,\xi)| \, e^{-\varphi(\xi)} \, dv(\xi)$$

$$\leqslant C\|f\|_{BO} \int_{\mathbb{C}^n} \left(\sum_{j=1}^n |\xi_j - z_j| + 1 \right) |g(\xi)||K(z,\xi)| \, e^{-2\varphi(\xi)} \, dv(\xi).$$

Applying lemma 2.4 of [9] to the holomorphic functions $\psi_j(\xi) = (\xi_j - z_j)g(\xi)K(\xi,z)$, it follows that

$$\left| \int_{\mathbb{C}^n} |(\xi_j - z_j)g(\xi)K(\xi,z)| \, e^{-2\varphi(\xi)} \, dv(\xi) \right|^p$$

$$\leqslant C \int_{\mathbb{C}^n} |(\xi_j - z_j)g(\xi)K(\xi,z)|^p \, e^{-2p\varphi(\xi)} \, dv(\xi)$$

for $z \in \mathbb{C}^n$. Similarly,

$$\left| \int_{\mathbb{C}^n} |g(\xi)K(\xi,z)| \, e^{-2\varphi(\xi)} \, dv(\xi) \right|^p \leqslant C \int_{\mathbb{C}^n} |g(\xi)K(\xi,z)|^p \, e^{-2p\varphi(\xi)} \, dv(\xi)$$

for $z \in \mathbb{C}^n$. Therefore,

$$\int_{\mathbb{C}^n} |H_f(g)(z)|^p \, e^{-p\varphi(z)} \, dv(z)$$

$$\leqslant C\|f\|_{BO} \int_{\mathbb{C}^n} e^{-p\varphi(z)} \, dv(z) \int_{\mathbb{C}^n} (|\xi - z|^p + 1) \, |g(\xi)|^p |K(z,\xi)|^p \, e^{-2p\varphi(\xi)} \, dv(\xi)$$

$$= C\|f\|_{BO} \int_{\mathbb{C}^n} |g(\xi)|^p \, e^{-2p\varphi(\xi)} \, dv(\xi) \int_{\mathbb{C}^n} (|\xi - z|^p + 1) \, |K(z,\xi)|^p \, e^{-p\varphi(z)} \, dv(z)$$

$$\leqslant C\|f\|_{BO}\|g\|_{p,\varphi}^p,$$

which gives the first assertion for $0 < p < 1$.

For the second assertion, suppose that $f \in VO$. For any $\varepsilon > 0$, we claim that there exists some $h \in C(\mathbb{C}^n)$ with compact support such that

$$\|f - h\|_{BO} < \varepsilon. \tag{3.12}$$

Since $f \in VO$, there is an $r > 0$ such that $\omega(f)(z) < \varepsilon$ whenever $|z| \geqslant r$. For such z, we have

$$\left| f(z) - f\left(\frac{r}{|z|} z \right) \right| < \varepsilon \left(1 + (|z| - r) \right)$$

and so

$$|f(z)| < \sup_{|\xi|=r} |f(\xi)| + \varepsilon(|z| + 1).$$

Dividing the both sides above by $|z|$, we obtain an $R > r + 2$ such that

$$\frac{|f(z)|}{|z|} < 2\varepsilon \tag{3.13}$$

whenever $|z| \geqslant R - 2$. Set

$$s(z) = \begin{cases} 1, & 0 \leqslant |z| < R; \\ \dfrac{1}{R}(2R - |z|) & R \leqslant t < 2R; \\ 0, & |z| \geqslant 2R. \end{cases}$$

Then $\omega(s) \leqslant 1/R$. Define $h(z) = f(z)s(z)$. For $|z| \leqslant R - 1$, $\omega(f - h)(z) = 0$. For $|z| \geqslant 2R + 1$, $\omega(f - h)(z) = \omega(f)(z) < \varepsilon$. For $R - 1 < |z| < 2R + 1$ and $w \in B(z, 1)$, using (3.13), we get

$$
\begin{aligned}
&|(f(z) - h(z)) - (f(w) - h(w))| \\
&\quad \leqslant |f(w)| \, |s(|w|) - s(|z|)| + (1 - s(|z|)) \, |f(w) - f(z)| \\
&\quad \leqslant |f(w)| \omega(s)(|z|) + \omega(f)(z) \leqslant |f(w)| \frac{1}{R} + \omega(f)(z) \\
&\quad = \frac{|f(w)|}{|w|} \frac{|w|}{R} + \omega(f)(z) \leqslant 2\varepsilon \frac{2R + 2}{R} + \varepsilon < 7\varepsilon.
\end{aligned}
$$

This completes the proof of the inequality in (3.12). Therefore, by the first assertion,

$$\|H_f - H_h\|_{F_\alpha^p \to L_\alpha^q} \leqslant C \|f - h\|_{BO} < C\varepsilon.$$

Since h has compact support, H_h is compact from F_φ^p to L_φ^p, and hence H_f is also compact from F_φ^p to L_φ^p. The proof is completed. $\qquad\square$

4. Fredholm theory

In this section, we prove that (1.1) is a necessary and sufficient condition for Toeplitz operators with symbols in VMO to be Fredholm and discuss some of the consequences of this result. We also show that the dual of F_φ^p can be identified with F_φ^∞ when $1 < p < 1$, which allows us to deal with Fredholmness of Toeplitz operators on Fock spaces with small exponents.

We start by recalling some of the basic theory of Fredholm operators that are needed in our proof. For operators on Banach spaces, all is well known and can be easily found in the literature, while for operators acting on more general vector spaces, the theory is significantly less developed. Indeed, in the context of Toeplitz operators on Hardy, Bergman or Fock spaces, to our knowledge, their Fredholm theory has been previously studied only in the setting of Banach spaces.

A linear mapping T on a topological vector space X is said to be Fredholm if

$$\dim \ker T < \infty \quad \text{and} \quad \dim X/T(X) < \infty.$$

When X is a Banach space, it is well known that T is Fredholm if and only if $T + K(X)$ is invertible in the Calkin algebra $B(X)/K(X)$, where $B(X)$ and $K(X)$ stand for the spaces of bounded and compact operators, respectively. From this, it follows that an operator on a Banach space is Fredholm if and only if there are bounded operators A and B on X such that

$$AT = I + K_1 \quad \text{and} \quad TB = I + K_2$$

for some compact operators K_1 and K_2 acting on X. Because two Toeplitz operators often commute modulo compact operators, the previous characterization for their Fredholmness is almost tailor-made for large classes of symbols.

These characterizations of Fredholm operators are not true in general if X is not a Banach space. However, an adequate theory can still be developed for quasi-Banach spaces under some additional conditions, which is important in certain PDE problems; see, e.g. [11]. A pair $(X, \|\cdot\|)$ is said to be a quasi-Banach space if $\|\cdot\|$ satisfies all the properties of a norm except for the triangle inequality and if there is a constant $C > 0$ such that

$$\|x + y\| \leqslant C(\|x\| + \|y\|)$$

for all $x, y \in X$. Observe that all generalized Fock spaces F_φ^p are quasi-Banach spaces. We now define an additional property for quasi-Banach spaces as in [13].

DEFINITION 4.1. A quasi-Banach space X is said to be dual rich if for all nonzero vectors $x \in X$, there is a continuous linear functional x^* such that $x^*(x) = 1$.

As an example, we mention that every Banach space is dual rich, and so are ℓ^p with $0 < p < 1$, while none of the $L^p(\mathbb{C}^n, dv)$ spaces with $0 < p < 1$ is dual rich; see [13].

In order to see that generalized Fock spaces are all dual rich, we describe the dual of F_φ^p when $0 < p < 1$. For this reason, define

$$F_\varphi^\infty = L_\varphi^\infty \cap H(\mathbb{C}^n),$$

where L_φ^∞ consists of all Lebesgue measurable functions f on $\mathbb{C}^n$ for which

$$\|f\|_{\infty,\varphi} = \operatorname{ess\,sup}\{|f(z)|\,\mathrm{e}^{-\varphi(z)} : z \in \mathbb{C}^n\} < \infty.$$

For $g \in F_\varphi^\infty$, we define a linean functional $F_g : F_\varphi^p \to \mathbb{C}$ by

$$F_g f = \int_{\mathbb{C}^n} f\bar{g}\,\mathrm{e}^{-2\varphi}\,\mathrm{d}v.$$

If $f \in F_\varphi^p$, then there is a constant $C > 0$ such that

$$|F_g f| = \left| \int_{\mathbb{C}^n} \left(f\,\mathrm{e}^{-\varphi} \right)\overline{\left(g\,\mathrm{e}^{-\varphi} \right)}\mathrm{d}v \right| \leqslant C\|f\|_{\infty,\varphi}\|f\|_{1,\varphi} \leqslant C\|f\|_{\infty,\varphi}\|f\|_{p,\varphi},$$

where the last inequality follows from the inclusion $F_\varphi^p \subset F_\varphi^1$. Thus, F_g is bounded. Define $\ell : F_\varphi^\infty \to (F_\varphi^p)^*$ by $\ell(g) = F_g$.

PROPOSITION 4.2. *Let $0 < p < 1$. Then $(F_\varphi^p)^* = F_\varphi^\infty$ under the pairing*

$$\langle f, g \rangle = \int_{\mathbb{C}^n} f(w)\overline{g(w)}\,\mathrm{e}^{-2\varphi(w)}\,\mathrm{d}v(w),$$

that is, the mapping ℓ is bijective.

Proof. We already verified that $\ell(g) \in (F_\varphi^p)^*$ and $\|\ell(g)\| \leqslant C\|g\|_{\infty,\varphi}$ for all $g \in F_\varphi^\infty$. Suppose next that F is a bounded linear functional on F_φ^p. Define a function g on $\mathbb{C}^n$ by $g(w) = \overline{F(K_w)}$. We claim that

$$g \in F_\varphi^\infty \quad \text{with } \|g\|_{\infty,\varphi} \leqslant C\|F\| \tag{4.1}$$

and

$$F(f) = \int_{\mathbb{C}^n} f(w)\overline{g(w)}\,\mathrm{e}^{-2\varphi(w)}\,\mathrm{d}A(w) \tag{4.2}$$

for all $f \in F_\varphi^p$.
   By lemma 3.1,

$$|g(w)| \leqslant \|F\|\|K_w\|_{p,\varphi} \simeq \|F\|\,\mathrm{e}^{\varphi(w)}$$

and so $g \in L_\varphi^\infty$. Next we show that g is holomorphic. For $z, w \in \mathbb{C}^n$, write $w = (w_1, w')$, where $w' \in \mathbb{C}^{n-1}$ and $\Delta w_1 \in \mathbb{C}$ with $|\Delta w_1| < \frac{1}{2}$. Then, by Cauchy's

estimate (see theorem I.1.6 of [**14**]), we get

$$\left| \frac{K\left((w_1 + \Delta w_1, w'), z\right) - K((w_1, w'), z)}{\Delta w_1} \right|^p$$

$$\leqslant \sup_{0 \leqslant t \leqslant 1} \left| \frac{\partial K}{\partial w_1}((w_1 + t\Delta w_1, w'), z) \right|^p$$

$$\leqslant C \int_{B(w,1)} |K(\xi, z)|^p \, dv(\xi),$$

where the constant C is independent of w and z. Furthermore, the estimate for the Bergman kernel in (3.1) tells us that

$$\int_{\mathbb{C}^n} \left\{ \int_{B(w,1)} |K(\xi, z)|^p \, dv(\xi) \right\} e^{-p\varphi(z)} \, dv(z) \leqslant C \sup_{\xi \in B(w,1)} e^{p\varphi(\xi)} < \infty.$$

Because of this and

$$\lim_{\Delta w_1 \to 0} \frac{K\left((w_1 + \Delta w_1, w'), z\right) - K((w_1, w'), z)}{\Delta w_1} = \frac{\partial K}{\partial w_1}(w, z),$$

we can apply the Lebesgue Dominated Theorem to get

$$\lim_{\Delta w_1 \to 0} \left\| \frac{K\left((w_1 + \Delta w_1, w'), \cdot\right) - K((w_1, w'), \cdot)}{\Delta w_1} - \frac{\partial K}{\partial w_1}(w, \cdot) \right\|_{p,\varphi} = 0.$$

This implies, for $F \in (F_\varphi^p)^*$,

$$\frac{\partial g}{\partial w_1}(w) = \lim_{\Delta w_1 \to 0} \frac{\overline{F(K(\cdot, (w_1 + \Delta w_1, w')))} - \overline{F(K(\cdot, w))}}{\Delta w_1} = \overline{F\left(\frac{\partial K}{\partial w_1}(w, \cdot)\right)}.$$

Similarly, $\partial g/\partial w_j(w)$ exists for $2 \leqslant j \leqslant n$. Therefore, $g \in H(\mathbb{C}^n)$ and (4.1) is proved.

It remains to prove (4.2). Let $f \in F_\varphi^p$. As in (2.7), for an r-lattice $\{w_n^{(r)}\}$ with $r > 0$, we define for $z \in \mathbb{C}^n$

$$S_{r,R}(f)(z) = \sum_j K\left(z, w_j^{(r)}\right) \int_{B_{j,r} \cap B(0,R)} f(w) \, e^{-2\varphi(w)} \, dv(w), \qquad (4.3)$$

where $R > 0$ is sufficiently large. The right-hand side of (4.3) is only a finite sum of holomorphic functions, so $S_{r,R}(f) \in H(\mathbb{C}^n)$. We claim that

$$\lim_{r \to 0} F\left(S_{r,R}(f)\right) = \int_{|w| \leqslant R} f(w) F(K(\cdot, w)) \, e^{-2\varphi(w)} \, dv(w). \qquad (4.4)$$

To see this, write $\nabla_w K(w, z) = (\partial K/\partial w_1, \partial K/\partial w_2, \ldots, \partial K/\partial w_n)$. Applying Cauchy's estimates again, we obtain the inequality

$$\sup_{w \in B(0,R)} |\nabla_w K(w, z)|^p \leqslant C \int_{B(0,R+1)} |K(z, w)|^p \, dv(w).$$

Then, for $w \in B(w_j^{(r)}, r)$ with $B(w_j^{(r)}, r) \cap B(0, R) \neq \emptyset$, we have

$$\left\| K\left(\cdot, w_j^{(r)}\right) - K(\cdot, w) \right\|_{p,\varphi}^p$$

$$\leqslant C r^p \int_{\mathbb{C}^n} \sup_{w \in B\left(w_j^{(r)}, r\right)} |\nabla_w K(w, z)|^p \, e^{-p\varphi(z)} \, dv(z)$$

$$\leqslant C r^p \int_{B(0, R+2)} \|K(\cdot, w)\|_{p,\varphi}^p \, dv(w)$$

$$= C r^p.$$

This implies, when $r \to 0$,

$$\left| F\left(S_{r,R}(f)\right) - \int_{|w| \leqslant R} f(w) F(K(\cdot, w)) \, e^{-2\varphi(w)} \, dv(w) \right|$$

$$= \left| \sum_j \int_{B_{j,r} \cap B(0,R)} \left[F\left(K\left(\cdot, w_j^{(r)}\right) - K(\cdot, w)\right) \right] f(w) \, e^{-2\varphi(w)} \, dv(w) \right|$$

$$\leqslant C \|F\| \sum_j \left\| K\left(\cdot, w_j^{(r)}\right) - K(\cdot, w) \right\|_{p,\varphi} \int_{B_{j,r} \cap B(0,R)} |f(w)| \, e^{-2\varphi(w)} \, dv(w)$$

$$\leqslant C r \|F\| \sup_{w \in B(0,R)} |f(w)| \to 0,$$

and hence (4.4) follows. Furthermore, by (4.1),

$$\int_{\mathbb{C}^n} |f(z) \overline{g(z)}| \, e^{-2\varphi(z)} \, dv(z) \leqslant \|g\|_{\infty,\varphi} \int_{\mathbb{C}^n} \left| f(z) \, e^{-\varphi(z)} \right| \, dv(z) \leqslant C \|F\| \|f\|_{p,\varphi} < \infty.$$

Notice that $g(\cdot) = \overline{F(K(\cdot, w))}$, applying the Lebesgue Dominated Theorem again to get

$$\lim_{R \to \infty} \int_{|w| \leqslant R} f(w) F(K(\cdot, w)) \, e^{-2\varphi(w)} \, dv(w) = \int_{\mathbb{C}^n} f(w) F(K(\cdot, w)) \, e^{-2\varphi(w)} \, dv(w).$$

(4.5)

Therefore, by (2.6), (2.8), (4.4) and (4.5), we have

$$F(f) = \lim_{R \to \infty} F(P(f \chi_{B(0,R)}))$$

$$= \lim_{R \to \infty} \lim_{r \to 0} F(S_{r,R}(f))$$

$$= \lim_{R \to \infty} \int_{|w| \leqslant R} f(w) F(K(\cdot, w)) \, e^{-2\varphi(w)} \, dv(w)$$

$$= \int_{\mathbb{C}^n} f(w) F(K(\cdot, w)) \, e^{-2\varphi(w)} \, dA(w),$$

which is (4.2), and the theorem is proved. $\qquad\square$

COROLLARY 4.3. *If $0 < p < 1$, then the generalized Fock space F_φ^p is a dual rich quasi-Banach space.*

The following result is needed in the next section when we characterize Fredholm operators on F_φ^p for $0 < p < 1$.

THEOREM 4.4. *A bounded linear operator on a dual rich quasi-Banach space X is Fredholm if and only if it has a regularizer; that is, there exists a bounded linear operator S on X such that $ST - I$ and $TS - I$ are both compact on X.*

Proof. See § 3.5.1 of [13].   □

We are now ready to prove our main result.

THEOREM 4.5. *Let $f \in VMO^1$ and $0 < p < \infty$. Then the Toeplitz operator T_f is Fredholm on F_φ^p if and only if*

$$0 < \liminf_{|z|\to\infty} |\widetilde{f}(z)| \leqslant \limsup_{|z|\to\infty} |\widetilde{f}(z)| < \infty. \tag{4.6}$$

Proof. According to the decomposition $VMO^1 = VO + VA^1$, there are functions $f_1 \in VO$ and $f_2 \in VA^1$ such that

$$f = f_1 + f_2. \tag{4.7}$$

For $f_2 \in VA^1$ and $R > 0$ fixed, we have

$$\lim_{z\to\infty} \frac{1}{|B(z,R)|} \int_{B(z,R)} |f_2| \, dv = 0,$$

which means that $|f_2| \, dv$ is vanishing (p,p)-Fock Carleson measure. Thus, by theorem 2.7 of [6], we have

$$\lim_{z\to\infty} \widetilde{|f_2|}(z) = 0, \tag{4.8}$$

and hence

$$\liminf_{|z|\to\infty} |\widetilde{f}(z)| = \liminf_{|z|\to\infty} |\widetilde{f_1}(z)|, \quad \limsup_{|z|\to\infty} |\widetilde{f}(z)| = \limsup_{|z|\to\infty} |\widetilde{f_1}(z)|. \tag{4.9}$$

And also, (4.8) and Fubini's theorem imply

$$\sup_{w\in B(z,r)} |\langle T_{f_2} k_z, k_w \rangle| \leqslant \sup_{w\in B(z,r)} \widetilde{|f_2|}(z)\widetilde{|f_2|}(w) \to 0$$

as $z \to \infty$. Thus, by theorem 3.2 of [9], the Toeplitz operator T_{f_2} is compact on F_φ^p for $0 < p < \infty$. So, T_f is Fredholm if and only if T_{f_1} is Fredholm. Therefore, we need to prove the statement only for symbols in VO.

Now suppose that T_{f_1} is Fredholm on F_φ^p for some $0 < p < \infty$. It is trivial that $\widetilde{f}_1(z) = \langle T_{f_1} k_z, k_z \rangle \in L^\infty(\mathbb{C}^n)$, equivalently

$$\limsup_{z \to \infty} |\widetilde{f}_1(z)| < \infty. \tag{4.10}$$

If $\liminf_{z \to \infty} |\widetilde{f}_1(z)| > 0$ were not true, we would have some sequence $\{z_k\}$ in $\mathbb{C}^n$ such that $\lim_{k \to \infty} z_k = \infty$ and $\lim_{k \to \infty} \widetilde{f}_1(z_k) = 0$. By lemma 3.4, $\lim_{j \to \infty} \|T_{\widetilde{f}_1}(k_{z_j})\|_{p,\varphi} = 0$, and hence, for any bounded operator G on F_φ^p, we have

$$\lim_{j \to \infty} \|G T_{\widetilde{f}_1}(k_{z_j})\|_{p,\varphi} = 0. \tag{4.11}$$

On the other hand, by theorem 3.3, we know that $T_{\widetilde{f}_1}$ is also Fredholm on F_φ^p, and so we can apply corollary 4.3 and theorem 4.4 to conclude that there is a bounded linear operator G on F_φ^p such that

$$G T_{\widetilde{f}_1} = I + K,$$

where I is the identity operator and K is some compact operator on F_φ^p. Therefore,

$$\lim_{j \to \infty} \|G T_{\widetilde{f}_1}(k_{z_j})\|_{p,\varphi} \geqslant \liminf_{j \to \infty} \|k_{z_j}\|_{p,\varphi} - \lim_{j \to \infty} \|K(k_{z_j})\|_{p,\varphi} = \liminf_{j \to \infty} \|k_{z_j}\|_{p,\varphi} > 0,$$

which contradicts (4.11). This completes the proof of the necessity condition.

Conversely, suppose that $f = f_1 + f_2 \in VO + VA^1$ and that $\widetilde{f}_1$ satisfies (4.6). Lemma 3.2 and (4.9) tell us that $\widetilde{f}_1 \in VO \cap L^\infty$. Proposition 9 of [**2**] implies that there is some $g \in VO \cap L^\infty$ such that

$$\lim_{z \to \infty} \widetilde{f}_1(z) g(z) = 1.$$

Therefore, $\widetilde{f}_1(z) g(z) - 1 \in VO \cap L^\infty$ and $(\widetilde{f}_1(z) g)\widetilde{}(z) - 1 \to 0$ as $z \to \infty$, and so, by theorem 3.3, the Toeplitz operator $T_{\widetilde{f}_1 g - 1}$ is compact on F_φ^p. Also, by theorem 3.8, the Hankel operator H_g is compact from F_φ^p to L_φ^p. Thus,

$$T_{\widetilde{f}_1} T_g = P M_{\widetilde{f}_1} P M_g = P M_{\widetilde{f}_1} \left[I - (I - P) \right] M_g = T_{\widetilde{f}_1 g} - P M_{f_1} H_g$$
$$= I + T_{\widetilde{f}_1 g - 1} - P M_{\widetilde{f}_1} H_g = I + K_1,$$

where $K_1 = T_{\widetilde{f}_1 g - 1} - P M_{f_1} H_g$ is compact on F_φ^p. Similarly, $T_g T_{\widetilde{f}_1} = I + K_2$ for some compact operator K_2 acting on F_φ^p. We conclude that $T_{\widetilde{f}_1}$ is Fredholm, and by what was said above, this means that T_f is Fredholm. The proof is complete. $\square$

COROLLARY 4.6. *Let $0 < p < \infty$ and $f \in VMO^1$. If (4.6) holds, then*

$$\sigma_{\mathrm{ess}}(T_f) = \bigcap_{R > 0} \overline{\widetilde{f}(\mathbb{C}^n \setminus B(0, R))} \tag{4.12}$$

and the essential spectrum $\sigma_{\mathrm{ess}}(T_f)$ is connected.

Proof. The previous theorem gives the description in (4.12) and the connectedness follows from this because $\widetilde{f}$ is continuous. $\square$

REMARK 4.7. We give an example that shows that (4.12) fails if $\tilde{f}$ is replaced by f in it. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$
f(z) = \begin{cases} \dfrac{1}{\sqrt{1 - (|z| - 2m)}} \cdot \dfrac{z}{m|z|} - \dfrac{1}{m} & \text{if } 2m \leqslant |z| < 2m + 1 \\ 0 & \text{otherwise.} \end{cases}
$$

Then for $|z| > 2$, we have

$$
\int_{B(z,1)} |f| \, \mathrm{d}A \leqslant \int_{|z|-1}^{|z|+1} r f(r) \, \mathrm{d}r \int_{-\arcsin \frac{1}{|z|}}^{\arcsin \frac{1}{|z|}} \mathrm{d}\theta \simeq \int_{|z|-1}^{|z|+1} |f(r)| \, \mathrm{d}r
$$

$$
\leqslant \int_{2[m/2]}^{2[m/2]+1} f(r) \, \mathrm{d}r \to 0
$$

as $|z| \to \infty$. Therefore, $T_{|f|}$ is compact and so is T_f. Thus, $\sigma_{\mathrm{ess}}(T_f) = \{0\}$. On the other hand, $f(\mathbb{C} \setminus B(0, R)) = \mathbb{C}$ for all $R > 0$, and hence $\sigma_{\mathrm{ess}}(T_f)$ cannot coincide with the right side of (4.12) if $\tilde{f}$ is replaced by f in it.

In order to state one more consequence of our main result, we recall the definition of block Toeplitz operators. Let $N \in \mathbb{N}$ and $0 < p < \infty$. Suppose that $a = (a_{jk})_{1 \leqslant j, k \leqslant N}$ with each $a_{jk} \in BMO^1$. The block Toeplitz operator T_a is defined on $F_{\varphi,N}^p$ by

$$
T_a f = \left(\sum_{k=1}^{N} T_{a_{jk}} f_k \right)_{1 \leqslant j \leqslant N}^{T} = \begin{pmatrix} T_{a_{11}} & T_{a_{12}} & \cdots & T_{a_{1N}} \\ \vdots & \vdots & & \vdots \\ T_{a_{N1}} & T_{a_{N2}} & \cdots & T_{a_{NN}} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix},
$$

where

$$
f \in F_{\varphi,N}^p = \{ (f_1, \ldots, f_N)^T : f_k \in F_\varphi^p \}
$$

and $F_{\varphi,N}^p$ is the closed subspace of $L_{\varphi,N}^p = \{ (f_1, \ldots, f_N)^T : f_k \in L_\varphi^p \}$ equipped with the norm

$$
\|f\|_{p,\varphi,N} = \sum_{k=1}^{N} \|f_k\|_{p,\varphi}.
$$

As in [2], we can use the standard Banach algebra techniques to obtain the following characterization for Fredholmness of block Toeplitz operators. Compared with theorem 22 of [2], we observe that in addition to more general weights, we can also deal with more general symbols in the following theorem.

THEOREM 4.8. *Let $1 < p < \infty$ and $a \in VMO_{N \times N}^1$ with $\widetilde{a}_{jk} \in L^\infty$ for $1 \leqslant j, k \leqslant N$. Then the block Toeplitz operator T_a is Fredholm on $F_{\alpha,N}^p$ if and only if*

$$
0 < \liminf_{|z| \to \infty} |\widetilde{\det a}(z)| \quad \text{and} \quad \limsup_{|z| \to \infty} |\widetilde{\det a}(z)| < \infty. \tag{4.13}
$$

Proof. If f, g are in VMO^1 with $\tilde{f}, \tilde{g} \in L^\infty$, then, T_f and T_g are both bounded and

$$T_f T_g = T_{fg} - PM_f H_g = T_{fg} + K$$

for some compact operator K by theorem 3.7. Therefore, as in lemma 21 of [2], it follows that $\det T_a - T_{\det a}$ is compact, and hence T_a is Fredholm if and only if $\det T_a$ is Fredholm, which is equivalent to $T_{\tilde{a}}$ being Fredholm by what we already proved. Thus, (4.13) is both necessary and sufficient for T_a to be Fredholm by theorem 4.5. □

5. Open problems

An immediate question related to our main result is what can be said about the Fredholm properties of Toeplitz operators on (more general) doubling Fock spaces F_ϕ^p. The Hilbert space case $F_\phi^p(\mathbb{C})$ was recently studied in [1]. The general case $0 < p < \infty$ and $n \in \mathbb{N}$ is much more difficult because the techniques of the present paper are no longer sufficient; however, there are a number of methods and inequalities for doubling Fock spaces that may prove fruitful and we hope to deal with the general case in a future publication.

It is also worth noting that here we have described Fredholmness of Toeplitz operators but are currently unable to say anything about their Fredholm index. In fact, to our knowledge, the index formula is only known for Toeplitz operators on Fock spaces with standard weights $\alpha > 0$ and with weights of the form $|z|^\beta$ where $\beta > 0$ (see [1]). In both of these cases, the space has a nice basis and an explicit formula for the reproducing kernel, which allows for a reduction to an index computation of a simpler operator. New ideas are required to deal with Fock spaces that lack these nice properties.

Another major challenge is the case of matrix-valued symbols. In theorem 4.8, we have merely reduced the study to the scalar-valued case; more general cases, where one needs to deal with block Toeplitz operators directly, are considerably more difficult, but the results known for Hardy and Bergman spaces may offer some clues as to what could be expected.

Finally, in our present paper, we have used and further developed the theory of Hankel operators. Their theory is also important in its own right and there are natural further questions about their compactness, for example, which arise from our work. Another aspect of Hankel operators, which is completely different, and perhaps surprising, from many other function spaces, such as Hardy and Bergman spaces, is the property that if H_f is compact, then $H_{\bar{f}}$ is also compact. This was recently proved for standard weighted Fock spaces F_α^p in [5] and it would be interesting to know whether this property remains true for Hankel operators acting on other weighted Fock spaces.

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