1

An Introduction to Affine Lie Algebras and the Associated Groups

The aim of this chapter is first to set some basic notation and preliminaries (to be used throughout the book) and then recall the definition of affine Kac–Moody Lie algebras and their basic representation theory and to study the associated groups and their flag varieties.

In Section 1.1 we recall the basic notation and preliminaries centered around schemes, varieties, ind-schemes, ind-group schemes, representable functors, quasi-coherent sheaves and vector bundles over ind-schemes. We also recall the Yoneda Lemma (cf. Lemma 1.1.1). *The notation set here will implicitly be used throughout the book.*

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and let G be the connected, simply-connected complex algebraic group with Lie algebra $\mathfrak{g}$.

In Section 1.2 we recall the definition of the associated affine Kac–Moody Lie algebra $\tilde{\mathfrak{g}}$ and its completion $\hat{\mathfrak{g}}$ and their various subalgebras, including the standard Cartan $\hat{\mathfrak{h}}$, standard Borel $\hat{\mathfrak{b}}$ and standard maximal parabolic subalgebra $\hat{\mathfrak{p}}$. Our $\tilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}$ do *not* include the degree derivation. Then we define their Verma and generalized Verma modules and give an explicit construction of integrable highest-weight modules $\mathscr{H}(\lambda_c)$ (cf. Definition 1.2.6). Further, we show that this explicit construction exhausts all the integrable highest-weight modules of $\hat{\mathfrak{g}}$ and, moreover, these modules are irreducible (cf. Theorem 1.2.10). We also define the affine Weyl group and its action on the Cartan subalgebra of $\mathfrak{g}$ (by affine transformations).

In Section 1.3 we define the loop group $G((t))$ (without the central extension) associated to the Lie algebra $\hat{\mathfrak{g}}$ and its various subgroups, e.g., $G[[t]]$, $G[t^{-1}]$. We define the affine group scheme $\bar{G}[[t]]$ which is a non-noetherian scheme and ind-affine group schemes $\bar{G}((t))$ and $\bar{G}[t^{-1}]$ (cf. Definition 1.3.1). They respectively represent the functors $G(R[[t]]), G(R((t)))$ and $G(R[t^{-1}])$

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from the category **Alg** of $\mathbb{C}$-algebras to the category of groups (cf. Lemma 1.3.2). In particular, $\bar{G}[[t]]$, $\bar{G}((t))$ and $\bar{G}[t^{-1}]$ have $\mathbb{C}$-points $G[[t]]$, $G((t))$ and $G[t^{-1}]$, respectively. Then we study the associated infinite Grassmannian $X_G = G((t))/G[[t]]$. Consider the functor $\mathscr{X}_G^o \colon R \in \mathbf{Alg} \rightsquigarrow G(R((t)))/G(R[[t]])$ and let its sheafification be denoted by $\mathscr{X}_G$. We first take $G = \mathrm{SL}_N$ and prove that $\mathscr{X}_{\mathrm{SL}_N}$ is represented by an ind-projective scheme $\bar{X}_{\mathrm{SL}_N}$ using the lattice construction (cf. Theorem 1.3.8). Moreover, $\bar{X}_{\mathrm{SL}_N}(\mathbb{C}) = X_{\mathrm{SL}_N}$. We further observe that the ind-group scheme $\overline{\mathrm{SL}}_N((t))$ acts on the ind-scheme $\bar{X}_{\mathrm{SL}_N}$ (cf. Definition 1.3.10). Then we prove that the product map $\overline{\mathrm{SL}}_N([t^{-1}])^- \times \overline{\mathrm{SL}}_N[[t]] \to \overline{\mathrm{SL}}_N((t))$ is an isomorphism onto an open subset of $\overline{\mathrm{SL}}_N((t))$, where $\overline{\mathrm{SL}}_N([t^{-1}])^-$ is the ind-scheme theoretic kernel of the evaluation homomorphism $\overline{\mathrm{SL}}_N([t^{-1}]) \to \mathrm{SL}_N$, $t^{-1} \mapsto 0$ (cf. Corollary 1.3.15). This last result is generalized for any connected reductive G in Lemma 1.3.16. This allows us to realize the infinite Grassmannian X_G as the $\mathbb{C}$-points of an ind-projective scheme $\bar{X}_G$ which represents the functor $\mathscr{X}_G$ (cf. Proposition 1.3.18). The projection $\pi : \bar{G}((t)) \to \bar{X}_G$ is a locally trivial principal $\bar{G}[[t]]$-bundle and $\bar{X}_G$ is an ind-projective scheme as proved in Corollary 1.3.19. This result is extended to $\bar{X}_G$ replaced by $\bar{G}((t))/\mathcal{P}$ for any parahoric subgroup $\mathcal{P} \subset \bar{G}[[t]]$ in Exercise 1.3.E.11.

We prove the following general result (cf. Theorem 1.3.22).

Theorem *Let $\mathscr{G}$ be an ind-affine group scheme filtered by (affine) finite type schemes over $\mathbb{C}$ and let $\mathscr{G}^{\mathrm{red}}$ be the associated reduced ind-affine group scheme. Assume that the canonical ind-group morphism $i : \mathscr{G}^{\mathrm{red}} \to \mathscr{G}$ induces an isomorphism of the associated Lie algebras. Then i is an isomorphism of ind-groups, i.e., $\mathscr{G}$ is a reduced ind-scheme.*

The basic idea of the proof involves considering the completion $\hat{\mathscr{G}}$ of $\mathscr{G}$ at the identity e, which is a formal group. Further, the formal groups in characteristic zero are determined by their Lie algebras. Moreover, the Lie algebras of $\mathscr{G}$ and $\hat{\mathscr{G}}$ are isomorphic. Thus, by assumption, we get that $\hat{\mathscr{G}}$ is isomorphic with the completion $\hat{\mathscr{G}}^{\mathrm{red}}$ of $\mathscr{G}^{\mathrm{red}}$ at e (and hence the completions of $\mathscr{G}$ and $\mathscr{G}^{\mathrm{red}}$ at any $\mathbb{C}$-point are isomorphic). From the isomorphism of the completions of $\mathscr{G}$ and $\mathscr{G}^{\mathrm{red}}$ at any $\mathbb{C}$-point, we conclude that $\mathscr{G}$ and $\mathscr{G}^{\mathrm{red}}$ themselves are isomorphic.

As a consequence of the above theorem, we get that the ind-affine group scheme $\bar{G}[t^{-1}]$ is reduced and hence so is $\bar{G}[t^{-1}]^-$ (cf. Theorem 1.3.23). In particular, the infinite Grassmannian $\bar{X}_G$ is a reduced ind-scheme. Moreover, $\bar{G}[[t]]$ is reduced. Thus, so is $\bar{G}((t))$ (cf. Remark 1.3.26(b)). It is shown that the ind-scheme $\bar{X}_G$ coincides with the ind-variety X_G^r defined via the representation theory (cf. Proposition 1.3.24).

We show that for any algebraic group H with a surjective algebraic group homomorphism $H \to \mathbb{C}^*$, $\bar{H}[t]$ is *not* reduced (cf. Example 1.3.25 and Remark 1.3.26(a)).

In Section 1.4 we study the central extension(s) of the ind-group scheme $\bar{G}((t))$. We define the adjoint representation of $G(R((t)))$ in Definition 1.4.2. The projective representation of $\mathfrak{g} \otimes \mathbb{C}((t))$ in any integrable highest-weight module $\mathscr{H}(\lambda_c)$ integrates to a projective representation of $G(R((t)))$ (cf. Proposition 1.4.3 and Theorem 1.4.4). The projective representation of the loop group $G(R((t)))$ in any $\mathscr{H}(\lambda_c) \otimes R$ gives rise to a central extension $\bar{\hat{G}}_{\lambda_c} \to \bar{G}((t))$, which is a $\mathbb{G}_m$-principal bundle, where $\bar{\hat{G}}_{\lambda_c}$ is a reduced ind-group scheme (cf. Definition 1.4.5 and Proposition 1.4.12). In particular, the projective representation of $G(R((t)))$ in $\mathscr{H}(\lambda_c) \otimes R$ lifts to an actual representation of $\bar{\hat{G}}_{\lambda_c}$ in $\mathscr{H}(\lambda_c)$ (cf. Corollary 1.4.7). Further, the central extension $\bar{\hat{G}}_{\lambda_c} \to \bar{G}((t))$ splits over $\bar{G}[[t]]$ as well as $\bar{G}[t^{-1}]^-$ (cf. Theorem 1.4.11).

1.1 Preliminaries and Notation

Unless otherwise explicitly stated, we take the base field to be the field of complex numbers $\mathbb{C}$. Though the bulk of the content of this book generalizes easily to any algebraically closed field of characteristic 0. The identity map of a set X is denoted by I_X , I_X or Id_X (or when no confusion is likely, by I , I or Id itself).

By *schemes* we mean quasi-compact (i.e., finite union of open affine subschemes) separated schemes over $\mathbb{C}$ but not necessarily of finite type over $\mathbb{C}$ (cf. (Mumford, 1988, §II.6, Definition 3), though quasi-compactness is not assumed here). Let $\mathfrak{S}$ be the category of schemes and morphisms between them. For a fixed scheme $S \in \mathfrak{S}$, let $\mathfrak{S}_S$ be the category of S-schemes whose objects are morphisms $f : T \to S$ (with target S) and the set of morphisms $\mathrm{Mor}\,(f, f')$ $(f' : T' \to S)$ consists of morphisms $h : T \to T'$ making the following triangle commutative:

$$\begin{array}{ccc} T & \xrightarrow{\quad h \quad} & T' \\ & {\scriptstyle f}\searrow \qquad \swarrow{\scriptstyle f'} & \\ & S. & \end{array}$$

By a *variety* we mean a reduced scheme which is of finite type over $\mathbb{C}$. We do *not* require varieties to be irreducible. When we talk of *points of a variety or scheme* X, we always mean closed points, i.e., points in $X(\mathbb{C})$ (see below).

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By an *ind-scheme* $X = (X_n)_{n\geq 0}$ we mean a collection of schemes X_n together with closed embeddings $i_n \colon X_n \hookrightarrow X_{n+1}$ for all $n \geq 0$. We thus think of X_n as a closed subscheme of X_{n+1}. Let $Y = (Y_n)_{n\geq 0}$ be another ind-scheme with closed embeddings $j_n \colon Y_n \hookrightarrow Y_{n+1}$. By a *morphism* $f \colon X \to Y$ we mean a sequence of non-negative integers $(m(0) \leq m(1) \leq m(2) \leq \cdots)$ and a collection of morphisms $f_n \colon X_n \to Y_{m(n)}$ (for all $n \geq 0$) such that the following diagram is commutative:

$$\begin{array}{ccc} X_n & \xrightarrow{\quad f_n \quad} & Y_{m(n)} \\ {\scriptstyle i_n}\big\downarrow & & \big\downarrow{\scriptstyle j_{m(n+1)-1}\circ\cdots\circ j_{m(n)}} \\ X_{n+1} & \xrightarrow{\quad f_{n+1} \quad} & Y_{m(n+1)}. \end{array}$$

If $f' \colon X \to Y$ is another morphism with the underlying sequence $(m'(0) \leq m'(1) \leq m'(2) \leq \cdots)$, then we say that f and f' are *equivalent* if the following diagram is commutative for all $n \geq 0$ (assuming $m(n) \leq m'(n)$, otherwise we reverse the arrow in the following diagram to $Y_{m'(n)} \to Y_{m(n)}$):

$$\begin{array}{ccc} X_n & \xrightarrow{\quad f_n \quad} & Y_{m(n)} \\ & {\scriptstyle f'_n}\searrow \quad \swarrow{\scriptstyle j_{m'(n)-1}\circ\cdots\circ j_{m(n)}} & \\ & Y_{m'(n)}. & \end{array}$$

We do not distinguish between two equivalent morphisms. This allows us to talk about *isomorphisms of ind-schemes*.

Let $\bar{X} := \cup_{n\geq 0} X_n$ endowed with the direct limit Zariski topology, where X_n is identified as a closed subspace of X_{n+1} via i_n. Then, a morphism $f \colon X \to Y$ clearly gives rise to a continuous map $\bar{f} \colon \bar{X} \to \bar{Y}$ which only depends upon the equivalence class of f.

A scheme X can be thought of as an ind-scheme by taking $X_n = X$. We call an ind-scheme $X =(X_n)_{n\geq 0}$ of *ind-finite type* if each scheme X_n is of finite type over $\mathbb{C}$. If each X_n is a projective (resp. affine) scheme over $\mathbb{C}$, we call X an *ind-projective scheme* (resp. *ind-affine scheme*). If each X_n is a variety we call X an *ind-variety*. If each X_n is a projective (resp. affine) variety we call X an *ind-projective variety* (resp. *ind-affine variety*). An ind-scheme X is called *irreducible* if under the (direct limit) Zariski topology on $\bar{X}$, it is an irreducible space.

A morphism $f \colon X \to Y$ between ind-schemes is called a *closed embedding* (also called a *closed immersion*) if for each $n \geq 0$, $f_n \colon X_n \to Y_{m(n)}$ is a closed embedding, $\bar{f}(\bar{X})$ is closed in $\bar{Y}$ and $\bar{f} \colon \bar{X} \to \bar{f}(\bar{X})$ is a homeomorphism

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under the subspace topology on $\bar{f}(\bar{X})$. In this case we also say that X is a *closed ind-subscheme* of Y.

Let **Alg** be the category of commutative algebras over $\mathbb{C}$ with identity (which are not necessarily finitely generated) and all $\mathbb{C}$-algebra homomorphisms between them. Also, let **Set** be the category of sets. For any ind-scheme X, define the covariant functor

$$h_X : \mathbf{Alg} \to \mathbf{Set}, \quad R \rightsquigarrow \mathrm{Mor}(\mathrm{Spec}\, R, X),$$

where Mor is the set of all the morphisms. The functor h_X extends to a contravariant functor

$$\tilde{h}_X : \mathfrak{S} \to \mathbf{Set}, \quad Y \rightsquigarrow \mathrm{Mor}(Y, X).$$

Recall the Yoneda Lemma (cf. (Mumford, 1988, §II.6, Proposition 2) for schemes; its extension to ind-schemes is straightforward).

Lemma 1.1.1 *For any ind-schemes X, Y,*

$$\mathrm{Mor}(X, Y) \simeq \mathrm{Hom}(h_X, h_Y),$$

where Hom denotes the set of natural transformations. Hence, h is a fully faithful functor from the category of ind-schemes to the category of functors from **Alg** *to* **Set**.

By *R-points of an ind-scheme* we mean

$$X(R) := \mathrm{Mor}(\mathrm{Spec}\, R, X). \tag{1}$$

Then, $X(\mathbb{C})$ are the *closed points* of X.

Let **Var** be the category of ind-varieties and morphisms between them. Then, the functor

$$\mathbf{Var} \to \mathbf{Set}, \quad X \rightsquigarrow X(\mathbb{C}),$$

is a faithful functor, i.e., for $X, Y \in \mathbf{Var}$,

$$\mathrm{Mor}(X, Y) \to \mathrm{Maps}(X(\mathbb{C}), Y(\mathbb{C})) \ \text{ is injective} \tag{2}$$

(cf. (Mumford, 1988, §II.6, p. 162)).

We sometimes abuse the notation and denote ind-scheme X by $X(\mathbb{C})$.

By an *affine algebraic group* we mean an affine algebraic group of finite type over $\mathbb{C}$.

An ind-scheme $X = (X_n)_{n \geq 0}$ is called an *ind-group scheme* if it is equipped with morphisms

$$\mu : X \times X \to X, \ \tau : X \to X \text{ and } \epsilon : \mathrm{Spec}\, \mathbb{C} \to X$$

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playing the role of multiplication, inverse and the identity element, respectively. Thus, they are required to satisfy the following three conditions:

(a) Associativity: $\mu \circ (\mu \times I_X) = \mu \circ (I_X \times \mu) \colon X^3 \to X$.

(b) Identity: The two morphisms $\mu \circ (I_X \times \epsilon)$ and $\mu \circ (\epsilon \times I_X) \colon \operatorname{Spec} \mathbb{C} \times X \to X$ coincide with I_X.

(c) Inverse: The morphism $\mu \circ (I_X, \tau) \colon X \to X$ coincides with the composite morphism $X \to \operatorname{Spec} \mathbb{C} \xrightarrow{\epsilon} X$.

In this book we only consider ind-affine group schemes, i.e., ind-group schemes $X = (X_n)_{n \geq 0}$ such that each X_n is an affine scheme. *So, by ind-group schemes, we will always mean ind-affine group schemes.*

For an ind-group scheme X and any $R \in \mathbf{Alg}$, $X(R)$ is clearly an abstract group given by the multiplication μ_R, inverse τ_R and the identity ϵ_R. If an ind-group scheme X is an ind-variety, then we call X an *ind-group variety*.

Let $X = (X_n)_{n \geq 0}$ be an ind-scheme. By a *quasi-coherent sheaf* $\mathcal{F}$ over X, we mean a collection of quasi-coherent sheaves $\mathcal{F}_n$ over X_n together with an isomorphism of $\mathcal{O}_{X_n}$-modules:

$$\theta_n \colon \mathcal{F}_n \simeq i_n^*(\mathcal{F}_{n+1}),$$

for all $n \geq 0$, where $i_n \colon X_n \to X_{n+1}$ is the closed embedding.

If each $\mathcal{F}_n$ is a locally free $\mathcal{O}_{X_n}$-module of rank r, then we call $\mathcal{F}$ a *rank-r vector bundle* over X. If $r = 1$, then, of course, $\mathcal{F}$ is called a *line bundle*.

For a quasi-coherent sheaf $\mathcal{F}$ over X, define

$$H^p(X, \mathcal{F}) = \varprojlim_n H^p(X_n, \mathcal{F}_n),$$

where the map $H^p(X_{n+1}, \mathcal{F}_{n+1}) \to H^p(X_n, \mathcal{F}_n)$ is defined as the composite

$$H^p(X_{n+1}, \mathcal{F}_{n+1}) \to H^p(X_{n+1}, i_{n*}\mathcal{F}_n) \simeq H^p(X_n, \mathcal{F}_n),$$

where the first map is obtained from the $\mathcal{O}_{X_{n+1}}$-module map $\mathcal{F}_{n+1} \to i_{n*}(\mathcal{F}_n)$ via the adjoint of the isomorphism θ_n (cf. (Hartshorne, 1977, Chap. II, §5)) and the second isomorphism is obtained from the closed embedding i_n (cf. (Hartshorne, 1977, Chap. III, Lemma 2.10)).

If an ind-group scheme Γ acts on ind-scheme X, then by a Γ-*equivariant vector bundle* $\mathcal{V}$ over X we mean a vector bundle $\mathcal{V}$ over X with an isomorphism of vector bundles $\phi \colon \mu^*(\mathcal{V}) \simeq \pi_X^*(\mathcal{V})$ over $\Gamma \times X$ satisfying the standard cocycle condition as in Mumford, Fogarty and Kirwan (2002, Definition 1.6), where $\pi_X \colon \Gamma \times X \to X$ is the projection and $\mu \colon \Gamma \times X \to X$ is the action map. For a Γ-equivariant vector bundle $\mathcal{V}$ over X, there is a natural action of $\Gamma(\mathbb{C})$ on $H^p(X,\mathcal{V})$ as follows (also see Definition B.22).

Take $\gamma\colon \operatorname{Spec}\mathbb{C} \to \Gamma$. This gives rise to a morphism $\mu_\gamma\colon X \to X$ by restricting μ to $\operatorname{Spec}\mathbb{C} \times X$ via γ and identifying $\operatorname{Spec}\mathbb{C} \times X$ with X. Thus, we get a canonical map

$$H^p(X,\mathcal{V}) \to H^p(X,\mu_\gamma^*\mathcal{V}) \simeq H^p(X,\mathcal{V}),$$

where the second isomorphism is obtained by restricting the isomorphism ϕ to $\operatorname{Spec}\mathbb{C} \times X \simeq X$. This is the required action of $\Gamma(\mathbb{C})$ on $H^p(X,\mathcal{V})$.

A covariant functor $\mathscr{F}$ from **Alg** $\to$ **Set** is called a *representable functor* if there exists an ind-scheme X such that there is a natural equivalence of functors between $\mathscr{F}$ and h_X. By Lemma 1.1.1, if such an X exists, then it is unique up to an isomorphism. Of course, we can extend this definition for any contravariant functor $\mathfrak{S} \to$ **Set**.

For any $S \in \mathfrak{S}$ and any ind-scheme $X \to S$, define the contravariant functor

$$\tilde{h}_{X/S}\colon \mathfrak{S}_S \to \mathbf{Set},\ \ (Y \to S) \rightsquigarrow \operatorname{Mor}_S(Y,X).$$

Then, a contravariant functor $\mathcal{F}\colon \mathfrak{S}_S \to$ **Set** is called *representable by an ind-scheme X over S* if there is a natural equivalence of functors between $\mathcal{F}$ and $\tilde{h}_{X/S}$.

For a projective variety X and an affine algebraic group G, any $\mathbb{C}$-analytic G-bundle over X has a unique algebraic G-bundle structure and any analytic morphism between G-bundles is an algebraic morphism (cf. (Serre, 1958, §6.3)). The same is true for vector bundles.

1.2 Affine Lie Algebras

For a more exhaustive treatment of the theory, we refer to the standard text (Kac, 1990).

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. Choose a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. Let $\Delta^+ \subset \mathfrak{h}^*$ be the set of positive roots (i.e., the roots for the subalgebra $\mathfrak{b}$) and let $\Delta = \Delta^+ \sqcup \Delta^-$ be the set of all the roots of $\mathfrak{g}$, where $\Delta^- := -\Delta^+$. Let $\{\alpha_1,\ldots,\alpha_\ell\} \subset \Delta^+$ be the set of simple roots and let $\{\alpha_1^\vee,\ldots,\alpha_\ell^\vee\} \subset \mathfrak{h}$ be the set of corresponding simple coroots, where $\ell := \dim\mathfrak{h}$ is the rank of $\mathfrak{g}$. Let $\langle\cdot,\cdot\rangle$ be the invariant (symmetric, nondegenerate) bilinear form on $\mathfrak{g}$ normalized so that the induced form on the dual space $\mathfrak{h}^*$ satisfies $\langle\theta,\theta\rangle = 2$ for the highest root θ of $\mathfrak{g}$. *Unless otherwise*

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stated, we will always take the invariant form on $\mathfrak{g}$ *to be normalized as above.* For any $\alpha \in \Delta$, let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ denote the root space corresponding to the root α.

Definition 1.2.1 Let $\mathfrak{g}$ be (as above) a finite-dimensional simple Lie algebra over $\mathbb{C}$ and let $\mathcal{A} := \mathbb{C}[t, t^{-1}]$, resp. $K = \mathbb{C}((t)) := \mathbb{C}[[t]][t^{-1}]$ be the algebra of Laurent polynomials, resp. the field of Laurent power series. Define the *affine Kac–Moody Lie algebra* (for short *affine Lie algebra*)

$$\tilde{\mathfrak{g}} := \left(\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}\right) \oplus \mathbb{C}C, \tag{1}$$

under the bracket

$$[x[t^m] + zC, x'[t^{m'}] + z'C] = [x, x'][t^{m+m'}] + m\delta_{m,-m'} \langle x, x'\rangle C, \tag{2}$$

for $z, z' \in \mathbb{C}, m, m' \in \mathbb{Z}$ and $x, x' \in \mathfrak{g}$, where $x[P]$ denotes $x \otimes P$.

We will be particularly interested in the following 'completion' $\hat{\mathfrak{g}}$ of $\tilde{\mathfrak{g}}$ defined by

$$\hat{\mathfrak{g}} := \mathfrak{g} \otimes_{\mathbb{C}} K \oplus \mathbb{C}C, \tag{3}$$

under the bracket

$$[x[P] + zC, x'[P'] + z'C] = [x, x'][PP'] + \operatorname*{Res}_{t=0}\left((dP)P'\right)\langle x, x'\rangle C, \tag{4}$$

for $P, P' \in K$, $z, z' \in \mathbb{C}$ and $x, x' \in \mathfrak{g}$, where $\operatorname*{Res}_{t=0}$ denotes the coefficient of $t^{-1}dt$.

Clearly, $\tilde{\mathfrak{g}}$ is a Lie subalgebra of $\hat{\mathfrak{g}}$.

The Lie algebra $\hat{\mathfrak{g}}$ admits a derivation d defined by

$$d(x[P]) = x\left[t\left(\frac{dP}{dt}\right)\right], \; d(C) = 0, \text{ for } P \in K \text{ and } x \in \mathfrak{g}. \tag{5}$$

Clearly, d keeps $\tilde{\mathfrak{g}}$ stable. Thus, we have semidirect product Lie algebras $\mathbb{C}d \ltimes \hat{\mathfrak{g}}$ and $\mathbb{C}d \ltimes \tilde{\mathfrak{g}}$.

Define the (formal) *loop algebra*

$$\mathfrak{g}((t)) := \mathfrak{g} \otimes_{\mathbb{C}} K, \tag{6}$$

under the bracket

$$[x[P], x'[P']] = [x, x'][PP'], \text{ for } P, P' \in K, \text{ and } x, x' \in \mathfrak{g}. \tag{7}$$

Then, $\hat{\mathfrak{g}}$ can be viewed as a 1-dimensional central extension of $\mathfrak{g}((t))$:

$$0 \to \mathbb{C}C \to \hat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g}((t)) \to 0, \tag{8}$$

where the Lie algebra homomorphism π is defined by $\pi(x[P]) = x[P]$, for $P \in K$ and $x \in \mathfrak{g}$, and $\pi(C) = 0$. As proved by Garland (1980, Theorem 3.14) and also independently by V. Chari (unpublished), the above is a universal central extension of $\mathfrak{g}((t))$ (see also Kac (1990, Exercises 3.14 and 7.8)). (For a geometric proof, see Kumar (1985, Corollary 1.9(c)).)

Definition 1.2.2 (Some subalgebras of $\hat{\mathfrak{g}}$) The Lie algebra $\mathfrak{g}$ is embedded in $\hat{\mathfrak{g}}$ as the subalgebra $\mathfrak{g} \otimes t^0$. Define the (standard) *Cartan subalgebra* of $\hat{\mathfrak{g}}$:

$$\hat{\mathfrak{h}} := \mathfrak{h} \otimes t^0 \oplus \mathbb{C}C, \tag{1}$$

the (standard) *Borel subalgebra*:

$$\hat{\mathfrak{b}} := \mathfrak{g} \otimes (t\mathbb{C}[[t]]) \oplus \mathfrak{b} \otimes t^0 \oplus \mathbb{C}C, \tag{2}$$

and the (standard) *maximal parabolic subalgebra*

$$\hat{\mathfrak{p}} := \mathfrak{g} \otimes \mathbb{C}[[t]] \oplus \mathbb{C}C. \tag{3}$$

Also, define the following subalgebras of $\hat{\mathfrak{g}}$:

$$\hat{\mathfrak{g}}_+ := \mathfrak{g} \otimes (t\mathbb{C}[[t]]), \; \hat{\mathfrak{g}}_- := \mathfrak{g} \otimes (t^{-1}\mathbb{C}[t^{-1}]), \; \hat{\mathfrak{l}} := \mathfrak{g} \otimes t^0 \oplus \mathbb{C}C. \tag{4}$$

Then, $\hat{\mathfrak{g}}_+$ is an ideal of $\hat{\mathfrak{p}}$ and we have the *Levi decomposition* (as vector spaces):

$$\hat{\mathfrak{p}} = \hat{\mathfrak{l}} \oplus \hat{\mathfrak{g}}_+. \tag{5}$$

Also, as vector spaces:

$$\hat{\mathfrak{g}} = \hat{\mathfrak{p}} \oplus \hat{\mathfrak{g}}_-. \tag{6}$$

We can similarly define $\tilde{\mathfrak{b}}, \tilde{\mathfrak{g}}_+, \tilde{\mathfrak{g}}_-, \tilde{\mathfrak{p}}$.

Finally, define the 3-dimensional subalgebra of $\hat{\mathfrak{g}}$:

$$\mathfrak{r} := \mathfrak{g}_\theta \otimes t^{-1} \oplus \mathfrak{g}_{-\theta} \otimes t \oplus \mathbb{C}(C - \theta^\vee), \tag{7}$$

where $\mathfrak{g}_\theta$ is the root space corresponding to the highest root θ and $\theta^\vee \in \mathfrak{h}$ is the coroot corresponding to θ.

Let $X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $Y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ be the standard basis of $s\ell_2$. Take any $x_\theta \in \mathfrak{g}_\theta$ and $y_\theta \in \mathfrak{g}_{-\theta}$ satisfying $\langle x_\theta, y_\theta \rangle = 1$.

The following lemma is trivial to verify using the commutation relations in $s\ell_2$.

Lemma 1.2.3 *The Lie algebra $\mathfrak{r}$ defined above is isomorphic with the Lie algebra $s\ell_2$ under an isomorphism $\gamma : s\ell_2 \to \mathfrak{r}$ taking $X \mapsto y_\theta \otimes t$, $Y \mapsto x_\theta \otimes t^{-1}$ and $H \mapsto C - \theta^\vee$.*

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Definition 1.2.4 (a) Let $\mathfrak{s}$ be a Lie algebra and let V be an $\mathfrak{s}$-module. Then V is called a *locally finite* $\mathfrak{s}$-module if, for any $v \in V$, there exists a finite-dimensional $\mathfrak{s}$-submodule $V_v \subset V$ containing v.

In particular, a linear transformation $T : V \to V$ (for a vector space V) is called *locally finite* if, for any $v \in V$, there exists a finite-dimensional T-stable subspace V_v containing v. Similarly, T is called *locally nilpotent* if, for any $v \in V$, there exists $n_v \in \mathbb{Z}_{\geq 1}$ such that $T^{n_v}(v) = 0$.

(b) A representation V of $\hat{\mathfrak{g}}$ (or $\tilde{\mathfrak{g}}$) is called *integrable* if V is a locally finite $\mathfrak{g}$-module as well as a locally finite $\mathfrak{r}$-module.

Clearly, any submodule of an integrable module is integrable and so is any quotient.

(c) A representation V of $\hat{\mathfrak{g}}$ is called a *highest-weight module* if V contains a nonzero vector $v_+ \in V$ satisfying the following two properties:

(c_1) The line $\mathbb{C}v_+$ is stable under the action of $\hat{\mathfrak{b}}$.
(c_2) v_+ generates the $\hat{\mathfrak{g}}$-module V, i.e., the only $\hat{\mathfrak{g}}$-submodule of V containing v_+ is the whole of V.

For a Lie algebra $\mathfrak{s}$, let $U(\mathfrak{s})$ denote its *enveloping algebra*.

Any highest-weight $\hat{\mathfrak{g}}$-module V decomposes into homogeneous components:

$$V = \oplus_{d \in \mathbb{Z}_+} V_d, \;\text{ where } V_d := U_d\left(\mathfrak{g} \otimes \mathbb{C}[t^{-1}]\right) \cdot v_+, \;\mathbb{Z}_+ := \mathbb{Z}_{\geq 0},$$

$x[n]$ denotes $x[t^n]$ and $U_d\left(\mathfrak{g} \otimes \mathbb{C}[t^{-1}]\right)$ is the span of $x_1[n_1] \dots x_k[n_k] \in U(\hat{\mathfrak{g}})$ with $n_i \leq 0$ and $\sum_{i=1}^{k} n_i = -d$.

In exactly the same way we can define the highest-weight modules for the Lie algebra $\tilde{\mathfrak{g}}$, where we replace the Borel subalgebra $\hat{\mathfrak{b}}$ of $\hat{\mathfrak{g}}$ by the standard Borel subalgebra

$$\tilde{\mathfrak{b}} := \mathfrak{g} \otimes (t\mathbb{C}[t]) \oplus \mathfrak{b} \otimes t^0 \oplus \mathbb{C}C. \tag{1}$$

Since $\hat{\mathfrak{g}} = \hat{\mathfrak{p}} \oplus \hat{\mathfrak{g}}_-$ (cf. identity (6) of Definition 1.2.2) and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{p}} \oplus \hat{\mathfrak{g}}_-$, it is easy to see that any highest-weight module of $\hat{\mathfrak{g}}$ is also a highest-weight module for $\tilde{\mathfrak{g}}$, where

$$\tilde{\mathfrak{p}} := (\mathfrak{g} \otimes \mathbb{C}[t]) \oplus \mathbb{C}C. \tag{2}$$

Any quotient module of a highest-weight module is clearly a highest-weight module.

(d) (Verma modules) For any $\hat{\lambda} \in \hat{\mathfrak{h}}^*$, define the *Verma module*

$$\hat{M}(\hat{\lambda}) := U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{b}})} \mathbb{C}_{\hat{\lambda}}, \tag{3}$$

where $U(\hat{\mathfrak{b}})$ acts on $U(\hat{\mathfrak{g}})$ via right multiplication and $\mathbb{C}_{\hat{\lambda}}$ is the 1-dimensional $\hat{\mathfrak{b}}$-module so that the commutator $[\hat{\mathfrak{b}}, \hat{\mathfrak{b}}]$ of course acts trivially on $\mathbb{C}_{\hat{\lambda}}$ and $\hat{\mathfrak{h}}$ acts via the character $\hat{\lambda}$. (Observe that $\hat{\mathfrak{b}} = \hat{\mathfrak{h}} \oplus [\hat{\mathfrak{b}}, \hat{\mathfrak{b}}]$.) The action of $U(\hat{\mathfrak{g}})$ on $\hat{M}(\hat{\lambda})$ is via left multiplication on the first factor.

Clearly, $\hat{M}(\hat{\lambda})$ is a highest-weight $\hat{\mathfrak{g}}$-module. Further, any highest-weight $\hat{\mathfrak{g}}$-module is a quotient of $\hat{M}(\hat{\lambda})$ for some $\hat{\lambda} \in \hat{\mathfrak{h}}^*$.

In exactly the same way, for any $\hat{\lambda} \in \hat{\mathfrak{h}}^*$, we can define the Verma module $\tilde{M}(\hat{\lambda})$ of $\tilde{\mathfrak{g}}$. Then, the canonical map $i\colon \tilde{M}(\hat{\lambda}) \to \hat{M}(\hat{\lambda})$ (induced from the inclusion $\tilde{\mathfrak{g}} \hookrightarrow \hat{\mathfrak{g}}$) is an isomorphism. In particular, the $\tilde{\mathfrak{g}}$-module structure on $\tilde{M}(\lambda)$ extends to a $\hat{\mathfrak{g}}$-module structure.

Similarly, we define the *generalized Verma module* $\hat{M}(V,c)$ for any $\mathfrak{g}$-module V and any $c \in \mathbb{C}$ as follows:

$$\hat{M}(V,c) := U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{p}})} I_c(V) = U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{p}})} I_c(V), \tag{4}$$

where $\hat{\mathfrak{p}}$ (resp. $\tilde{\mathfrak{p}}$) is defined by identity (3) of Definition 1.2.2 (resp. identity (2) of Definition 1.2.4), $U(\hat{\mathfrak{p}})$ acts on $U(\hat{\mathfrak{g}})$ via right multiplication and $I_c(V)$ is the vector space V on which $\hat{\mathfrak{p}}$ acts via $(x[P] + zC) \cdot v = P(0)x \cdot v + zcv$, for $P \in \mathbb{C}[[t]]$, $x \in \mathfrak{g}$, $v \in V$, $z \in \mathbb{C}$. Here $P(0)$ denotes the constant term of P. To prove the second equality in (4), use identity (6) of Definition 1.2.2 and the analogous identity for $\tilde{\mathfrak{g}}$.

Let V be a highest-weight $\mathfrak{g}$-module generated by a highest-weight vector $v_+ \neq 0 \in V$ of weight $\lambda \in \mathfrak{h}^*$ (i.e., the line $\mathbb{C}v_+$ is stable under $\mathfrak{b}$, v_+ generates V as a $\mathfrak{g}$-module and the action of $\mathfrak{b}$ on v_+ is via the weight λ). Then, for any $c \in \mathbb{C}$, there is a unique $\hat{\mathfrak{g}}$-module map

$$\pi\colon \hat{M}(\lambda_c) \longrightarrow \hat{M}(V,c),$$

taking $1 \otimes 1_{\lambda_c} \mapsto 1 \otimes v_+$, where $\lambda_c \in \hat{\mathfrak{h}}^*$ is defined by $\lambda_{c|\mathfrak{h}} = \lambda$ and $\lambda_c(C) = c$. Since V is a highest-weight $\mathfrak{g}$-module (by assumption), $U(\mathfrak{g}) \cdot v_+ = I_c(V)$. Thus, π is surjective. In particular, in this case $\hat{M}(V,c)$ is a highest-weight $\hat{\mathfrak{g}}$-module.

Lemma 1.2.5 *For any locally finite $\mathfrak{g}$-module V and any $c \in \mathbb{C}$, $\hat{M}(V,c)$ is locally finite as a $\mathfrak{g}$-module.*

Proof Recall the decomposition $\hat{\mathfrak{g}} = \hat{\mathfrak{p}} \oplus \hat{\mathfrak{g}}_-$ from identity (6) of Definition 1.2.2. Then, by the Poincaré–Birkhoff–Witt (PBW) theorem,

$$U(\hat{\mathfrak{g}}) = U(\hat{\mathfrak{p}}) \otimes_{\mathbb{C}} U(\hat{\mathfrak{g}}_-)$$

as vector spaces, and hence the inclusion

$$\iota\colon U(\hat{\mathfrak{g}}_-) \otimes_{\mathbb{C}} I_c(V) \longrightarrow U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{p}})} I_c(V)$$

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is an isomorphism of vector spaces. We next claim that ι is an isomorphism of $\mathfrak{g}$-modules, where $\mathfrak{g}$ acts on $U(\hat{\mathfrak{g}}_-)$ via the adjoint action: $(\operatorname{ad} x)a = xa - ax$, for $x \in \mathfrak{g}$, $a \in U(\hat{\mathfrak{g}}_-)$, and $\mathfrak{g}$ acts on $U(\hat{\mathfrak{g}}_-) \otimes_{\mathbb{C}} I_c(V)$ via the standard tensor product action. (Of course, $\mathfrak{g}$ acts on the range of ι via its standard embedding $\mathfrak{g} \hookrightarrow \hat{\mathfrak{g}}$.) To prove the claim, for $x \in \mathfrak{g}$, $a \in U(\hat{\mathfrak{g}}_-)$ and $v \in I_c(V)$, we have

$$\begin{aligned}\iota(x \cdot (a \otimes v)) &= \iota\big((\operatorname{ad} x)a \otimes v\big) + \iota(a \otimes x \cdot v)\\ &= (\operatorname{ad} x)a \otimes v + a \otimes x \cdot v\\ &= (xa - ax) \otimes v + ax \otimes v\\ &= xa \otimes v\\ &= x \cdot \iota(a \otimes v).\end{aligned}$$

This proves that ι is a $\mathfrak{g}$-module isomorphism. Now, by assumption, the action of $\mathfrak{g}$ on $I_c(V)$ is locally finite and it is easy to see that the adjoint action of $\mathfrak{g}$ on $U(\hat{\mathfrak{g}}_-)$ is locally finite. This proves the lemma. □

Definition 1.2.6 Let $D \subset \mathfrak{h}^*$ be the set of *dominant integral weights* for $\mathfrak{g}$, i.e.,

$$D := \big\{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z}_+ \text{ for all the simple coroots } \alpha_i^\vee\big\}.$$

For any $\lambda \in D$, let $V(\lambda)$ be the finite-dimensional irreducible $\mathfrak{g}$-module with highest weight λ.

Define the set of dominant integral weights $\hat{D}$ for $\hat{\mathfrak{g}}$ as follows:

$$\hat{D} = \big\{\hat{\lambda} \in \hat{\mathfrak{h}}^* : \hat{\lambda}_{|\mathfrak{h}} \in D \quad \text{and} \quad \hat{\lambda}(C) - \hat{\lambda}(\theta^\vee) \in \mathbb{Z}_+\big\}.$$

We will denote $\hat{\lambda} \in \hat{\mathfrak{h}}^*$ by λ_c, where $\lambda := \hat{\lambda}_{|\mathfrak{h}}$ and $c = \hat{\lambda}(C)$.

For any $\hat{\lambda} = \lambda_c \in \hat{D}$, define the $\hat{\mathfrak{g}}$-module

$$\mathscr{H}(\lambda_c) := \frac{\hat{M}(V(\lambda), c)}{U\big(\hat{\mathfrak{g}}\big) \cdot \Big((x_\theta[t^{-1}])^{c-\lambda(\theta^\vee)+1} \otimes v_+\Big)},$$

where x_θ is a nonzero element of $\mathfrak{g}_\theta$ and v_+ is a nonzero vector in the unique line $\mathbb{C}v_+ \subset V(\lambda)$ stabilized by $\mathfrak{b}$.

We prove that $\mathscr{H}(\lambda_c)$ is $\hat{\mathfrak{g}}$-integrable, for which we need the following general result.

Lemma 1.2.7 (a) *Let $\mathfrak{s}$ be any Lie algebra and let $x \in \mathfrak{s}$. Define*

$$\mathfrak{s}_x := \{y \in \mathfrak{s} : (\operatorname{ad} x)^{n_y} y = 0, \quad \textit{for some } n_y \in \mathbb{N}\},$$

where $\mathbb{N} := \mathbb{Z}_{\geq 1}$. Then, $\mathfrak{s}_x$ is a Lie subalgebra of $\mathfrak{s}$.

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(b) *For any representation* (V, π) *of* $\mathfrak{s}$ *and* $x \in \mathfrak{s}$, *define* $V_x = \{v \in V : \pi(x)^{n_v} v = 0$, *for some* $n_v \in \mathbb{N}\}$. *Then* V_x *is a* $\mathfrak{s}_x$*-submodule of* V.

(c) *Let* (V, π) *be a representation of* $\mathfrak{s}$ *such that* $\mathfrak{s}$ *is generated (as a Lie algebra) by the set* $F_V = \{x \in \mathfrak{s} : \operatorname{ad} x$ *acting on* $\mathfrak{s}$ *is locally finite and* $\pi(x)$ *is locally finite*$\}$. *Then*

- (c_1) $\mathfrak{s}$ *is spanned over* $\mathbb{C}$ *by* F_V. *In particular, if* $\mathfrak{s}$ *is generated by the set* F *of its* ad *locally finite vectors, then* F *spans* $\mathfrak{s}$.
- (c_2) *If* $\dim \mathfrak{s} < \infty$, *then any* $v \in V$ *lies in a finite-dimensional* $\mathfrak{s}$*-submodule of* V.

Proof (a) follows immediately from the Leibnitz formula (i.e., $\operatorname{ad} x$ is a derivation)

$$(\operatorname{ad} x)^n [y, z] = \sum_{j=0}^{n} \binom{n}{j} \left[(\operatorname{ad} x)^j\, y, (\operatorname{ad} x)^{n-j}\, z\right].$$

For a locally finite $T : V \to V$, we can define an automorphism $\exp T : V \to V$ in the usual manner:

$$\exp T = I + \sum_{n=1}^{\infty} \frac{T^n}{n!}. \tag{1}$$

Then,

$$\exp(kT) = (\exp T)^k, \qquad \text{for any } k \in \mathbb{Z}. \tag{2}$$

In an associative algebra R, we have the identity (for any $a, b \in R$ and $k \in \mathbb{N}$)

$$(\operatorname{ad} a)^k\, b = \sum_{r=0}^{k} (-1)^r \binom{k}{r} a^{k-r} b a^r, \tag{3}$$

where $\operatorname{ad} a : R \to R$ is defined by

$$(\operatorname{ad} a) b = ab - ba.$$

To obtain (3), apply the Binomial Theorem to $(L_a - R_a)^k$ for the two commuting operators L_a and R_a given respectively by $L_a b = ab$, $R_a b = ba$.

From (3) it is easy to see that for two linear maps $T, S : V \to V$ such that T is locally finite and $\{(\operatorname{ad} T)^n\, S, n \in \mathbb{N}\}$ spans a finite-dimensional subspace of $\operatorname{End} V$, we have

$$(\exp T) S \exp(-T) = \sum_{n \geq 0} \frac{(\operatorname{ad} T)^n}{n!} (S) \tag{4}$$

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as operators on V, where ad T on the right-hand side is to be thought of as an operator on the associative algebra End V (of all the linear operators of V).

Similar to identity (3), considering the Binomial Theorem for the operator $L_x^n = (\operatorname{ad} x + R_x)^n$, we obtain in any associative algebra R and any elements $x, a \in R$,

$$x^n a = \sum_{j=0}^{n} \binom{n}{j} \big((\operatorname{ad} x)^j\, a\big) x^{n-j}.$$

Applying the above identity to v, the (b)-part follows.

We first show that for $a, x \in F_V$ and $t \in \mathbb{C}$, $(\exp(t \operatorname{ad} a))\, x \in F_V$: Since π is a Lie algebra representation, for any $y, z \in \mathfrak{s}$ and $n \in \mathbb{Z}_+$,

$$\pi\big((\operatorname{ad} y)^n\, z\big) = \big(\operatorname{ad} \pi(y)\big)^n\, \pi(z), \tag{5}$$

as elements of $\operatorname{End}(V)$. In particular, for $a, x \in F_V$,

$$\begin{aligned} \pi\big((\exp(\operatorname{ad} a))x\big) &= \big(\exp(\operatorname{ad} \pi(a))\big)\pi(x) \\ &= \exp(\pi a)\, \pi(x)\ \exp(-\pi a), \quad \text{by (4).} \end{aligned} \tag{6}$$

(Observe that, since $a \in F_V$, $\pi(a)$ is locally finite and, by (5), $\{(\operatorname{ad} \pi(a))^n \pi(x) : n \in \mathbb{N}\}$ is finite dimensional.) This shows that $\pi\big((\exp(t \operatorname{ad} a))x\big)$ is locally finite. Taking V to be the adjoint representation, we see that $\big(\exp(t \operatorname{ad} a)\big)\, x \in F_V$.

Let $\mathfrak{s}_V \subset \mathfrak{s}$ be the $\mathbb{C}$-span of F_V. Since

$$\operatorname*{limit}_{t \to 0} \frac{\big(\exp(t \operatorname{ad} a)\big)x - x}{t} = [a, x],$$

we see that $[a, x] \in \mathfrak{s}_V$ (for $a, x \in F_V$). In particular, $\mathfrak{s}_V$ is a Lie subalgebra of $\mathfrak{s}$. This proves (c_1). Now (c_2) follows from (c_1) by the PBW theorem. □

Proposition 1.2.8 *For any $\lambda_c \in \hat{D}$, the $\hat{\mathfrak{g}}$-module $\mathscr{H}(\lambda_c)$ is an integrable highest-weight $\hat{\mathfrak{g}}$-module.*

By Exercise 1.1.E.4, $\mathscr{H}(\lambda_c)$ is nonzero.

Proof We have already seen in Definition 1.2.4(d) that $\mathscr{H}(\lambda_c)$ is a highest-weight $\hat{\mathfrak{g}}$-module. By Lemma 1.2.5, it is locally finite as a $\mathfrak{g}$-module. So, to prove that it is integrable, it suffices to show that it is locally finite as an $\mathfrak{r}$-module.

Apply Lemma 1.2.7(b) in the case $\mathfrak{s} = \tilde{\mathfrak{g}}$, $x = x_\theta[t^{-1}]$ and $V = \mathscr{H}(\lambda_c)$. By Exercise 1.1.E.1, $\mathfrak{s}_x = \mathfrak{s}$. Moreover, clearly $1 \otimes v_+ \in V_x$ and, by Lemma 1.2.7(b), V_x is an $\mathfrak{s}_x = \mathfrak{s}$ submodule of V. Further, the $\mathfrak{s}$-submodule of V generated by $1 \otimes v_+$ is the whole of V. To prove this, observe that the canonical map $j\colon \tilde{M}(\lambda_c) \to \hat{M}(\lambda_c)$ is an isomorphism and, moreover, the canonical map $\pi : \hat{M}(\lambda_c) \to \hat{M}(V(\lambda), c)$ is surjective (cf. Definition 1.2.4(d)). Thus, $V_x = V$, i.e., $x_\theta[t^{-1}]$ acts locally nilpotently on $\mathscr{H}(\lambda_c)$. By the same argument we see that $x_{-\theta}[t]$ acts locally nilpotently on V. Now, any $s\ell_2$-module L such that X and Y act locally nilpotently on L is a locally finite $s\ell_2$-module. This follows, e.g., by Lemma 1.2.7(c_2). Thus, in view of Lemma 1.2.3, the proposition is proved. □

A $(\mathbb{C}d \ltimes \hat{\mathfrak{g}})$-module V is called *integrable* if it is integrable as a $\hat{\mathfrak{g}}$-module. It is called a *highest-weight* $(\mathbb{C}d \ltimes \hat{\mathfrak{g}})$-module if there exists a line $\mathbb{C}v_+ \subset V$ which is stable under $\mathbb{C}d \ltimes \hat{\mathfrak{b}}$ and v_+ generates V as a $(\mathbb{C}d \ltimes \hat{\mathfrak{g}})$-module, where $\hat{\mathfrak{b}}$ is defined by identity (2) of Definition 1.2.2. (The notion of a highest-weight $(\mathbb{C}d \ltimes \tilde{\mathfrak{g}})$-module can, of course, be defined similarly.) With this definition we recall the following important theorem from Kumar (2002, Corollaries 2.2.6, 3.2.10 and Theorem 13.1.3).

Theorem 1.2.9 *Any integrable highest-weight* $(\mathbb{C}d \ltimes \tilde{\mathfrak{g}})$*-module is irreducible.*

Theorem 1.2.10 *Any integrable highest-weight* $\hat{\mathfrak{g}}$*-module is isomorphic with a unique* $\mathscr{H}(\lambda_c)$*,* $\lambda_c \in \hat{D}$*.*

Thus, $\lambda_c \mapsto \mathscr{H}(\lambda_c)$ *sets up a bijective correspondence between* $\hat{D}$ *and the set of isomorphism classes of integrable highest-weight* $\hat{\mathfrak{g}}$*-modules.*

Moreover, $\mathscr{H}(\lambda_c)$ *is an irreducible* $\hat{\mathfrak{g}}$*-module.*

Proof Take an integrable highest-weight $\hat{\mathfrak{g}}$-module V. Let $\mathbb{C}v_+ \subset V$ be a line stable under $\hat{\mathfrak{b}}$ such that the $\hat{\mathfrak{g}}$-submodule generated by v_+ is the whole of V. Let $\lambda_c \in \hat{\mathfrak{h}}^*$ be the character by which $\hat{\mathfrak{b}}$ acts on the line $\mathbb{C}v_+$. Since V is integrable, the $\mathfrak{g}$-submodule V^o generated by v_+ is finite dimensional and so is the $\mathfrak{r}$-submodule V' generated by v_+. Since the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ keeps the line $\mathbb{C}v_+$ stable, from the representation theory of $\mathfrak{g}$ applied to V^o, we get $\lambda \in D$ and $V^o \simeq V(\lambda)$ as $\mathfrak{g}$-modules (cf. (Serre, 1966, Théorème 1 and Proposition 3(d), Chapitre VII)). Moreover, from the $s\ell_2$-representation theory (cf. (Serre, 1966, Corollaire 1, Chapitre IV)) and Lemma 1.2.3, $\lambda_c(C - \theta^\vee) \in \mathbb{Z}_+$, i.e., $\lambda_c \in \hat{D}$.

Since $\hat{\mathfrak{g}}_+$ annihilates v_+ and hence V^o, we get a surjective $\hat{\mathfrak{g}}$-module map

$$\phi \colon \hat{M}\big(V(\lambda), c\big) \to V,$$

taking $I_c(V(\lambda)) \xrightarrow{\sim} V^o$ isomorphically as a $\hat{\mathfrak{l}} = (\mathfrak{g} \oplus \mathbb{C}C)$-module.

Again, using the $s\ell_2$-representation theory (cf. (Serre, 1966, Corollaire 1, Chapitre IV)) and Lemma 1.2.3,

$$(x_\theta[t^{-1}])^{c-\lambda(\theta^\vee)+1} \cdot v_+ = 0 \quad \text{in } V'.$$

Thus, ϕ factors through (as a surjective $\hat{\mathfrak{g}}$-module map)

$$\bar{\phi}: \mathscr{H}(\lambda_c) \to V.$$

For any $\mathfrak{g}$-module L and any $c \in \mathbb{C}$, define the action of d on

$$\hat{M}(L,c) \simeq U(\hat{\mathfrak{g}}_-) \otimes_{\mathbb{C}} I_c(L)$$

via its standard derivation action on $U(\hat{\mathfrak{g}}_-)$ induced from the action on $\hat{\mathfrak{g}}_-$ given in identity (5) of Definition 1.2.1 (d acts trivially on $I_c(L)$). This action of d turns the $\hat{\mathfrak{g}}$-module $\hat{M}(L,c)$ into a $(\mathbb{C}d \ltimes \hat{\mathfrak{g}})$-module. Clearly, this $(\mathbb{C}d \ltimes \hat{\mathfrak{g}})$-module structure on $\hat{M}(V(\lambda),c)$ descends to a $(\mathbb{C}d \ltimes \hat{\mathfrak{g}})$-module structure on the quotient $\mathscr{H}(\lambda_c)$, making it an integrable and a highest-weight $(\mathbb{C}d \ltimes \hat{\mathfrak{g}})$-module (cf. Definition 1.2.4(d) and Proposition 1.2.8); in particular, an integrable and highest-weight $(\mathbb{C}d \ltimes \tilde{\mathfrak{g}})$-module. Thus, by Theorem 1.2.9, it is an irreducible $(\mathbb{C}d \ltimes \tilde{\mathfrak{g}})$-module, and hence an irreducible $(\mathbb{C}d \ltimes \hat{\mathfrak{g}})$-module. We next show that it is irreducible as a $\hat{\mathfrak{g}}$-module.

Let $N \subset \mathscr{H}(\lambda_c)$ be a nonzero $\hat{\mathfrak{g}}$-submodule. Consider the decomposition

$$\mathscr{H}(\lambda_c) = \oplus_{i \in \mathbb{Z}_+} \mathscr{H}(\lambda_c)_i,$$

where

$$\mathscr{H}(\lambda_c)_i := \{v \in \mathscr{H}(\lambda_c) : d \cdot v = -iv\}. \tag{1}$$

Observe that for any $n \in \mathbb{Z}$ and $x \in \mathfrak{g}$,

$$x[t^n] \cdot \mathscr{H}(\lambda_c)_i \subset \mathscr{H}(\lambda_c)_{i-n}. \tag{2}$$

For any nonzero $v \in \mathscr{H}(\lambda_c)$, $v = \sum v_i$ with $v_i \in \mathscr{H}(\lambda_c)_i$, set $|v| = \sum i :$ $v_i \neq 0$. Choose a nonzero $v^o \in N$ such that $|v^o| \leq |v|$ for all nonzero $v \in N$. Then,

$$x[t^n] \cdot v^o = 0 \quad \text{for all } n \geq 1 \text{ and } x \in \mathfrak{g}. \tag{3}$$

For, otherwise, $|x[t^n]{\cdot}v^o| < |v^o|$, which contradicts the choice of v^o. If $|v^o| > 0$, take a nonzero component $v^o_{i_o}$ with $i_o > 0$. By (1) and (2),

$$x[t^n] \cdot v^o_{i_o} = 0 \quad \text{for all } n \geq 1 \text{ and } x \in \mathfrak{g}.$$

In particular, by the PBW theorem, the $(\mathbb{C}d \ltimes \hat{\mathfrak{g}})$-submodule of $\mathscr{H}(\lambda_c)$ generated by $v^o_{i_o}$ is proper, which contradicts the irreducibility of $\mathscr{H}(\lambda_c)$ as a $(\mathbb{C}d \ltimes \hat{\mathfrak{g}})$-module. Thus, $|v^o| = 0$, i.e., $v^o \in \mathscr{H}(\lambda_c)_0$ and hence, by the PBW theorem, the $(\mathbb{C}d \ltimes \hat{\mathfrak{g}})$-submodule of $\mathscr{H}(\lambda_c)$ generated by v^o is the same as the $\hat{\mathfrak{g}}$-submodule of $\mathscr{H}(\lambda_c)$ generated by v^o. Hence, $N = \mathscr{H}(\lambda_c)$, proving the irreducibility of $\mathscr{H}(\lambda_c)$ as a $\hat{\mathfrak{g}}$-module.

From the irreducibility of $\mathscr{H}(\lambda_c)$ as a $\hat{\mathfrak{g}}$-module, we get that $\bar{\phi}$ is an isomorphism.

So, to complete the proof of the theorem, it suffices to show that for $\lambda_c \neq \mu_{c'} \in \hat{D}$, $\mathscr{H}(\lambda_c)$ and $\mathscr{H}(\mu_{c'})$ are nonisomorphic as $\hat{\mathfrak{g}}$-modules.

Define the $\mathfrak{g}$-submodule

$$\mathscr{H}(\lambda_c)^o = \left\{v \in \mathscr{H}(\lambda_c) : \hat{\mathfrak{g}}_+ \cdot v = 0\right\}.$$

Then, clearly, as a $\mathfrak{g}$-submodule of $\mathscr{H}(\lambda_c)$,

$$1 \otimes V(\lambda) = \mathscr{H}(\lambda_c)^o_0 \text{ and } \mathscr{H}(\lambda_c)^o = \bigoplus_{i \geq 0} \mathscr{H}(\lambda_c)^o_i. \tag{4}$$

We claim that, for any $i > 0$,

$$\mathscr{H}(\lambda_c)^o_i = 0. \tag{5}$$

For, if not, the $\hat{\mathfrak{g}}$-submodule of $\mathscr{H}(\lambda_c)$ generated by $\mathscr{H}(\lambda_c)^o_i$ would be proper (again use the PBW theorem), contradicting the irreducibility of the $\hat{\mathfrak{g}}$-module $\mathscr{H}(\lambda_c)$. Thus,

$$\mathscr{H}(\lambda_c)^o = 1 \otimes V(\lambda).$$

So, if $\mathscr{H}(\lambda_c)$ and $\mathscr{H}(\mu_{c'})$ are isomorphic as $\hat{\mathfrak{g}}$-modules, then the $\mathfrak{g}$-modules $V(\lambda)$ and $V(\mu)$ are isomorphic, i.e., $\lambda = \mu$. Moreover, the action of C on $\mathscr{H}(\lambda_c)$ and $\mathscr{H}(\mu_{c'})$ is by the same scalar, i.e., $c = c'$. Thus $\lambda_c = \mu_{c'}$, proving the theorem completely. □

Definition 1.2.11 Recall from the beginning of this section that $\langle \cdot, \cdot \rangle$ is the invariant normalized form on $\mathfrak{g}$. Extend this to an invariant symmetric bilinear form on $\hat{\mathfrak{g}}$, still denoted by $\langle \cdot, \cdot \rangle$, as follows:

$$\langle x[P], y[Q] \rangle = \operatorname*{Res}_{t=0} (t^{-1} PQ) \langle x, y \rangle, \text{ for } x, y \in \mathfrak{g} \text{ and } P, Q \in K, \quad \langle C, \hat{\mathfrak{g}} \rangle = 0.$$

This form clearly descends to a bilinear form on the loop algebra $\mathfrak{g}((t)) = \mathfrak{g} \otimes K$. It is easy to see that this form on $\mathfrak{g}((t))$ is nondegenerate.

Definition 1.2.12 Let W be the Weyl group of $\mathfrak{g}$. Then W can be realized as the subgroup of Aut($\mathfrak{h}$) generated by the *simple reflections* $\{s_1, \dots, s_\ell\}$, where

$$s_i(h) := h - \alpha_i(h)\, \alpha_i^\vee, \text{ for } h \in \mathfrak{h}. \tag{1}$$

Then W is a Coxeter group with Coxeter generators $\{s_1, \dots, s_\ell\}$.

The dual representation of W in $\mathfrak{h}^*$ is explicitly given by

$$s_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\, \alpha_i, \text{ for } \lambda \in \mathfrak{h}^*. \tag{2}$$

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Let $Q^\vee \in \mathfrak{h}$ be the coroot lattice of $\mathfrak{g}$:

$$Q^\vee := \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i^\vee. \tag{3}$$

Since $\alpha_i(\alpha_j^\vee) \in \mathbb{Z}$ (cf. (Serre, 1966, Chap. V, §11)), $s_i Q^\vee \subset Q^\vee$. Thus, W keeps $Q^\vee$ stable.

Define the *affine Weyl group* to be the semidirect product

$$\hat{W} := W \ltimes Q^\vee. \tag{4}$$

For $q \in Q^\vee$, we denote the corresponding element of $\hat{W}$ by τ_q. By definition, $\hat{W}$ acts on $\mathfrak{h}$ via affine transformations, where W acts linearly on $\mathfrak{h}$ via the standard action (1) and τ_q acts on $\mathfrak{h}$ via translation:

$$\tau_q(h) = q + h. \tag{5}$$

Consider the element $s_0 \in \hat{W}$ defined by

$$s_0 = \tau_{\theta^\vee}\gamma_\theta, \tag{6}$$

where (as in Definition 1.2.2) $\theta^\vee$ is the coroot corresponding to the highest root θ and $\gamma_\theta \in W$ is the reflection through the root plane θ, i.e., $\gamma_\theta h = h - \theta(h)\theta^\vee$.

The following well-known result can be found, e.g., in (Kumar, 2002, Propositions 13.1.7, 1.3.21 and the identity (13.1.1.7)).

Lemma 1.2.13 *The affine Weyl group $\hat{W}$ is a Coxeter group with Coxeter generators $\{s_0, s_1, \ldots, s_\ell\}$. In particular, for any $\hat{w} \in \hat{W}$, we have the notion of its length $\ell(\hat{w})$.*

The Coxeter relations among $\{s_i\}_{1\le i\le \ell}$ together with the following relations provide a complete set of relations for $\hat{W}$:

(a) $s_0^2 = 1$,

(b) $(s_0 s_i)^{m_i} = 1$, for all $1 \le i \le \ell$,

where $m_i = 2, 3, 4, 6$ or ∞ according as $\alpha_i(\theta^\vee)\,\theta(\alpha_i^\vee) = 0, 1, 2, 3$ or ≥ 4, respectively.

1.2.E Exercises

(1) For any root vector $x_\beta \in \mathfrak{g}_\beta$ and $n \in \mathbb{Z}$, show that $\mathrm{ad}(x_\beta[t^n]) : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ is a locally nilpotent transformation.

(2) Show that for any highest-weight $\tilde{\mathfrak{g}}$-module V, its $\tilde{\mathfrak{g}}$-module structure extends to a $\hat{\mathfrak{g}}$-module structure.

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(3) For any $\lambda_c \in \hat{D}$, show that the line $\mathbb{C}\big(x_\theta[t^{-1}]^{c-\lambda(\theta^\vee)+1} \otimes v_+\big)$ inside $\hat{M}(V(\lambda), c)$ is stable under the action of $\mathfrak{b}$ and is annihilated by $\hat{\mathfrak{g}}_+$. Thus, the line is stable under $\hat{\mathfrak{b}}$.
(4) Show that, for any $\lambda_c \in \hat{D}$, $\mathscr{H}(\lambda_c)$ is nonzero. *Hint:* Use Exercise 3.
(5) Show that, for any $\lambda_c \in \hat{D}$, the line $\mathbb{C}v_+ \subset \mathscr{H}(\lambda_c)$ is the unique line stable under $\hat{\mathfrak{b}}$. Hence, any $\hat{\mathfrak{g}}$-module endomorphism of $\mathscr{H}(\lambda_c)$ is the identity map up to a scalar multiple. Moreover, $\mathbb{C}v_+ \subset \mathscr{H}(\lambda_c)$ is the unique line annihilated by $\hat{\mathfrak{u}} := (\mathfrak{g} \otimes t\mathbb{C}[[t]]) \oplus \mathfrak{u}$, where $\mathfrak{u}$ is the nil-radical of $\mathfrak{b}$.
(6) Show that, for any $\lambda_c \in \hat{D}$, $\hat{M}(\lambda_c)$ has a unique proper maximal $\hat{\mathfrak{g}}$-submodule. Hence, $\mathscr{H}(\lambda_c)$ is the unique irreducible quotient of $\hat{M}(\lambda_c)$.
(7) For any $f \in K = \mathbb{C}((t))$, any root vector $x_\beta \in \mathfrak{g}_\beta$, and any $\lambda_c \in \hat{D}$, show that $x_\beta[f]$ acts locally nilpotently on $\mathscr{H}(\lambda_c)$.

1.3 Loop Groups and Infinite Grassmannians

We follow the convention from Section 1.1.

As in Definition 1.2.1, let $K = \mathbb{C}((t)) = \mathbb{C}[[t]][t^{-1}]$ be the field of Laurent power series.

For any commutative $\mathbb{C}$-algebra R with identity and affine scheme X over $\mathbb{C}$, let $X(R)$ denote the R-points of X. Then, $X(R)$ can be identified with the set of all the $\mathbb{C}$-algebra homomorphisms $f : \mathbb{C}[X] \to R$, where $\mathbb{C}[X]$ is the affine coordinate ring of X (cf. (Mumford, 1988, §II.6, Definition 1 and §II.2, Theorem 1)). We want to realize $G(K)$, $G(\mathbb{C}[t^{-1}])$ as $\mathbb{C}$-points of ind-affine group schemes and $G(\mathbb{C}[[t]])$ as $\mathbb{C}$-points of an affine group scheme.

Recall first that $\mathrm{Spec}(\mathbb{C}[y_1, y_2, \dots])(\mathbb{C}) = \mathbb{C}[[t]]$, where an element $\sum_{n\geq 0} a_n t^n \in \mathbb{C}[[t]]$ corresponds to the unique algebra homomorphism $\mathbb{C}[y_1, y_2, \dots] \to \mathbb{C}$ taking y_n to a_n.

Definition 1.3.1 Let G be any affine algebraic group (of finite type over $\mathbb{C}$). Take a faithful representation $i : G \hookrightarrow \mathrm{SL}_N \subset M_N$, where M_N is the vector space of $N \times N$ matrices over $\mathbb{C}$, and let $I_G \subset S(M_N^*)$ be the radical ideal of G inside M_N. For any $1 \leq i, j \leq N$ and an integer n, define the linear function

$$y_n^{i,j} : M_N((t)) := M_N \otimes_{\mathbb{C}} \mathbb{C}((t)) \to \mathbb{C}, \;\; E \otimes \left(\sum_{n\in\mathbb{Z}} a_n t^n\right) \mapsto y^{i,j}(E) a_n,$$

for $E \in M_N$ and $\sum_n a_n t^n \in \mathbb{C}((t))$, where $y^{i,j} : M_N \to \mathbb{C}$ is the linear function taking any $E \in M_N$ to its (i, j)th entry.

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For any $P \in \mathbb{C}[M_N] = S(M_N^*)$, let $\widehat{P}\colon M_N((t)) \to \mathbb{C}((t))$ be the (polynomial) function obtained from extending the scalars from $\mathbb{C}$ to $\mathbb{C}((t))$. Express $\widehat{P} = \sum_{m\in\mathbb{Z}} \widehat{P}_m t^m$. For any $d \geq 0$, restrict $\widehat{P}$ to $M_N \otimes t^{-d}\mathbb{C}[[t]]$ and denote this restriction by $\widehat{P}^{(d)}$. Then, $\widehat{P}_m^{(d)} = 0$ for $m << 0$ and $\widehat{P}_m^{(d)}$ are polynomial functions on $M_N \otimes t^{-d}\mathbb{C}[[t]]$.

Let $R_N^{(d)}$ be the polynomial ring in the variables $\{y_n^{i,j}\}_{n\geq -d;\, 1\leq i,j\leq N}$ and let $I_G^{(d)}$ be the ideal of $R_N^{(d)}$ generated by $\{\widehat{P}_m^{(d)} : m \in \mathbb{Z} \text{ and } P \in I_G\}$.

Consider the affine (though non-noetherian) scheme $\bar{G}(t^{-d}\mathbb{C}[[t]])$ associated to the ring $R_N^{(d)}/I_G^{(d)}$, i.e.,

$$\bar{G}(t^{-d}\mathbb{C}[[t]]) := \operatorname{Spec}\left(R_N^{(d)}/I_G^{(d)}\right).$$

In particular, taking $d = 0$, we get the affine scheme

$$\bar{G}[[t]] = \bar{G}(\mathbb{C}[[t]]) := \operatorname{Spec}\left(R_N^{(0)}/I_G^{(0)}\right).$$

Exactly similarly, we can define the scheme

$$\bar{G}\left(\sum_{p=0}^{d} \mathbb{C}t^{-p}\right) := \operatorname{Spec}\left(\left(\mathbb{C}[y_n^{i,j}]_{-d\leq n\leq 0;\, 1\leq i,j\leq N}\right)\Big/\left\langle\left(\widehat{P}_m^{(d)}\right)_{|M_N\otimes \sum_{p=0}^{d}\mathbb{C}t^{-p}} : m \in \mathbb{Z} \text{ and } P \in I_G\right\rangle\right).$$

Clearly, the inclusions (for any $d \geq 0$)

$$\bar{G}\left(t^{-d}\mathbb{C}[[t]]\right) \subset \bar{G}\left(t^{-d-1}\mathbb{C}[[t]]\right) \text{ and } \bar{G}\left(\sum_{p=0}^{d} \mathbb{C}t^{-p}\right) \subset \bar{G}\left(\sum_{p=0}^{d+1} \mathbb{C}t^{-p}\right),$$

under the above scheme structures, are closed embeddings. This gives rise to ind-schemes

$$\bar{G}((t)) := \left\{\bar{G}(t^{-d}\mathbb{C}[[t]])\right\}_{d\geq 0} \quad \text{and}$$
$$\bar{G}[t^{-1}] := \bar{G}\left(\mathbb{C}[t^{-1}]\right) = \left\{\bar{G}\left(\sum_{p=0}^{d} \mathbb{C}t^{-p}\right)\right\}_{d\geq 0}.$$

Observe that $\bar{G}((t))$ is an inductive limit of non-noetherian affine schemes with closed embedding in $\overline{\mathrm{SL}}_N((t))$, whereas $\bar{G}[t^{-1}]$ is an inductive limit of

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noetherian affine schemes (in fact, affine schemes of finite type over $\mathbb{C}$) with closed embedding in $\overline{\mathrm{SL}}_N[t^{-1}]$.

By virtue of the following Lemma 1.3.2, the (ind)-scheme structures on $\bar{G}[[t]]$, $\bar{G}((t))$ and $\bar{G}[t^{-1}]$ do not depend upon the choice of a faithful representation $G \hookrightarrow \mathrm{SL}_N$.

Lemma 1.3.2 *Let G be any affine algebraic group. Consider the covariant functors $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$ from* **Alg** *to* **Set** *by*

$$\mathscr{F}_1(R) = G(R[[t]]),$$

$$\mathscr{F}_2(R) = G(R((t))),$$

$$\mathscr{F}_3(R) = G(R[t^{-1}]).$$

Then all these are representable functors represented respectively by the scheme $\bar{G}[[t]]$ and ind-schemes $\bar{G}((t))$ and $\bar{G}[t^{-1}]$ (with the scheme structure given in Definition 1.3.1).

In particular, the (ind)-scheme structures on these do not depend upon the choice of a faithful representation $i : G \hookrightarrow \mathrm{SL}_N$. Moreover, the $\mathbb{C}$-points of $\bar{G}[[t]]$, $\bar{G}((t))$ and $\bar{G}[t^{-1}]$ coincide with $G[[t]] := G(\mathbb{C}[[t]])$, $G((t)) := G(\mathbb{C}((t)))$ and $G[t^{-1}] := G(\mathbb{C}[t^{-1}])$, respectively.

Further, $\bar{G}[[t]]$ is an affine group scheme, which is a closed subgroup scheme of $\overline{\mathrm{SL}}_N[[t]]$. Similarly, $\bar{G}((t))$ and $\bar{G}[t^{-1}]$ are ind-affine group schemes which are closed ind-subgroup schemes of $\overline{\mathrm{SL}}_N((t))$ and $\overline{\mathrm{SL}}_N[t^{-1}]$, respectively.

Proof We prove the lemma for $\mathscr{F}_1$; the proof for $\mathscr{F}_2$ and $\mathscr{F}_3$ is similar. Let R be a $\mathbb{C}$-algebra. We need to prove that there is a functorial identification

$$\mathrm{Mor}(\mathrm{Spec}\, R, \bar{G}[[t]]) \simeq G(R[[t]]). \tag{1}$$

As at the beginning of the section, since G is an affine variety, there is a canonical bijection

$$G(R[[t]]) \simeq \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[G], R[[t]]), \tag{2}$$

where $\mathrm{Hom}_{\mathrm{alg}}(-,-)$ denotes the set of $\mathbb{C}$-algebra homomorphisms. Further, since $\bar{G}[[t]]$ is an affine scheme, there is a canonical bijection

$$\mathrm{Mor}(\mathrm{Spec}\, R, \bar{G}[[t]]) \simeq \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[\bar{G}[[t]]], R). \tag{3}$$

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Combining (1)–(3), it suffices to prove that there is a canonical bijection

$$\mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[G], R[[t]]) \simeq \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[\bar{G}[[t]]], R). \tag{4}$$

The closed embedding $i: G \hookrightarrow M_N$ gives rise to the closed embedding

$$i_t: \bar{G}[[t]] \hookrightarrow \bar{M}_N[[t]],$$

where $\bar{M}_N[[t]]$ is the scheme Spec $\left(\mathbb{C}[y_n^{i,j}]_{n\geq 0;\, 1\leq i,j\leq N}\right)$.

Clearly, the analogue of (4) for G replaced by M_N is true under the map (following the notation in Definition 1.3.1)

$$\varphi_{M_N}: \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[M_N], R[[t]]) \to \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[\bar{M}_N[[t]]], R), \quad f \mapsto \bar{f},$$

where $f(y^{i,j}) = \sum_{n\geq 0} \bar{f}(y_n^{i,j})t^n$, for any $1 \leq i, j \leq N$. For any $P \in \mathbb{C}[M_N]$ and any $f \in \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[M_N], R[[t]])$, it is easy to see that

$$f(P) = \sum_{m\geq 0} \bar{f}(\widehat{P}_m^{(0)})t^m.$$

From this, it follows that the above bijection φ_{M_N} restricts to a bijection φ_G under the canonical embeddings induced by i:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[G], R[[t]]) & \overset{\varphi_G}{\simeq} & \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[\bar{G}[[t]]], R) \\ \downarrow \widehat{i} & & \downarrow \widehat{i_t} \\ \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[M_N], R[[t]]) & \underset{\varphi_{M_N}}{\simeq} & \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[\bar{M}_N[[t]]], R). \end{array}$$

This proves (4) and hence (1).

The ‘In particular’ part of the lemma follows since the functor $\mathscr{F}_1$ is independent of the choice of an embedding $G \hookrightarrow M_N$ (by (2)) and the representing scheme is unique (cf. Lemma 1.1.1).

To prove that $\bar{G}[[t]]$ is an affine group scheme, since $\mathscr{F}_1$ is representable by $\bar{G}[[t]]$, it suffices to observe (using (Mumford, 1988, Chapter II, §6, Proposition 2)) that the morphism $G \times G \to G$, $(g,h) \mapsto gh^{-1}$, induces a natural map $\mathscr{F}_1(R) \times \mathscr{F}_1(R) \to \mathscr{F}_1(R)$ for any $R \in$ **Alg**. It is a closed subgroup scheme of $\overline{\mathrm{SL}}_N[[t]]$ by construction. The proofs for $\bar{G}((t))$ and $\bar{G}[t^{-1}]$ are identical. □

Corollary 1.3.3 *Let G be any affine algebraic group.*

(a) Consider the morphism $\epsilon(\infty): \bar{G}[t^{-1}] \to G$ induced from the $\mathbb{C}$-algebra homomorphism $R[t^{-1}] \to R$, $t^{-1} \mapsto 0$.

Let $\bar{G}[t^{-1}]^-$ be the (ind)-scheme theoretic fiber of $\epsilon(\infty)$ over 1. Then, it represents the functor $G(R[t^{-1}])^-$: **Alg** $\to$ **Set** *defined as the kernel of the*

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homomorphism $\epsilon_R(\infty)\colon G(R[t^{-1}]) \to G(R)$ *induced from the* $\mathbb{C}$*-algebra homomorphism* $R[t^{-1}] \to R$, $t^{-1} \mapsto 0$.

Since $\bar{G}[t^{-1}] \hookrightarrow \overline{\mathrm{SL}}_N[t^{-1}]$ *is a closed embedding (cf. Definition 1.3.1), it is easy to see that* $\bar{G}[t^{-1}]^- \hookrightarrow \overline{\mathrm{SL}}_N[t^{-1}]^-$ *is a closed embedding.*

(b) Let $H \subset G$ *be a closed subgroup. Consider the morphism* $\epsilon(0)\colon \bar{G}[[t]] \to G$ *induced from the* $\mathbb{C}$*-algebra homomorphism* $R[[t]] \to R$, $t \mapsto 0$.

Let $\bar{G}[[t]]_H$ *be the scheme-theoretic inverse image of* H*. Then it represents the functor* $G(R[[t]])_H$ *defined as the inverse image of* $H(R)$ *under the homomorphism* $\epsilon_R(0)\colon G(R[[t]]) \to G(R)$.

Proof (a) By Lemma 1.3.2, $\bar{G}[t^{-1}]$ represents the functor $G(R[t^{-1}])$ (and, of course, G represents the functor $G(R)$). Now, by Exercise 1.3.E.6, $\bar{G}[t^{-1}]^-$ represents the functor $G(R[t^{-1}])^-$. This proves (a).

The proof of (b) is identical. □

Remark 1.3.4 Even though we do not need to, for any affine scheme X of finite type over $\mathbb{C}$, as in Definition 1.3.1 and Lemma 1.3.2, we can define an affine (non-noetherian) scheme $\bar{X}[[t]]$ which represents the covariant functor $\mathscr{F}_X\colon$ **Alg** $\to$ **Set** defined by

$$\mathscr{F}_X(R) = X(R[[t]]) \cong \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[X], R[[t]]).$$

In particular, the $\mathbb{C}$-points of $\bar{X}[[t]] = X(\mathbb{C}[[t]])$.

Exactly the same remark applies to $\mathbb{C}[[t]]$ replaced by $\mathbb{C}[t^{-1}]$ or $\mathbb{C}((t))$.

Definition 1.3.5 (Infinite Grassmannian) For any affine algebraic group G over $\mathbb{C}$, define the *infinite Grassmannian* $\mathscr{X}_G$ as the sheafification of the functor $\mathscr{X}_G^o\colon R \rightsquigarrow G(R((t)))/G(R[[t]])$ (cf. Lemma B.2). Observe that $\mathscr{X}_G^o$ satisfies condition (1) of Lemma B.2 since for any fppf R-algebra R', $R \to R'$ is injective (cf. (Matsumura, 1989, Theorem 7.5)).

In the following, we show that $\mathscr{X}_G$ is representable, represented by an ind-projective scheme $\bar{X}_G$ with $\mathbb{C}$-points $X_G := G((t))/G[[t]]$ for any connected reductive group G. We first consider the case $G = \mathrm{SL}_N$.

Definition 1.3.6 (Representing $\mathscr{X}_{\mathrm{SL}_N}$ by an ind-projective scheme) Denote $V = \mathbb{C}^N$. For any non-negative integer n, define the *nth special lattice functor* $\mathscr{Q}_n = \mathscr{Q}_n^N\colon$ **Alg** $\to$ **Set** by $\mathscr{Q}_n(R) =$ set of projective $R[[t]]$-submodules L^R of $R((t)) \otimes_{\mathbb{C}} V$ satisfying the following two conditions:

(a) $t^n L_o^R \subset L^R \subset t^{-n} L_o^R$, where $L_o^R := R[[t]] \otimes_{\mathbb{C}} V$.

(b) $\wedge^N_{R[[t]]}(L^R) \to \wedge^N_{R((t))}(R((t)) \otimes_{\mathbb{C}} V) \simeq R((t))$ has image, denoted by $\det L^R$, precisely equal to $R[[t]]$.

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Now, define the *special lattice functor* $\mathscr{Q} = \mathscr{Q}^N$ by

$$\mathscr{Q}(R) = \cup_{n\geq 0}\, \mathscr{Q}_n(R).$$

By Exercise 1.3.E.5, the functor $\mathscr{Q}$ is the sheafification $\mathscr{X}_{\mathrm{SL}_N}$ of the functor $R \rightsquigarrow \mathrm{SL}_N(R((t)))/\,\mathrm{SL}_N(R[[t]])$.

In particular, taking $R = \mathbb{C}$, define

$$Q_n := \mathscr{Q}_n(\mathbb{C}) \ \text{ and } \ Q := \mathscr{Q}(\mathbb{C}).$$

(In fact, any $\mathbb{C}[[t]]$-submodule $L^{\mathbb{C}}$ of $V((t)) := \mathbb{C}((t)) \otimes_{\mathbb{C}} V$ satisfying (a) is automatically $\mathbb{C}[[t]]$-free, being a submodule of a free module over a principal ideal domain (PID). Thus, Q_n consists of $\mathbb{C}[[t]]$-submodules L of $V((t))$ such that

$$t^n L_o \subset L \subset t^{-n} L_o, \quad \text{and} \quad \det(L) = \mathbb{C}[[t]],$$

where $L_o := \mathbb{C}[[t]] \otimes_{\mathbb{C}} V$. In fact, in the proof of Theorem 1.3.8, we will see that the condition $\det(L) = \mathbb{C}[[t]]$ can be replaced by the condition $\dim_{\mathbb{C}}(L/t^n L_o) = nN$.)

Recall that for any scheme X and any automorphism f of X, the fixed-point subset X^f acquires a canonical scheme structure as the inverse image subscheme of the diagonal $\Delta(X)$ under the morphism

$$\mathfrak{f}\colon X \to X \times X, \quad x \mapsto (x, f(x)).$$

Consider the complex vector space $V_n := t^{-n}L_o/t^n L_o$ of dimension $2nN$. Then multiplication by t induces a nilpotent endomorphism $\bar{t}_n$ of V_n and hence $1+\bar{t}_n$ is a unipotent automorphism of V_n. In particular, $1+\bar{t}_n$ induces an isomorphism (denoted by the same symbol) of the Grassmannian $\mathrm{Gr}(nN, 2nN)$ of nN-dimensional subspaces of the $2nN$-dimensional space V_n. Let $\bar{F}_n = \bar{F}_n^N := \mathrm{Gr}(nN, 2nN)^{1+\bar{t}_n}$ denote its fixed-point projective scheme and let $F_n := \bar{F}_n(\mathbb{C})$ be the $\mathbb{C}$-points of $\bar{F}_n$. Then clearly the map $i_n\colon Q_n \to F_n \subset \mathrm{Gr}(nN, 2nN)$ given by $L \mapsto L/t^n L_o$ is a bijection.

It is easy to see that the inclusion $\theta_n\colon \mathrm{Gr}(nN, 2nN) \hookrightarrow \mathrm{Gr}((n+1)N, 2(n+1)N)$ is a closed embedding, where (denoting $t^k V := t^k \otimes V$) the map θ_n takes $V' \subset t^{-n}L_o/t^n L_o \simeq t^{n-1}V \oplus t^{n-2}V \oplus \cdots \oplus t^{-n}V$ to $t^n V \oplus V'$. Moreover, it is easy to see that θ_n restricts to a closed embedding $\bar{\theta}_n\colon \bar{F}_n \hookrightarrow \bar{F}_{n+1}$.

By virtue of the following lemma, we have a bijection $\beta\colon X_{\mathrm{SL}_N} \to Q^N$.

Extending the scalar, the group $\mathrm{SL}_N((t))$ clearly acts on $V((t))$.

Lemma 1.3.7 *The map*

$$\beta\colon X_{\mathrm{SL}_N} \to Q^N, \ \ g\,\mathrm{SL}_N[[t]] \mapsto gL_o, \ \textit{for} \ \ g \in \mathrm{SL}_N((t)),$$

is a bijection.

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Proof Let $g \in \mathrm{SL}_N((t))$. It is easy to see that there exists some n (depending upon g) such that

$$t^n L_o \subset g L_o \subset t^{-n} L_o. \tag{1}$$

Of course, gL_o is t-stable. We next calculate the dimension of $gL_o/t^n L_o$.

By the Bruhat decomposition (cf. (Kumar, 2002, Corollary 13.2.10)), we may assume that g is an algebraic group homomorphism $\mathbb{C}^* \to D$, where D is the diagonal subgroup of SL_N. Write

$$g(t) = \begin{pmatrix} t^{n_1} & & O \\ & \ddots & \\ O & & t^{n_N} \end{pmatrix}, \quad \text{for} \quad t \in \mathbb{C}^* \quad \text{and} \quad n_i \in \mathbb{Z}.$$

Then, since $\operatorname{Im} g \subset \mathrm{SL}_N$, we get $\Sigma n_i = 0$. Now

$$\dim \left(gL_o/t^n L_o\right) = \sum_{i=1}^{N} (n - n_i) = Nn - \Sigma n_i = Nn.$$

This proves that $gL_o \in Q_n$.

Conversely, take $L \in Q_n$. Since $\mathscr{O} := \mathbb{C}[[t]]$ is a PID and $t^k L_o$ is $\mathscr{O}$-free of rank N (for any $k \in \mathbb{Z}$), we get that L is $\mathscr{O}$-free of rank N. Further, $K \otimes_{\mathscr{O}} L \to V((t))$ is an isomorphism, where $K = \mathbb{C}((t))$. Let $\{e_1, \ldots, e_N\}$ be the standard $\mathbb{C}$-basic of V and take a $\mathscr{O}$-basis $\{v_1, \ldots, v_N\}$ of L. Now, define the K-linear automorphism g of $V((t))$ by $ge_i = v_i$ $(1 \le i \le N)$. We prove that $\det g$ is a unit of $\mathscr{O}$: write $\det g = t^k u$, where $k \in \mathbb{Z}$ and u is a unit of $\mathscr{O}$. Consider the K-linear automorphism α of $V((t))$ defined by

$$\begin{aligned} \alpha e_i &= e_i, \ \text{for} \ 1 \le i < N \\ &= t^{-k} u^{-1} e_N, \ \text{for} \ i = N. \end{aligned}$$

Then $\det(g\alpha) = 1$ and $t^{n+|k|} L_o \subset (g\alpha) L_o \subset t^{-n-|k|} L_o$. Hence, by the earlier part of the proof, we get

$$\dim \left(\frac{g\alpha(L_o)}{t^{n+|k|} L_o}\right) = (n + |k|)N. \tag{2}$$

On the other hand,

$$\begin{aligned} \dim \left(\frac{g\alpha(L_o)}{t^{n+|k|} L_o}\right) &= \dim \left(\frac{gL_o}{t^n L_o}\right) + |k|N + k \\ &= Nn + |k|N + k, \ \text{since} \ L \in Q_n. \end{aligned} \tag{3}$$

Combining (2) and (3), we get $k = 0$. Hence, $g\alpha(L_o) = gL_o = L$. This proves the surjectivity of β. The injectivity of β is clear. This proves the lemma. □

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Theorem 1.3.8 *For any fixed $N \geq 1$ and $n \geq 0$, the nth special lattice functor $\mathscr{Q}_n = \mathscr{Q}_n^N$ (defined in Definition 1.3.6) is representable, represented by a projective scheme $\bar{H}_n$ (with $\mathbb{C}$-points Q_n), which is a closed subscheme of $\bar{F}_n$ (defined in Definition 1.3.6). Moreover, the inclusion $\bar{H}_n \hookrightarrow \bar{F}_n$ induces an isomorphism of the corresponding reduced schemes $\bar{H}_n^{red} \xrightarrow{\sim} \bar{F}_n^{red}$.*

Further, the canonical morphism $\bar{H}_n \to \bar{H}_{n+1}$ (induced from the inclusion of the functors $\mathscr{Q}_n \subset \mathscr{Q}_{n+1}$) is a closed embedding. Thus, we get an ind-projective scheme $\bar{H} = (\bar{H}_n)_{n\geq 0}$ representing the functor $\mathscr{Q}$, with $\mathbb{C}$-points $Q^N := \bigcup_{n\geq 0} Q_n^N$. Through the bijection β of Lemma 1.3.7, we get the $\mathbb{C}$-points of $\bar{H}$ to be X_{SL_N}.

Thus, by Exercise 1.3.E.5, $\bar{H}$ also represents the functor $\mathscr{X}_{\mathrm{SL}_N}$. In particular, $\mathscr{X}_{\mathrm{SL}_N}(\mathbb{C}) = X_{\mathrm{SL}_N}$.

We denote the ind-scheme $\bar{H}$ by $\bar{X}_{\mathrm{SL}_N}$. Thus, $\bar{X}_{\mathrm{SL}_N}$ represents the functor $\mathscr{X}_{\mathrm{SL}_N}$.

Proof By Eisenbud and Harris (2000, Exercise VI-18), $\mathrm{Gr}(nN, 2nN)$ represents the functor

$$\begin{aligned} R &\rightsquigarrow \mathrm{Gr}(nN, 2nN; R) \\ &:= \text{ set of } R\text{-module direct summands of } R^{2nN} \text{ of rank } nN. \end{aligned}$$

Thus, following the notation of Definition 1.3.6 and Exercise 1.3.E.3, the functor represented by the scheme $\bar{F}_n$ is given by

$$\begin{aligned} \mathscr{F}_n(R) &= \mathrm{Gr}(nN, 2nN; R)^{1+\bar{t}_n} \\ &= \text{set of } R\text{-module direct summands } \tilde{L}^R \text{ of } \frac{t^{-n}L_o^R}{t^n L_o^R} \text{ of rank } nN, \\ &\quad\ \text{which are } (1+\bar{t}_n)\text{-stable.} \end{aligned}$$

Taking the inverse image L^R of $\tilde{L}^R$ under $t^{-n}L_o^R \to t^{-n}L_o^R/t^n L_o^R$, we get that L^R satisfies (a) of Definition 1.3.6 and $\tilde{L}^R$ is $(1+\bar{t}_n)$-stable if and only if L^R is an $R[[t]]$-submodule of $t^{-n}L_o^R$. Further, L^R is a projective $R[[t]]$-module if and only if $\tilde{L}^R$ is an R-module direct summand of $t^{-n}L_o^R/t^n L_o^R$ (cf. Exercise 1.3.E.1).

We next show that when the $\mathbb{C}$-algebra R is a field $k \supset \mathbb{C}$,

$$\mathscr{F}_n(k) = \mathscr{Q}_n(k), \quad \text{for any} \quad n \geq 0. \tag{1}$$

Take any $k[[t]]$-submodule L^k satisfying condition (a) of Definition 1.3.6. By the Elementary Divisor Theorem for free modules over a PID, we get that there exists a $k[[t]]$-basis $\{v_1, \ldots, v_N\}$ of L_o^k such that

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$\{t^{-n+d_1}v_1, \ldots, t^{-n+d_N}v_N\}$ is a basis of L^k, for some $d_i \geq 0$. Now, condition (b) of Definition 1.3.6 is equivalent to the condition

$$\sum_{i=1}^{N} -n + d_i = 0. \tag{2}$$

Further,

$$\dim_k \tilde{L}^k = \sum_{i=1}^{N}(2n - d_i) = 2nN - \sum d_i. \tag{3}$$

Comparing (2) and (3), we see that condition (b) is equivalent to the condition that $\dim_k \tilde{L}^k = nN$. This proves (1).

We next show that for any $R \in \mathbf{Alg}$,

$$\mathscr{Q}_n(R) \subset \mathscr{F}_n(R), \quad \text{for any} \quad n \geq 0. \tag{4}$$

Let L^R be a projective $R[[t]]$-submodule satisfying conditions (a) and (b) of Definition 1.3.6. Taking $\mathbb{C}$-algebra homomorphisms $\varphi\colon R \to k$ (where k is a field) and considering $L^k := k[[t]] \otimes_{R[[t]]} L^R$ and using Exercise 1.3.E.2, we get (from the case that R is a field proved earlier) that $\tilde{L}^R$ is of rank nN over R, proving (4).

Conversely, assume that $R \in \mathbf{Alg}$ has no nonzero nilpotents. In this case, we prove that

$$\mathscr{F}_n(R) \subset \mathscr{Q}_n(R). \tag{5}$$

Take $\tilde{L}^R \in \mathscr{F}_n(R)$. Let $R_\mathfrak{p}$ be the localization of R at a prime ideal $\mathfrak{p}$ of R. Since a projective module over a local ring is free (cf. (Matsumura, 1989, Theorem 2.5)), we get that $L^{R_\mathfrak{p}} := R_\mathfrak{p}[[t]] \otimes_{R[[t]]} L^R$ is an $R_\mathfrak{p}[[t]]$-free module of rank N ($R_\mathfrak{p}[[t]]$ is a local ring by Exercise 1.3.E.13). Thus, $\det(L^{R_\mathfrak{p}}) \subset R_\mathfrak{p}((t))$ is given by $t^{-nN}P(t) \cdot R_\mathfrak{p}[[t]]$, where $P(t) \in R_\mathfrak{p}[[t]]$. Now, take any $\mathbb{C}$-algebra homomorphism $\varphi\colon R_\mathfrak{p} \to k$ to a field k. From the case when R is a field proved above as in (1), we get that the image $P^k(t)$ of $P(t)$ in $k[[t]]$ (via φ) is t^{nN} times a unit of $k[[t]]$. Since this is true for any φ and R has no nonzero nilpotents, we get by using Atiyah and Macdonald (1969, Proposition 1.8) that $P(t)$ is t^{nN} times a unit of $R_\mathfrak{p}[[t]]$. (In general, if we allow R to have nilpotents, $P(t)$ would be of the form

$$P(t) = a_0 + a_1 t + \cdots + a_{nN-1}t^{nN-1} + a_{nN}t^{nN} + \cdots, \tag{6}$$

where $a_0, a_1, \ldots, a_{nN-1}$ are nilpotents in $R_\mathfrak{p}$ and a_{nN} is a unit of $R_\mathfrak{p}$.) Reverting to the case when R has no nonzero nilpotents, from the above, we get that

$$\det(L^{R_\mathfrak{p}}) = R_\mathfrak{p}[[t]]. \tag{7}$$

Let $M := \det(L^R) \subset R((t))$. Then, from condition (a) of Definition 1.3.6, we get

$$M = t^{-nN} M_o, \text{ where } M_o \text{ is a finitely generated ideal of } R[[t]]. \tag{8}$$

Take $Q(t) = \sum_{i\geq 0} b_i t^i \in M_o$. Then $Q(t)$, considered as an element of $R_\mathfrak{p}[[t]]$, belongs to $t^{nN}\det(L^{R_\mathfrak{p}}) = t^{nN} R_\mathfrak{p}[[t]]$ for any prime ideal $\mathfrak{p}$ of R (by (7)). Thus, $b_1 = \cdots = b_{nN-1} = 0$ as elements of $R_\mathfrak{p}$ (for any $\mathfrak{p}$). Thus, $b_1 = \cdots = b_{nN-1} = 0$ as elements of R (since R has no nonzero nilpotents). Hence, $M_o \subset t^{nN} R[[t]]$. Since $t^n L_o^R \subset L^R$, we have $t^{2nN} R[[t]] \subset M_o$. Consider the quotient R-module

$$A := \frac{t^{nN} R[[t]]}{M_o} \simeq \frac{t^{nN} R[[t]]/t^{2nN} R[[t]]}{M_o/t^{2nN} R[[t]]}.$$

Applying Atiyah and Macdonald (1969, Proposition 3.8) to the R-module A and using (7), we get $A = 0$, i.e., $\det(L^R) = R[[t]]$. Thus, L^R satisfies condition (b) of Definition 1.3.6, proving $L^R \in \mathscr{Q}_n(R)$ by Exercise 1.3.E.1. This proves (5).

Now, we analyze the failure of (5) for general $R \in$ **Alg**. Take any affine open subset $\mathrm{Spec}(S) \subset \bar{F}_n$, for a finitely generated $\mathbb{C}$-algebra S. The inclusion gives rise to the element $\tilde{L}_o^S \in \mathrm{Mor}(\mathrm{Spec}(S), \bar{F}_n) = \mathscr{F}_n(S)$ and hence a projective $S[[t]]$-module L_o^S satisfying (a) of Definition 1.3.6. Take an affine open cover $\{\mathrm{Spec}(S_i)\}_i$ of $\mathrm{Spec}(S)$ so that the $S_i[[t]]$-module

$$L_o^{S_i} := S_i[[t]] \otimes_{S[[t]]} L_o^S \quad \text{is free.} \tag{9}$$

This is possible by Exercise 1.3.E.4. Thus we get, from the proof of (5) given above (see specifically (6)), that

$$\det(L_o^{S_i}) = t^{-nN} P_i(t) S_i[[t]] \subset S_i((t)), \tag{10}$$

where $P_i(t)$ is of the form

$$P_i(t) = a_0^i + a_1^i t + \cdots + a_{nN-1}^i t^{nN-1} + a_{nN}^i t^{nN} + \cdots, \\ \text{for some nilpotents } a_0^i, a_1^i, \ldots, a_{nN-1}^i \text{ in } S_i. \tag{11}$$

The nilpotent ideal

$$J_{S_i} = \langle a_0^i, a_1^i, \ldots, a_{nN-1}^i \rangle \subset S_i$$

clearly does not depend upon the choice of the representative $P_i(t)$. In particular, these ideals descend to give a nilpotent ideal $J_S \subset S$. Taking an affine open cover of $\bar{F}_n$ by $\mathrm{Spec}\, S$, we get a nilpotent ideal sheaf $\mathscr{J} \subset \mathscr{O}_{\bar{F}_n}$. Now,

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define the closed subscheme $\bar{H}_n$ of $\bar{F}_n$ given by the ideal sheaf $\mathscr{J}$. Thus, their reduced subschemes are isomorphic:

$$\bar{H}_n^{\text{red}} \simeq \bar{F}_n^{\text{red}}. \tag{12}$$

We next prove that the scheme $\bar{H}_n$ represents the functor $\mathscr{Q}_n$, i.e., for any $R \in \mathbf{Alg}$, there is a natural isomorphism

$$\mathscr{Q}_n(R) \xrightarrow{\sim} \text{Mor}(\text{Spec}(R), \bar{H}_n) \hookrightarrow \text{Mor}(\text{Spec}\, R, \bar{F}_n) =: \bar{F}_n(R). \tag{13}$$

By (4), we have an inclusion $\mathscr{Q}_n(R) \subset \mathscr{F}_n(R)$. We claim that the image lands inside $\text{Mor}(\text{Spec}(R), \bar{H}_n)$.

Take $L^R \in \mathscr{Q}_n(R)$. Then, by definition,

$$\det(L^R) = R[[t]]. \tag{14}$$

The element L^R gives rise to a morphism $\tilde{L}^R\colon \text{Spec}(R) \to \bar{F}_n$. Since $\bar{F}_n$ is a scheme of finite type over $\mathbb{C}$, we can assume that R is a finitely generated $\mathbb{C}$-algebra. Take 'small enough' affine open covers $\{\text{Spec}(R_i)\}_i$ of Spec R and $\{\text{Spec}(S_i)\}_i$ of $\bar{F}_n$ such that $\tilde{L}^R$ restricts to $\text{Spec}(R_i) \to \text{Spec}\, S_i$ (i.e., gives a $\mathbb{C}$-algebra homomorphism $f_i\colon S_i \to R_i$) and the $S_i[[t]]$-module $L_o^{S_i}$ is free, where $L_o^{S_i}$ is defined by (9). Thus, by (10), $\det(L_o^{S_i}) = t^{-nN} P_i(t) S_i[[t]]$, where $P_i(t) = \sum_{d \geq 0} a_d^i t^d$ is of the form (11). In particular,

$$\det\left(L^{R_i} := R_i[[t]] \otimes_{S_i[[t]]} L_o^{S_i}\right) = t^{-nN} f_i(P_i(t)) R_i[[t]], \tag{15}$$

where $f_i(P_i(t))$ is obtained from $P_i(t)$ by applying f_i to all the coefficients. But $L^{R_i} = R_i[[t]] \otimes_{R[[t]]} L^R$. Hence, by (14),

$$\det(L^{R_i}) = R_i[[t]]. \tag{16}$$

Comparing (15) and (16), we get

$$f_i(a_d^i) = 0, \quad \text{for all} \quad 0 \leq d < nN, \tag{17}$$

i.e., the homomorphism f_i factors through $S_i/\langle a_0^i, a_1^i, \ldots, a_{nN-1}^i \rangle$. This shows that, from the definition of $\bar{H}_n$, $\tilde{L}^R\colon \text{Spec}(R) \to \bar{H}_n$. Hence, the image of $\mathscr{Q}_n(R)$ (inside $\bar{F}_n(R)$) lands inside $\bar{H}_n(R)$.

Conversely, take $\tilde{L}^R \in \text{Mor}(\text{Spec}\, R, \bar{H}_n)$ and let L^R be the corresponding projective $R[[t]]$-submodule of $R((t)) \otimes_{\mathbb{C}} V$ (which satisfies condition (a) of Definition 1.3.6). Then, by the above calculation (see (15) and (17)), for a 'small enough' open cover $\{\text{Spec}(R_i)\}_i$ of $\text{Spec}(R)$ (since $\tilde{L}^R$ has image inside $\bar{H}_n$), $\det(L^{R_i}) = b_i(t) R_i[[t]]$, for some $b_i(t) = \sum_{d \geq 0} \alpha_d^i t^d \in R_i[[t]]$, where $L^{R_i} := R_i[[t]] \otimes_{R[[t]]} L^R$. Considering $\mathbb{C}$-algebra homomorphisms $\varphi\colon R_i \to k$

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(for a field k), from the case when R is a field proved earlier, we get that $\varphi(\alpha_0^i) \neq 0$. Since this is true for any φ, we get that α_0^i is a unit of R_i, i.e.,

$$\det(L^{R_i}) = R_i[[t]]. \tag{18}$$

By (8), $\det(L^R) = t^{-nN} M_o$, for some finitely generated ideal M_o of $R[[t]]$. Since the image of M_o in $R_i[[t]]$ equals $t^{nN} R_i[[t]]$ by (18) (for an affine open cover $\{\mathrm{Spec}(R_i)\}_i$ of $\mathrm{Spec}\, R$), we first conclude that $M_o \subset t^{nN} R[[t]]$ (and, of course, from the definition of L^R, $M_o \supset t^{2nN} R[[t]]$). Moreover, considering the quotient R-module

$$A := \frac{t^{nN} R[[t]]}{M_o} \simeq \frac{t^{nN} R[[t]]/t^{2nN} R[[t]]}{M_o/t^{2nN} R[[t]]}$$

and using Atiyah and Macdonald (1969, Proposition 3.8) together with the equation (18), we get $\det(L^R) = R[[t]]$. Thus, L^R satisfies condition (b) of Definition 1.3.6 as well, i.e., $L^R \in \mathcal{Q}_n(R)$, proving $\mathrm{Mor}(\mathrm{Spec}\, R, \bar{H}_n) \subset \mathcal{Q}_n(R)$. Thus, $\mathcal{Q}_n(R) \simeq \bar{H}_n(R)$ and hence $\mathcal{Q}_n(R)$ is a representable functor represented by the scheme $\bar{H}_n$ for all $n \geq 0$.

Finally, we have the following commutative diagram of schemes and morphisms between them:

$$\begin{array}{ccc} \bar{H}_n & \xhookrightarrow{\;i_n\;} & \bar{F}_n \\ {\scriptstyle j_n}\downarrow & & \downarrow{\scriptstyle \overline{\theta}_n} \\ \bar{H}_{n+1} & \xhookrightarrow{\;i_{n+1}\;} & \bar{F}_{n+1}, \end{array} \tag{$\mathcal{D}$}$$

where the morphism $j_n \colon \bar{H}_n \to \bar{H}_{n+1}$ is induced from the canonical inclusion of functors $\mathcal{Q}_n(R) \hookrightarrow \mathcal{Q}_{n+1}(R)$. Since i_n, i_{n+1} are closed embeddings (by definition, $\bar{H}_n \subset \bar{F}_n$ is a closed subscheme) and $\overline{\theta}_n$ is a closed embedding as seen in Definition 1.3.6, we get that j_n is a closed embedding. This completes the proof of the theorem. □

The following example shows that the inclusion $\mathcal{Q}_n(R) \subset \mathcal{F}_n(R)$ (cf. (4) of the proof of Theorem 1.3.8) is proper for some $R \in \mathbf{Alg}$ already for $n = 1$, $N = 2$. In particular, by Theorem 1.3.8, the scheme $\bar{F}_1^2$ is not reduced.

Example 1.3.9 Let $R = \mathbb{C}[\epsilon]/\langle \epsilon^2 \rangle$. Consider the element $g \in \mathrm{GL}_2(R((t)))$ given by

$$g = \begin{pmatrix} \epsilon t^{-1} + 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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Its inverse is $\begin{pmatrix} -\epsilon t^{-1}+1 & 0 \\ 0 & 1 \end{pmatrix}$. Clearly, $tL_o^R \subset gL_o^R \subset t^{-1}L_o^R$. Now $\Lambda^2_{R[[t]]}(gL_o^R) = (\epsilon t^{-1}+1)R[[t]] \neq R[[t]]$.

But

$$\frac{gL_o^R}{tL_o^R} \simeq \frac{(\epsilon t^{-1}+1)R[[t]]}{tR[[t]]} \oplus \frac{R[[t]]}{tR[[t]]}$$

is a free R-module of rank 2. To show this, observe that, as an R-module,

$$\frac{(\epsilon t^{-1}+1)R[[t]]}{tR[[t]]} \simeq \frac{(\epsilon+t)R[[t]]}{(\epsilon+t)(\epsilon-t)R[[t]]} \simeq \frac{R[[t]]}{(\epsilon-t)R[[t]]},$$

where the last isomorphism follows since $\epsilon+t$ is not a zero divisor in $R[[t]]$ as can be seen by multiplying it by $\epsilon - t$.

Define an R-module map

$$\theta\colon R \to R[[t]]/(t-\epsilon)R[[t]] \quad \text{by} \quad r \mapsto r + (t-\epsilon)R[[t]].$$

It is clearly surjective. Moreover, it is injective since if $r + (t-\epsilon)P(t) = 0$, for some $P(t) \in R[[t]]$, then $(t+\epsilon)r + t^2P(t) = 0$. But $t^2P(t)$ has no 't-term,' thus $r = 0$.

Further, it is easy to see that gL_o^R/tL_o^R is an R-module direct summand of $t^{-1}L_o^R/tL_o^R$.

Definition 1.3.10 Recall that $\overline{\mathrm{SL}_N}((t))$ represents the functor $\mathrm{SL}_N(R((t)))$ (cf. Lemma 1.3.2) and $\bar{X}_{\mathrm{SL}_N}$ represents the functor $\mathscr{X}_{\mathrm{SL}_N}$ (cf. Theorem 1.3.8). Also, it is easy to see that the sheafification of the functor $\mathrm{SL}_N(R((t))) \times \mathscr{X}^o_{\mathrm{SL}_N}$ is $\mathrm{SL}_N(R((t))) \times \mathscr{X}_{\mathrm{SL}_N}$ (since $\mathrm{SL}_N(R((t)))$ is representable), where $\mathscr{X}^o_{\mathrm{SL}_N}(R) := \mathrm{SL}_N(R((t)))/\,\mathrm{SL}_N(R[[t]])$. Thus, the multiplication

$$\mathrm{SL}_N(R((t))) \times \mathscr{X}^o_{\mathrm{SL}_N}(R) \to \mathscr{X}^o_{\mathrm{SL}_N}(R), \ \ (g, h\bar{o}_R) \mapsto gh\bar{o}_R,$$

gives rise to a $\mathbb{C}$-space functor morphism

$$\mathrm{SL}_N(R((t))) \times \mathscr{X}_{\mathrm{SL}_N} \to \mathscr{X}_{\mathrm{SL}_N},$$

where $\bar{o}_R$ is the base point of $\mathrm{SL}_N(R((t)))/\,\mathrm{SL}_N(R[[t]])$. This, in turn, gives rise to a morphism of ind-schemes

$$\mu\colon \overline{\mathrm{SL}}_N((t)) \times \bar{X}_{\mathrm{SL}_N} \to \bar{X}_{\mathrm{SL}_N}.$$

We define the 'basic' line bundle $\mathscr{L}$ on $\bar{X}_{\mathrm{SL}_N}$ as follows.

Definition 1.3.11 For any $n \geq 0$, let $\hat{\mathscr{L}}_n$ be the dual of the tautological line bundle over $\mathrm{Gr}(nN, 2nN)$. Recall that the fiber of $\hat{\mathscr{L}}_n$ over any $V' \in \mathrm{Gr}(nN, 2nN)$ is the dual $\Lambda^{nN}(V')^*$. Let $\mathscr{L}_n$ be the pull-back line bundle over $\bar{H}_n$ via the embedding $\bar{i}_n\colon \bar{H}_n \to \mathrm{Gr}(nN, 2nN)$, which is the composite of

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$i_n\colon \bar{H}_n \to \bar{F}_n \hookrightarrow \mathrm{Gr}(nN, 2nN)$ (cf. Definition 1.3.6 and the proof of Theorem 1.3.8). It is easy to see that $\hat{\mathscr{L}}_{n+1}$ restricts to $\hat{\mathscr{L}}_n$ under the embedding θ_n (cf. Definition 1.3.6). Thus, from the commutative diagram $\mathscr{D}$ in the proof of Theorem 1.3.8, $\mathscr{L}_{n+1}$ restricts to $\mathscr{L}_n$ under the embedding $j_n\colon \bar{H}_n \hookrightarrow \bar{H}_{n+1}$. Hence, we get the 'basic' line bundle $\mathscr{L}$ on $\bar{X}_{\mathrm{SL}_N}$.

It is easy to see that the action of $\overline{\mathrm{SL}}_N[[t]]$ on $\bar{X}_{\mathrm{SL}_N}$ (cf. Definition 1.3.10) canonically lifts to its action on $\mathscr{L}$.

Definition 1.3.12 Let $V_n^- \subset V_n := \frac{t^{-n}L_o}{t^n L_o}$ be the subspace $t^{-1}V \oplus \cdots \oplus t^{-n}V$ under the identification

$$V_n \simeq t^{n-1}V \oplus \cdots \oplus t^0 V \oplus t^{-1}V \oplus \cdots \oplus t^{-n}V.$$

Define a section $\hat{\sigma}_n$ of $\hat{\mathscr{L}}_n$ over $\mathrm{Gr}(nN, 2nN)$ by defining $\hat{\sigma}_n(L)$ as the linear form

$$\hat{\sigma}_n(L)\colon \Lambda^{nN}(L) \to \Lambda^{nN}\left(\frac{L_o}{t^n L_o}\right), \quad \text{for any } L \in \mathrm{Gr}(nN, 2nN),$$

induced from the linear map (obtained from the inclusion $L \subset V_n$):

$$L \to \frac{V_n}{V_n^-} \simeq \frac{L_o}{t^n L_o}.$$

We identify $\Lambda^{nN}(L_o/t^n L_o)$ with $\mathbb{C}$ under the basis

$$\Big((t^{n-1}e_1) \wedge \ldots \wedge (t^{n-1}e_N)\Big)$$
$$\wedge \Big((t^{n-2}e_1) \wedge \ldots \wedge (t^{n-2}e_N)\Big) \wedge \ldots \wedge (e_1 \wedge \ldots \wedge e_N),$$

where $\{e_1, \ldots, e_N\}$ is the standard basis of $V = \mathbb{C}^N$.

It is easy to see that $\hat{\sigma}_{n+1}$ restricts to $\hat{\sigma}_n$ under the embedding $\theta_n\colon \mathrm{Gr}(nN, 2nN) \hookrightarrow \mathrm{Gr}((n+1)N, 2(n+1)N)$.

Pulling back the sections $\hat{\sigma}_n$ via the embeddings $\bar{i}_n\colon \bar{H}_n \hookrightarrow \bar{F}_n \subset \mathrm{Gr}(nN, 2nN)$, we get a section σ of the line bundle $\mathscr{L}$ over $\bar{X}_{\mathrm{SL}_N}$ (cf. diagram ($\mathscr{D}$) in the proof of Theorem 1.3.8).

Let $Z(\sigma) \subset \bar{X}_{\mathrm{SL}_N}$ be the zero set of the section σ.

Lemma 1.3.13 *The open ind-subscheme $\bar{X}_{\mathrm{SL}_N} \backslash Z(\sigma)$ represents the functor*

$$R \rightsquigarrow \mathscr{Q}(R) \backslash Z(\sigma_R),$$

where

$$\mathscr{Q}(R)\backslash Z(\sigma_R) := \Big\{L^R \in \mathscr{Q}(R) : i_{L^R} : \\ L^R \to \big(R((t)) \otimes_{\mathbb{C}} V\big)/\big(t^{-1}R[t^{-1}] \otimes V\big) \text{ is an isomorphism}\Big\},$$

and i_{L^R} *is induced from the inclusion* $L^R \subset R((t)) \otimes_{\mathbb{C}} V$.

Proof We need to prove that for any $R \in \mathbf{Alg}$,

$$\operatorname{Mor}\big(\operatorname{Spec} R, \bar{X}_{\mathrm{SL}_N}\backslash Z(\sigma)\big) \simeq \mathscr{Q}(R)\backslash Z(\sigma_R). \tag{1}$$

Take $f \in \operatorname{Mor}(\operatorname{Spec} R, \bar{X}_{\mathrm{SL}_N}) \simeq \mathscr{Q}(R)$ (by Theorem 1.3.8) and let $L^R = L^R(f) \in \mathscr{Q}_n(R)$ be the corresponding lattice (for some $n \geq 0$). To prove (1), we need to prove that $L^R \in \mathscr{Q}(R)\backslash Z(\sigma_R)$ if and only if

$$\operatorname{Im} f \subset \bar{X}_{\mathrm{SL}_N}\backslash Z(\sigma).$$

Since $\bar{X}_{\mathrm{SL}_N}$ is an ind-scheme filtered by schemes of finite type over $\mathbb{C}$, and for any maximal ideal $\mathfrak{m}$ of a finitely generated algebra S over $\mathbb{C}$, $S/\mathfrak{m} \simeq \mathbb{C}$ (cf. (Atiyah and Macdonald, 1969, Corollary 7.10)), to show (1), it suffices to show that for any $L^R \in \mathscr{Q}_n(R)$, where R is a finitely generated $\mathbb{C}$-algebra,

$$L^R \in \mathscr{Q}_n(R)\backslash Z(\sigma_R) \Longleftrightarrow (R/\mathfrak{m}) \otimes_R L^R \in Q_n\backslash Z(\sigma_{\mathbb{C}}) \\ \text{for all the maximal ideals } \mathfrak{m} \text{ of } R. \tag{2}$$

Now

$$\begin{aligned} &L^R \in \mathscr{Q}_n(R)\backslash Z(\sigma_R) \\ &\quad \Leftrightarrow i_{L^R} : L^R \to \frac{R((t)) \otimes V}{t^{-1}R[t^{-1}] \otimes V} \\ &\qquad \text{is an isomorphism by definition} \\ &\quad \Leftrightarrow \bar{i}_{L^R} : \frac{L^R}{t^n L^R_o} \to \frac{R((t)) \otimes V}{(t^n L^R_o) \oplus (t^{-1}R[t^{-1}] \otimes V)} \\ &\qquad \text{is an isomorphism} \\ &\quad \Leftrightarrow \bar{i}_{L^{R_\mathfrak{m}}} : \frac{L^{R_\mathfrak{m}}}{(R_\mathfrak{m} \otimes_R t^n L^R_o)} \to \frac{R_\mathfrak{m}((t)) \otimes V}{(t^n L^{R_\mathfrak{m}}_o) \oplus (t^{-1}R_\mathfrak{m}[t^{-1}] \otimes V)} \\ &\qquad \text{is an isomorphism for all the maximal ideals } \mathfrak{m} \subset R \\ &\qquad \text{by Atiyah and Macdonald (1969, Proposition 3.9)} \end{aligned}$$

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$$\Leftrightarrow \bar{i}_{L^{R/\mathfrak{m}}}: \frac{L^{R/\mathfrak{m}}}{t^n L_o^{R/\mathfrak{m}}} \to \frac{(R/\mathfrak{m})((t)) \otimes V}{(t^n L_o^{R/\mathfrak{m}}) \oplus ((t^{-1}(R/\mathfrak{m})[t^{-1}]) \otimes V)}$$

is an isomorphism by the Nakayama lemma

$$\Leftrightarrow i_{L^{R/\mathfrak{m}}}: L^{R/\mathfrak{m}} \to \frac{(R/\mathfrak{m})((t)) \otimes V}{(t^{-1}(R/\mathfrak{m})[t^{-1}]) \otimes V} \text{ is an isomorphism,} \qquad (3)$$

where $L^{R_\mathfrak{m}} := R_\mathfrak{m} \otimes_R L^R$ and $L^{R/\mathfrak{m}} := (R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}) \otimes_{R_\mathfrak{m}} L^{R_\mathfrak{m}} \simeq (R/\mathfrak{m}) \otimes_R L^R$. (Observe that since $L^R/t^n L_o^R$ is a finitely generated projective R-module by Exercise 1.3.E.1,

$$\left(\frac{R_\mathfrak{m}}{\mathfrak{m}R_\mathfrak{m}}\right) \otimes_{R_\mathfrak{m}} \frac{L^{R_\mathfrak{m}}}{R_\mathfrak{m} \otimes_R t^n L_o^R} \simeq \frac{L^{R/\mathfrak{m}}}{t^n L_o^{R/\mathfrak{m}}}.$$

Moreover, since $\mathfrak{m}$ is a finitely generated ideal, $(R/\mathfrak{m}) \otimes_R t^n L_o^R \simeq t^n L_o^{R/\mathfrak{m}} := t^n (R/\mathfrak{m})[[t]] \otimes_{\mathbb{C}} V$.)

The equivalence (3) is of course the same as the equivalence (2). This proves the lemma. □

Proposition 1.3.14 *The morphism $\mu_1: \overline{\mathrm{SL}}_N[t^{-1}]^- \to \bar{X}_{\mathrm{SL}_N}$, induced from the functor morphism $g \mapsto g \cdot \bar{o}_R$ for $g \in \mathrm{SL}_N(R[t^{-1}])^-$, has its image in $\bar{X}_{\mathrm{SL}_N} \backslash Z(\sigma)$, where $\overline{\mathrm{SL}}_N[t^{-1}]^-$ is defined in Corollary 1.3.3(a) and $\bar{o}_R$ is the base point of $\mathrm{SL}_N(R((t)))/\mathrm{SL}_N(R[[t]])$. Moreover, $\mu_1: \overline{\mathrm{SL}}_N[t^{-1}]^- \to \bar{X}_{\mathrm{SL}_N} \backslash Z(\sigma)$ is an isomorphism of ind-schemes.*

Proof By Corollary 1.3.3(a) and Lemma 1.3.13, it suffices to prove that for any $R \in \mathbf{Alg}$, the map

$$\mu_1(R): \mathrm{SL}_N(R[t^{-1}])^- \to \mathscr{Q}(R),\ g \mapsto g L_o^R,$$

gives a bijection onto $\mathscr{Q}(R) \backslash Z(\sigma_R)$. We first show that

$$\mathrm{Im}\,(\mu_1(R)) \subset \mathscr{Q}(R) \backslash Z(\sigma_R). \qquad (1)$$

We show that $g L_o^R \in \mathscr{Q}(R)$ for any $g \in \mathrm{SL}_N(R((t)))$. In fact, we show that for $g \in \mathrm{SL}_N(R((t)))$ and $L^R \in \mathscr{Q}^N(R)$, $g L^R \in \mathscr{Q}^N(R)$, i.e., $g L^R$ satisfies properties (a) and (b) of Definition 1.3.6 for some $n \geq 0$. Let $g \in \mathrm{SL}_N(t^{-d} R[[t]])$ (cf. Definition 1.3.1) and let $L^R \in \mathscr{Q}_m^N(R)$, for some $d, m \geq 0$. Then it is easy to see that

$$g L^R \subset t^{-m-d} L_o^R. \qquad (2)$$

Choosing $d' \geq 0$ such that $g^{-1} \in \mathrm{SL}_N(t^{-d'} R[[t]])$, we get from (2) (replacing g by g^{-1}): $g^{-1} L^R \subset t^{-m-d'} L_o^R$, which gives (since $t^m L_o^R \subset L^R$ by assumption)

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$$t^{3m+d'} L_o^R \subset gL^R. \tag{3}$$

Combining (3) and (2), we get that gL^R satisfies property (a).

Let $\beta^{L^R}: \Lambda^N_{R[[t]]}(L^R) \to R((t))$ be the map as in (b) of Definition 1.3.6 and let $e_1, \ldots, e_N$ be the standard basis of $V = \mathbb{C}^N$. Taking $v_j = \sum_{i=1}^N p_{ij} e_i \in L^R$, for $p_{ij} \in R((t))$, it is easy to see that

$$\begin{aligned}\beta^{gL^R}(gv_1 \wedge \ldots \wedge gv_N) &= \det g \cdot \beta^{L^R}(v_1 \wedge \ldots \wedge v_N) \\ &= \beta^{L^R}(v_1 \wedge \ldots \wedge v_N), \quad \text{since} \quad g \in \mathrm{SL}_N(R((t))).\end{aligned}$$

Thus, $\det(gL^R) = \det(L^R) = R[[t]]$, proving property (b).

Take $g \in \mathrm{SL}_N(R[t^{-1}])^-$. Then (1) is equivalent to showing that $i_{gL_o^R}: gL_o^R \to \frac{R((t))\otimes V}{t^{-1}R[t^{-1}]\otimes V}$ is an isomorphism. Since $g \in \mathrm{SL}_N(R[t^{-1}])$,

$$g^{-1}(R[t^{-1}] \otimes V) = R[t^{-1}] \otimes V. \tag{4}$$

Thus

$$g^{-1}(t^{-1}R[t^{-1}] \otimes V) = t^{-1}R[t^{-1}] \otimes V. \tag{5}$$

Hence, $i_{gL_o^R}$ is an isomorphism if and only if $i_{L_o^R}: L_o^R \to \frac{R((t))\otimes V}{g^{-1}\cdot(t^{-1}R[t^{-1}]\otimes V)}$ is an isomorphism, which follows from (5).

The injectivity of $\mu_1(R)$ is easy to see.

Finally, take $L^R \in \mathcal{Q}(R)\backslash Z(\sigma_R)$. We first show that the map (induced from the inclusion) $k: L^R \cap (R[t^{-1}] \otimes V) \to L^R/tL^R$ is an isomorphism of R-modules.

From the definition of $\mathcal{Q}(R)\backslash Z(\sigma_R)$,

$$R((t)) \otimes V = L^R \oplus (t^{-1}R[t^{-1}] \otimes V), \tag{6}$$

which gives $R((t)) \otimes V = tL^R \oplus (R[t^{-1}] \otimes V)$. Hence,

$$\begin{aligned}L^R &= L^R \cap (tL^R \oplus (R[t^{-1}] \otimes V)) \\ &= tL^R \oplus (L^R \cap (R[t^{-1}] \otimes V)), \text{ since } tL^R \subset L^R. \end{aligned} \tag{7}$$

From (7), we get that k is an isomorphism.

We further claim that the map (induced from the inclusion) $\ell: L^R \cap (R[t^{-1}] \otimes V) \to \frac{R[t^{-1}]\otimes V}{t^{-1}R[t^{-1}]\otimes V}$ is an isomorphism (of R-modules).

From (6), since $t^{-1}R[t^{-1}] \otimes V \subset R[t^{-1}] \otimes V$, we get

$$R[t^{-1}] \otimes V = \left(L^R \cap (R[t^{-1}] \otimes V)\right) \oplus \left(t^{-1}R[t^{-1}] \otimes V\right).$$

This proves that ℓ is an isomorphism.

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Since $\frac{R[t^{-1}]\otimes V}{t^{-1}R[t^{-1}]\otimes V}$ is a free R-module of rank N, by virtue of the isomorphism ℓ, we get an R-module basis $\{v_1, \ldots, v_N\}$ of $L^R \cap (R[t^{-1}] \otimes V)$, where

$$v_i = 1 \otimes e_i \pmod{t^{-1}R[t^{-1}] \otimes V}. \tag{8}$$

Since $L^R \in \mathscr{Q}(R)$, by the definition of $\mathscr{Q}$, $\det(L^R) = R[[t]]$. From this and the choice of v_i satisfying (8), we easily get that

$$\beta^{L^R}(v_1 \wedge \ldots \wedge v_N) = 1, \tag{9}$$

where $\beta^{L^R}\colon \Lambda^N_{R[[t]]}(L^R) \to R((t))$ is the map as in (b) of Definition 1.3.6. Define $g_o \in \mathrm{SL}_N(R[t^{-1}])^-$ as follows:

$$g_o e_i = v_i, \quad \text{for all } 1 \le i \le N. \tag{10}$$

For any $u_1, \ldots, u_N \in R((t)) \otimes V$ and $g \in M_N(R((t)))$, we have

$$gu_1 \wedge \ldots \wedge gu_N = \det g \cdot (u_1 \wedge \ldots \wedge u_N). \tag{11}$$

Thus, from (9)–(11), we get that $\det(g_o) = 1$ and hence, from (8), we get that indeed $g_o \in \mathrm{SL}_N(R[t^{-1}])^-$. From the definition of $\mu_1(R)$ and g_o, we get that

$$\mu_1(R)(g_o) = \sum_{i=1}^{N} R[[t]]v_i \subset L^R.$$

By the isomorphism k,

$$L^R = tL^R + \sum_{i=1}^{N} R[[t]]v_i.$$

Hence, by the Nakayama lemma (Atiyah and Macdonald, 1969, Corollary 2.7), we get that $\sum_{i=1}^N R[[t]]v_i = L^R$. (We have used here that any maximal ideal of $R[[t]]$ contains $tR[[t]]$, cf. Exercise 1.3.E.13.) Thus, $\mu_1(R)(g_o) = L^R$. This proves the surjectivity of $\mu_1(R)$ onto $\mathscr{Q}(R)\backslash Z(\sigma_R)$, proving the proposition. □

Following Definition 1.3.10, consider the morphism of ind-schemes:

$$\pi : \overline{\mathrm{SL}}_N((t)) \to \bar{X}_{\mathrm{SL}_N}, \text{ induced from } g \mapsto g\bar{o}_R \text{ for } g \in \mathrm{SL}_N(R((t))),$$

where $\bar{o}_R$ is the base point of $\mathrm{SL}_N(R((t)))/\mathrm{SL}_N(R[[t]])$. Let $\tilde{Z}(\sigma)$ be the inverse image of the zero set $Z(\sigma)$ under the above morphism, where σ is the section of the line bundle $\mathscr{L}$ over $\bar{X}_{\mathrm{SL}_N}$ as in Definition 1.3.12.

Corollary 1.3.15 *The morphism*

$$\overline{\mu}\colon \overline{\mathrm{SL}}_N[t^{-1}]^- \times \overline{\mathrm{SL}}_N[[t]] \to \overline{\mathrm{SL}}_N((t)), \text{ induced from } (g,h) \mapsto gh$$

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for $g \in \mathrm{SL}_N(R[t^{-1}])^-$ *and* $h \in \mathrm{SL}_N(R[[t]])$*, is an isomorphism onto its image* $\overline{\mathrm{SL}}_N((t))\backslash\tilde{Z}(\sigma)$ $\big($*which is an open subset of* $\overline{\mathrm{SL}}_N((t))\big)$.

In particular, π *is a locally trivial principal* $\overline{\mathrm{SL}}_N[[t]]$*-bundle.*

Proof From the representability of the functor $\mathscr{Q}(R)\backslash Z(\sigma_R)$ by $\bar{X}_{\mathrm{SL}_N}\backslash Z(\sigma)$ (cf. Lemma 1.3.13), representability of $\mathrm{SL}_N(R((t)))$ by $\overline{\mathrm{SL}}_N((t))$ (cf. Lemma 1.3.2) and Exercise 1.3.E.6, we get that $\overline{\mathrm{SL}}_N((t))\backslash\tilde{Z}(\sigma)$ represents the functor

$$R \rightsquigarrow \mathrm{SL}_N(R((t)))\backslash\tilde{Z}(\sigma_R)$$
$$:= \left\{g \in \mathrm{SL}_N(R((t))) : i_{gL_o^R} : gL_o^R \to \frac{R((t)) \otimes V}{t^{-1}R[t^{-1}] \otimes V} \text{ is an isomorphism}\right\}. \tag{1}$$

So, to prove the corollary, it suffices to show (by Lemma 1.3.2 and Corollary 1.3.3(a)) that the map

$$\overline{\mu}(R)\colon\ \mathrm{SL}_N(R[t^{-1}])^- \times \mathrm{SL}_N(R[[t]]) \to \mathrm{SL}_N(R((t))),\ (g,h) \mapsto gh,$$

gives a bijection onto $\mathrm{SL}_N(R((t)))\backslash\tilde{Z}(\sigma_R)$.

From (1) and (5) of the proof of Proposition 1.3.14,

$$\mathrm{Im}\,(\overline{\mu}(R)) \subset \mathrm{SL}_N(R((t)))\backslash\tilde{Z}(\sigma_R).$$

Conversely, take $g' \in \mathrm{SL}_N(R((t)))\backslash\tilde{Z}(\sigma_R)$. By Proposition 1.3.14, there exists $g \in \mathrm{SL}_N(R[t^{-1}])^-$ such that $gL_o^R = g'L_o^R$. But the isotropy of L_o^R in $\mathrm{SL}_N(R((t)))$ is precisely equal to $\mathrm{SL}_N(R[[t]])$. Thus, $g' = g \cdot h$, for some $h \in \mathrm{SL}_N(R[[t]])$. Hence, $\overline{\mu}(R)$ has image precisely equal to $\mathrm{SL}_N(R((t)))\backslash\tilde{Z}(\sigma_R)$. It is easy to see that $\overline{\mu}(R)$ is injective. This proves the first part of the corollary.

Of course, the assertion that π is a locally trivial principal $\overline{\mathrm{SL}}_N[[t]]$-bundle follows from the first part and Proposition 1.3.14 once we prove that $V = \bar{X}_{\mathrm{SL}_N}$, where $V := \cup_{g\in\mathrm{SL}_N((t))}\, g\left(\bar{X}_{\mathrm{SL}_N} \setminus Z(\sigma)\right)$. But clearly $V(\mathbb{C}) = \bar{X}_{\mathrm{SL}_N}(\mathbb{C}) = \mathrm{SL}_N((t))/\,\mathrm{SL}_N[[t]]$. Since V and $\bar{X}_{\mathrm{SL}_N}$ have the same $\mathbb{C}$-points and V is open in $\bar{X}_{\mathrm{SL}_N}$, we get that $V = \bar{X}_{\mathrm{SL}_N}$ (since any ind-scheme of ind-finite type has nonempty set of $\mathbb{C}$-points). □

Now, we are ready to show that $\mathscr{X}_G$ is represented by an ind-scheme (cf. Definition 1.3.5). We first prove the following lemma.

Lemma 1.3.16 *For any connected reductive group* G*, the morphism*

$$\overline{\mu}_G\colon \bar{G}[t^{-1}]^- \times \bar{G}[[t]] \to \bar{G}((t)),\ \textit{induced from the morphism}\ (g,h) \mapsto gh,$$

for $g \in G(R[t^{-1}])^-$ *and* $h \in G(R[[t]])$*, is an isomorphism onto an open subset of* $\bar{G}((t))$.

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Proof Take a faithful representation $j\colon G \hookrightarrow \mathrm{SL}_N$. This gives rise to the commutative diagram

$$\begin{array}{ccc} \bar{G}[t^{-1}]^- \times \bar{G}[[t]] & \xrightarrow{\overline{\mu}_G} & \bar{G}((t)) \\ \big\downarrow j_1 & & \big\downarrow j_2 \\ \overline{\mathrm{SL}}_N[t^{-1}]^- \times \overline{\mathrm{SL}}_N[[t]] & \xrightarrow[\overline{\mu}]{} & \overline{\mathrm{SL}}_N((t)), \end{array} \qquad (\mathscr{D})$$

where $\overline{\mu}$ is as in Corollary 1.3.15 and the vertical maps are induced from the inclusion j. Let $\overline{\mathrm{SL}}_N((t))\backslash\tilde{Z}(\sigma)$ be the open subset of $\overline{\mathrm{SL}}_N((t))$ as in Corollary 1.3.15. Then we assert that

$$\mathrm{Im}\,(\overline{\mu}_G) = j_2^{-1}(\overline{\mathrm{SL}}_N((t))\backslash\tilde{Z}(\sigma)); \tag{1}$$

in particular, $j_2^{-1}\big(\overline{\mathrm{SL}}_N((t))\backslash\tilde{Z}(\sigma)\big)$ does not depend upon the choice of j. Moreover, we show that $\bar{\mu}_G$ is an isomorphism onto $\mathrm{Im}\,(\overline{\mu}_G)$.

By (1) of Corollary 1.3.15 and Exercise 1.3.E.6 applied to $\bar{G}((t)) \underset{\overline{\mathrm{SL}}_N((t))}{\times} \big(\overline{\mathrm{SL}}_N((t))\backslash\tilde{Z}(\sigma)\big)$, the functor

$$\begin{aligned} R \rightsquigarrow\ & G(R((t)))\backslash\tilde{Z}_G(\sigma_R) \\ := & \Big\{ g \in G(R((t))) : i_{j_2(g)L_o^R} : \\ & j_2(g)L_o^R \to \frac{R((t)) \otimes V}{t^{-1}R[t^{-1}] \otimes V} \quad \text{is an isomorphism} \Big\} \end{aligned}$$

represents the open subscheme $j_2^{-1}\ (\overline{\mathrm{SL}}_N((t))\backslash\tilde{Z}(\sigma))$ of the ind-scheme $\bar{G}((t))$.

Moreover, $G(R[t^{-1}])^-$ (resp. $G(R[[t]])$) represents $\bar{G}[t^{-1}]^-$ (resp. $\bar{G}[[t]]$) by Corollary 1.3.3(a) (resp. Lemma 1.3.2). Thus, to prove the lemma, it suffices to show that for any $R \in \mathbf{Alg}$,

$$\overline{\mu}_G(R)\colon G(R[t^{-1}])^- \times G(R[[t]]) \to G(R((t))), \ \ (g_R, h_R) \mapsto g_R \cdot h_R$$

is a bijection onto $G(R((t)))\backslash\tilde{Z}_G(\sigma_R)$.

We have the following commutative diagram $(\mathscr{D}_R)$ (analogue of the diagram $\mathscr{D}$ for any R):

$$\begin{array}{ccc} G(R[t^{-1}])^- \times G(R[[t]]) & \xrightarrow{\overline{\mu}_G(R)} & G(R((t))) \\ \big\downarrow j_1(R) & & \big\downarrow j_2(R) \\ \mathrm{SL}_N(R[t^{-1}])^- \times \mathrm{SL}_N(R[[t]]) & \xrightarrow[\overline{\mu}(R)]{} & \mathrm{SL}_N(R((t))), \end{array} \qquad (\mathscr{D}_R)$$

where $\overline{\mu}(R)$ is bijective onto its image $\mathrm{SL}_N(R((t)))\backslash \tilde{Z}(\sigma_R)$ (cf. Proof of Corollary 1.3.15). From this, the injectivity of $\overline{\mu}_G(R)$ follows as well as

$$\mathrm{Im}\,(\overline{\mu}_G(R)) \subset G(R((t)))\backslash \tilde{Z}_G(\sigma_R). \tag{2}$$

To prove the converse, take $x_R \in G(R((t)))\backslash \tilde{Z}_G(\sigma_R)$. Then, by the proof of Corollary 1.3.15, there exists $g_R \in \mathrm{SL}_N(R[t^{-1}])^-$, $h_R \in \mathrm{SL}_N(R[[t]])$ such that

$$g_R \cdot h_R = j_2(R)(x_R). \tag{3}$$

Choose a polynomial representation W over $\mathbb{C}$ of SL_N with a vector $w_o \in W$ such that the scheme-theoretic isotropy subgroup $(\mathrm{SL}_N)_{w_o}$ of w_o in SL_N is precisely equal to G (cf. (Borel, 1991, Chap. II, Theorem 5.1 and §5.5)). Then, for any $S \in \mathbf{Alg}$, by Exercise 1.3.E.6 applied to $\mathrm{SL}_N \underset{W}{\times} w_o$ (for the map $\mathrm{SL}_N \to W$, $g \mapsto gw_o$), we get that

$$G(S) \text{ is precisely the isotropy of } w_o \text{ in } \mathrm{SL}_N(S). \tag{4}$$

Evaluating identity (3) at $w_o \in R((t)) \otimes W$ and applying (4) for $S = R((t))$, we get

$$g_R^{-1}(w_o) = h_R(w_o). \tag{5}$$

But $g_R^{-1}(w_o) - w_o \in t^{-1}R[t^{-1}] \otimes W$ and $h_R(w_o) \in R[[t]] \otimes W$. Thus, from (5), we get that

$$h_R(w_o) = w_o = g_R(w_o). \tag{6}$$

Thus, from (4), we get that $(g_R, h_R) \in \mathrm{Im}\,(j_1(R))$. This proves, in view of (2),

$$\mathrm{Im}\,(\overline{\mu}_G(R)) = G(R((t)))\backslash \tilde{Z}_G(\sigma_R),$$

proving the lemma. □

Remark 1.3.17 Lemma 1.3.16 remains valid, more generally, for any closed subgroup $H \subset \mathrm{SL}_N$ (in lieu of G) such that H is the scheme-theoretic stabilizer of a vector w_o in a polynomial representation W of SL_N.

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Proposition 1.3.18 *[Realizing $\mathscr{X}_G$ as an ind-scheme] Let G be any group H as in the above remark.*

(a) The functor $\mathscr{X}_G$ as in Definition 1.3.5 is represented by an ind-scheme denoted $\bar{X}_G$ with $\mathbb{C}$-points X_G.

In fact, $\bar{X}_G$ is an ind-projective variety if G is a connected, semisimple algebraic group (cf. Corollary 1.3.19 and Theorem 1.3.23).

(b) The morphism $\bar{G}[t^{-1}]^- \to \bar{X}_G$ induced by the functor morphism

$$G(R[t^{-1}])^- \to \mathscr{X}_G(R),\ g \mapsto g\bar{o}_R,\ \textit{for } g \in G(R[t^{-1}])^-,$$

is an open embedding, where $\bar{o}_R$ is the base point of $G(R((t)))/G(R[[t]])$. Moreover, $\{g_o\bar{G}[t^{-1}]^- \cdot \bar{o}\}_{g_o\in G((t))}$ provides an open cover of the ind-scheme $\bar{X}_G$.

(c) We have a morphism

$$\bar{G}((t)) \times \bar{X}_G \to \bar{X}_G$$

induced from the morphism of functors

$$G(R((t))) \times \mathscr{X}^o_G(R) \to \mathscr{X}^o_G(R) \subset \mathscr{X}_G(R),$$
$$(g, h\bar{o}_R) \mapsto gh\bar{o}_R \textit{ for } g, h \in G(R((t))).$$

Proof (a) Consider the subfunctor $h_{\bar{G}[t^{-1}]^-} \hookrightarrow \mathscr{X}_G$ which takes (for any $\mathbb{C}$-algebra R) $g \in h_{\bar{G}[t^{-1}]^-}(R) = G(R[t^{-1}])^-$ to $g\bar{o}_R \in \mathscr{X}^o_G(R) = G(R((t)))/G(R[[t]]) \subset \mathscr{X}_G(R)$. Then, $h_{\bar{G}[t^{-1}]^-}$ is an open subfunctor of $\mathscr{X}_G$ (cf. Exercise 1.3.E.7). Thus, for any $g_o \in G((t))$,

$$h_{g_o\bar{G}[t^{-1}]^-} \hookrightarrow \mathscr{X}_G,\ \ g_o g \mapsto g_o g\bar{o}_R \ \text{ for } g \in G(R[t^{-1}])^-$$

is an open subfunctor.

We next claim that the collection of subfunctors $\{h_{g_o\bar{G}[t^{-1}]^-}\}_{g_o\in G((t))}$ is an open covering of $\mathscr{X}_G$. To prove this, in view of Eisenbud and Harris (2000, Exercise VI-11), it suffices to show that for any field $k \supset \mathbb{C}$,

$$\cup_{g_o\in G((t))}\, g_o G(k[t^{-1}])^-\bar{o}_k = \mathscr{X}_G(k) = \mathscr{X}^o_G(k) = G(k((t)))/G(k[[t]]), \tag{1}$$

where the second equality in the above equation follows since k is a field (cf. Exercise 1.3.E.7).

To prove (1), equivalently, we need to prove

$$\cup_{g_o\in G((t))}\, g_o G(k[t^{-1}])^- \cdot G(k[[t]]) = G(k((t))). \tag{2}$$

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For any $g_o \in G((t))$, by Lemma 1.3.16, the functor $R \rightsquigarrow g_o G(R[t^{-1}])^- \cdot G(R[[t]])$ is an open subfunctor of $G(R((t)))$ represented by an open ind-subscheme of $\bar{G}((t))$ with $\mathbb{C}$-points $g_o G[t^{-1}]^- \cdot G[[t]]$. Consider the sheafification $\mathcal{F}$ of the functor

$$\mathcal{F}^o\colon R \rightsquigarrow \cup_{g_o \in G((t))} g_o G(R[t^{-1}])^- \cdot G(R[[t]]) \subset G(R((t))).$$

Then $\mathcal{F}$ is an open subfunctor of $G(R((t)))$ represented by an open ind-subscheme denoted V of $\bar{G}((t))$ (cf. Lemma 1.3.2 and Definition B.5(b)) with $\mathbb{C}$-points

$$\cup_{g_o \in G((t))} g_o G[t^{-1}]^- \cdot G[[t]] = G((t)).$$

Also, $\bar{G}((t))$ has $\mathbb{C}$-points $G((t))$. So, both V and $\bar{G}((t))$ have the same set of $\mathbb{C}$-points and hence $V = \bar{G}((t))$ (since any closed ind-subscheme of $\bar{G}((t))$ has nonempty set of $\mathbb{C}$-points, cf. Exercise 1.3.E.8). Thus, for any $\mathbb{C}$-algebra R,

$$V(R) = \bar{G}((t))(R) = G(R((t))). \tag{3}$$

But, k being a field,

$$G(k((t))) = V(k) = \mathcal{F}(k) = \mathcal{F}^o(k) = \cup_{g_o \in G((t))} g_o G(k[t^{-1}])^- \cdot G(k[[t]]),$$

where the first equality follows from (3) and the third equality follows since k is a field. This proves (2) and hence (1).

Recall from Lemma 1.3.16 that there is an isomorphism

$$\overline{\mu}_G\colon \bar{G}[t^{-1}]^- \times \bar{G}[[t]] \to \mathring{V},\ (g,h) \mapsto gh,$$

where $\mathring{V}$ is an open subset of $\bar{G}((t))$. For any $g_o \in G((t))$ this gives rise to an isomorphism

$$\overline{\mu}_G(g_o)\colon (g_o\bar{G}[t^{-1}]^-) \times \bar{G}[[t]] \to g_o\mathring{V},\ (g_o g,h) \mapsto g_o gh.$$

For any $d \geq 0$, recall the closed subscheme $\bar{G}(t^{-d}\mathbb{C}[[t]])$ of $\bar{G}((t))$ from Definition 1.3.1. Then, there exists a closed (affine) subscheme $(g_o\bar{G}[t^{-1}]^-)_d$ of $g_o\bar{G}[t^{-1}]^-$ (of finite type over $\mathbb{C}$) such that $\overline{\mu}_G(g_o)$ restricts to an isomorphism

$$(\overline{\mu}_G(g_o))_d\colon (g_o\bar{G}[t^{-1}]^-)_d \times \bar{G}[[t]] \to (g_o\mathring{V}) \cap \bar{G}(t^{-d}\mathbb{C}[[t]]).$$

Consider the subfunctor $\mathscr{X}_G^d$ of $\mathscr{X}_G$ (cf. Definition 1.3.5) defined as the sheafification of the functor $R \rightsquigarrow G(t^{-d}R[[t]])/G(R[[t]])$ (cf. Lemma B.2).

Then, parallel to the above proof, we get that the collection of subfunctors $\{h_{(g_o\bar{G}[t^{-1}]^-)_d}\}_{g_o \in G((t))}$ is an open covering of $\mathscr{X}_G^d$. Thus, by Eisenbud and Harris (2000, Theorem VI-14) (since a Zariski cover is an fppf cover by Stacks (2019, Tag 021N)), the functor $\mathscr{X}_G^d$ is represented by a scheme denoted $\bar{X}_G^d$ with $\mathbb{C}$-points $X_G^d = G(t^{-d}\mathbb{C}[[t]])/G[[t]]$. Moreover, the morphism $i_d\colon \mathscr{X}_G^d \to \mathscr{X}_G^{d+1}$ induced from the inclusion gives rise to a morphism $\bar{i}_d\colon \bar{X}_G^d \to \bar{X}_G^{d+1}$. Since $\bar{G}(t^{-d}\mathbb{C}[[t]])$ is a closed subscheme of $\bar{G}(t^{-d-1}\mathbb{C}[[t]])$, we get that $(g_o\bar{G}[t^{-1}]^-)_d$ is a closed subscheme of $(g_o\bar{G}[t^{-1}]^-)_{d+1}$. Moreover, $\{(g_o\bar{G}[t^{-1}]^-)_d\}_{g_o\in G((t))}$ provides an open cover of $\bar{X}_G^d$ (see the proof of (b) below). Thus, the morphism $\bar{i}_d\colon \bar{X}_G^d \to \bar{X}_G^{d+1}$ is a closed embedding. Now, it is easy to see that the ind-scheme

$$\bar{X}_G := (\bar{X}_G^0 \subset \bar{X}_G^1 \subset \bar{X}_G^2 \subset \cdots)$$

represents the functor $\mathscr{X}_G$, once we observe that $\cup_{d\geq 0}\mathscr{X}_G^d(R) = \mathscr{X}_G(R)$ for any $\mathbb{C}$-algebra R. This proves (a).

An alternative proof of (a). Fix an embedding $G \subset \mathrm{SL}_N$. Then, by Beilinson and Drinfeld (1994, lemma after Theorem 4.5.1), the functor $\mathscr{X}_G$ is a closed subfunctor of $\mathscr{X}_{\mathrm{SL}_N}$ (for more details of the proof, see Zhu (2017, Proposition 1.2.6)[1], where the notion of a closed subfunctor is parallel to that of an open subfunctor as in Definition B.5(b). Thus, by Theorem 1.3.8, $\mathscr{X}_G$ is a representable functor represented by a closed ind-subscheme of $\bar{X}_{\mathrm{SL}_N}$.

(b) Since $h_{\bar{G}[t^{-1}]^-} \hookrightarrow \mathscr{X}_G$ is an open subfunctor (as observed above), we have that $\bar{G}[t^{-1}]^-\bar{o} \subset \bar{X}_G$ is an open ind-subscheme from the representability of $\mathscr{X}_G$ by $\bar{X}_G$ and Definition B.5(b).

To prove that $\{g_o\bar{G}[t^{-1}]^- \cdot \bar{o}\}_{g_o\in G((t))}$ is an open cover of the ind-scheme $\bar{X}_G$, observe that $U := \cup_{g_o\in G((t))}\, g_o\bar{G}[t^{-1}]^- \cdot \bar{o}$ is an open subset of $\bar{X}_G$, $\bar{X}_G$ is a closed ind-subscheme of $\bar{X}_{\mathrm{SL}_N}$ (cf. Exercise 1.3.E.9) and $\bar{X}_{\mathrm{SL}_N}$ is of ind-finite type (cf. Theorem 1.3.8). Thus, any closed ind-subscheme of $\bar{X}_G$ has nonempty set of $\mathbb{C}$-points. Moreover,

$$U(\mathbb{C}) = \cup_{g_o\in G((t))}\, g_oG[t^{-1}]^- \cdot \bar{o} = \bar{X}_G(\mathbb{C}) = G((t)) \cdot \bar{o}.$$

(c) To prove (c), simply observe that the sheafification of the functor $h_{\bar{G}((t))} \times \mathscr{X}_G^o$ is $h_{\bar{G}((t))} \times \mathscr{X}_G$, since $h_{\bar{G}((t))}$ is representable. □

By the above Proposition 1.3.18 and Lemma 1.3.16, the following corollary follows easily.

Corollary 1.3.19 *Let G be any group H as in Remark 1.3.17. The projection $\pi : \bar{G}((t)) \to \bar{X}_G$ is a locally trivial principal $\bar{G}[[t]]$-bundle.*

Further, for any faithful representation $j\colon G \hookrightarrow \mathrm{SL}_N$, $\bar{X}_G \hookrightarrow \bar{X}_{\mathrm{SL}_N}$ is a closed embedding (cf. Exercise 1.3.E.9). Thus, by Theorem 1.3.8, $\bar{X}_G$ is an ind-projective scheme.

[1] I thank X. Zhu for these two references.

Definition 1.3.20 An ind-scheme $X = (X_n)_{n\geq 0}$ is called *reduced* if there exists an equivalent filtration $(Y_n)_{n\geq 0}$ of X (i.e., $\mathrm{Id}\colon X \to X$ is an isomorphism of ind-schemes, where the two copies of X are equipped with the two ind-scheme structures induced from the filtrations X_n and Y_n) such that each Y_n is a reduced scheme.

Lemma 1.3.21 *If $X = (X_n)_{n\geq 0}$ is a reduced ind-scheme, then $(X_n^{\mathrm{red}})_{n\geq 0}$ provides an equivalent filtration of X, where X_n^{red} is the corresponding reduced scheme (cf. (Hartshorne, 1977, Chap. II, Exercise 2.3)). (Of course, as a topological space, $X_n^{\mathrm{red}} = X_n$.)*

Proof Since X is a reduced ind-scheme, for any $n \geq 0$, there exists $k(n) \geq 0$ such that $i_n\colon Y_n \to X_{k(n)}$ is a closed embedding. But, since Y_n is reduced, i_n factors through a closed embedding $\bar{i}_n\colon Y_n \to X_{k(n)}^{\mathrm{red}}$. Thus, the identity map $\mathrm{Id}\colon X \to X^{\mathrm{red}}$ is a morphism of ind-schemes, where X^{red} denotes the ind-scheme obtained from the filtration $(X_n^{\mathrm{red}})_{n\geq 0}$.

Conversely, the closed embedding $X_n^{\mathrm{red}} \to X_n$ clearly shows that $\mathrm{Id}\colon X^{\mathrm{red}} \to X$ is a morphism of ind-schemes. This proves the lemma. $\square$

The following theorem's proof was briefly outlined by G. Faltings (personal communication dated January 26, 2017). B. Conrad provided a detailed proof of the theorem given below (personal communication dated January 27, 2017).

Recall that the Lie algebra $\mathrm{Lie}\,\mathscr{G}$ of an ind-affine group scheme $\mathscr{G}$ is, by definition, the kernel of the group homomorphism $\mathscr{G}(\mathbb{C}(\epsilon)) \to \mathscr{G}(\mathbb{C})$ induced by $\epsilon \mapsto 0$, where $\mathbb{C}(\epsilon) := \mathbb{C}[\epsilon]/\langle \epsilon^2\rangle$. By Corollary B.21, $\mathrm{Lie}\,\mathscr{G}$ is a Lie algebra.

Let $\mathscr{G} = (\mathscr{G}_n)_{n\geq 0}$ be an ind-affine group scheme. Then $\mathscr{G}^{\mathrm{red}} := (\mathscr{G}_n^{\mathrm{red}})_{n\geq 0}$ is again an ind-affine group scheme. This follows since the multiplication map $\mathscr{G}_n \times \mathscr{G}_m \to \mathscr{G}_l$ clearly restricts to $\mathscr{G}_n^{\mathrm{red}} \times \mathscr{G}_m^{\mathrm{red}} \to \mathscr{G}_l^{\mathrm{red}}$ and so does the morphism induced from the inverse.

Theorem 1.3.22 *Let $\mathscr{G} = (\mathscr{G}_n)_{n\geq 0}$ be an ind-affine group scheme filtered by (affine) finite-type schemes over $\mathbb{C}$ and let $\mathscr{G}^{\mathrm{red}} = (\mathscr{G}_n^{\mathrm{red}})_{n\geq 0}$ be the associated reduced ind-affine group scheme. Assume that the canonical ind-group morphism $i\colon \mathscr{G}^{\mathrm{red}} \to \mathscr{G}$ induces an isomorphism $\dot{i}_e\colon Lie(\mathscr{G}^{\mathrm{red}}) \xrightarrow{\sim} Lie\,\mathscr{G}$ of the associated Lie algebras. Then, i is an isomorphism of ind-groups, i.e., $\mathscr{G}$ is a reduced ind-scheme.*

Proof Let $A := \mathbb{C}[\mathscr{G}]$ be the affine coordinate ring of $\mathscr{G}$, i.e., $A := \varprojlim A_n$ with inverse limit topology, where each $A_n := \mathbb{C}[\mathscr{G}_n]$ is given the discrete topology. For any $\mathbb{C}$-algebra B, the set of morphisms $\mathrm{Spec}(B) \to \mathscr{G}$ coincides exactly with the set of continuous $\mathbb{C}$-algebra homomorphisms $A \to B$, where B has discrete topology.

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Let $\hat{A}_n$ denote the formal completion of A_n at the identity e, so $\{\hat{A}_n\}_{n\geq 1}$ is an inverse system of complete local noetherian rings with surjective transition maps $\hat{A}_{n+1} \to \hat{A}_n$ with closed kernels for the max-adic topologies by the Artin–Rees Lemma (cf. (Eisenbud, 1995, Lemma 7.15)). Define

$$\hat{A} := \varprojlim \hat{A}_n \tag{1}$$

to be the topological inverse limit of $\hat{A}_n$ equipped with their max-adic topologies. Each $\hat{A}_n$ is itself a topological inverse limit of Artinian local $\mathbb{C}$-algebras, so the same goes for $\hat{A}$. Concretely, viewing the local Artinian algebra quotients of each A_n supported at e as local Artinian algebra quotients of A also, we see that $\hat{A}$ is (as a topological algebra) the inverse limit of all these local Artinian $\mathbb{C}$-algebras. This makes $\hat{A}$ into a (local) pseudo-compact $\mathbb{C}$-algebra. (For an introduction to pseudo-compact rings, we refer to Demazure and Grothendieck (1970, Exp. VII_B).) Recall that the class of pseudo-compact $\mathbb{C}$-algebras includes all the complete local noetherian $\mathbb{C}$-algebras with residue field $\mathbb{C}$ and is stable under arbitrary topological inverse limits. The most basic example of a non-noetherian local pseudo-compact $\mathbb{C}$-algebra is the topological ring $\mathbb{C}[[X_i]]_{i\in I}$ of formal power series over $\mathbb{C}$ in an arbitrary infinite set $\{X_i\}_{i\in I}$ of variables, realized as the completion of the polynomial ring $\mathbb{C}[X_i]$ with respect to the system of ideals $(X_j : j \in J)^N + (X_i : i \notin J)$, for finite subsets $J \subset I$ and integers $N \geq 1$.

The tangent space at a $\mathbb{C}$-point x of a pseudo-compact $\mathbb{C}$-algebra is, by definition, the topological $\mathbb{C}$-linear dual of $\mathfrak{m}/\overline{\mathfrak{m}^2}$, where $\mathfrak{m}$ is the (necessarily open) maximal ideal at x and $\overline{\mathfrak{m}^2}$ is the closure of $\mathfrak{m}^2$. Further, any pseudo-compact $\mathbb{C}$-algebra is determined (including its topology) by its functor of points on local Artinian $\mathbb{C}$-algebras (viewed discretely). The continuous maps from $\hat{A}$ (as in (1)) to any discrete $\mathbb{C}$-algebra must factor through one of the $\hat{A}_n$ (even through some local Artinian $\mathbb{C}$-algebra quotient of A_n).

Since $\mathscr{G}$ is an ind-affine group scheme, we get that $\hat{A}$ is a 'Hopf algebra' (in the weaker sense that the coproduct lands in a completed tensor product over $\mathbb{C}$), which makes $\hat{A}$ into a connected formal group. (For a discussion of formal groups, we refer to Fontaine (1977, Chap. I, §9) and also Demazure and Grothendieck (1970, Exp. VII_B, §3).) Now, over a field of characteristic 0, any connected formal group is necessarily of the form $\mathbb{C}[[X_i]]$ as an underlying pseudo-compact $\mathbb{C}$-algebra (cf. (Fontaine, 1977, Chap. I, §9.6)).

In exactly the same way

$$\hat{A}^{\mathrm{red}} := \varprojlim \hat{A}_n^{\mathrm{red}}$$

is a formal group, where $A_n^{\mathrm{red}} := \mathbb{C}[\mathscr{G}_n^{\mathrm{red}}]$ and $\hat{A}_n^{\mathrm{red}}$ is the completion of A_n^{red} at e.

The Lie algebra Lie$(\hat{A})$ of any formal group $\hat{A}$ over $\mathbb{C}$ is defined as the set of points valued in $\mathbb{C}(\epsilon)$ based at e. It is exactly the tangent space of $\hat{A}$ at e.

Assertion I: The canonical map Lie$(\hat{A}) \to$ *Lie*$(\mathscr{G})$ *is an isomorphism and a similar result for Lie*$(\mathscr{G}^{\text{red}})$. *Thus, the canonical map*

$$Lie(\hat{A}^{\text{red}}) \to Lie(\hat{A}) \tag{2}$$

is an isomorphism.

This follows since the definition of Lie$(\hat{A})$ as $\mathbb{C}(\epsilon)$-points of $\hat{A}$ based at e forces it to factor through a local Artinian algebra quotient of some A_n (based at e). The proof for Lie$(\mathscr{G}^{\text{red}})$ is identical. So, Assertion I follows from the assumption Lie$(\mathscr{G}^{\text{red}}) \xrightarrow{\sim}$ Lie $\mathscr{G}$.

Assertion II: The canonical map $\hat{i}\colon \hat{A} \to \hat{A}^{\text{red}}$ *is surjective.*

We first recall the following general result:

Let $\xi\colon R' \to R$ *be a continuous homomorphism between pseudo-compact rings such that for every open ideal* J *of* R *(thus* R/J *is an Artinian ring) the map* $R' \to R/J$ *is surjective. Thus, the preimage* J' *of* J *in* R' *is an open ideal with* $R'/J' \simeq R/J$. *The map* $\xi\colon R' \to R$ *is then identified with the map*

$$\bar{\xi}\colon R' \to \varprojlim_J R'/J' \simeq \varprojlim_J R/J \simeq R,$$

for J *varying through the full family of open ideals of* R. *Then,* $\bar{\xi}$ *is surjective and hence so is* ξ *(cf. (Demazure and Grothendieck, 1970, Exp. VII*$_B$*, Cor. 0.2D(ii)(a))).*

We now come to the proof of Assertion II. By the above result, it suffices to show that for any open ideal J of $\hat{A}^{\text{red}}$, the map $\hat{i}_J\colon \hat{A} \to \hat{A}^{\text{red}}/J$ is surjective, where $\hat{i}_J$ is the map $\hat{i}$ followed by the projection $\hat{A}^{\text{red}} \to \hat{A}^{\text{red}}/J$. But, J being an open ideal, clearly

$$\hat{A}^{\text{red}}/J \simeq \mathbb{C}[\mathscr{G}_n^{\text{red}}]/J_n, \ \text{ for some } n \geq 1 \text{ and some ideal } J_n \text{ of } \mathbb{C}[\mathscr{G}_n^{\text{red}}].$$

Of course, the canonical map $\mathbb{C}[\mathscr{G}_n] \to \mathbb{C}[\mathscr{G}_n^{\text{red}}]/J_n$ is surjective and so is $A \to \mathbb{C}[\mathscr{G}_n]$. Hence, the canonical map $A \to \mathbb{C}[\mathscr{G}_n^{\text{red}}]/J_n$ is surjective. But, since $\hat{i}$ is a continuous map and J is an open ideal, we get that $\hat{i}_j$ is surjective, proving Assertion II.

Assertion III: The canonical map $\hat{i}\colon \hat{A} \to \hat{A}^{\text{red}}$ *is an isomorphism.*

We first recall the following (simple) general result obtained using the isomorphism of R below with $\mathbb{C}[[X_i]]$ and similarly for R'.

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Let R and R' be two connected formal groups over $\mathbb{C}$ and $f: R \to R'$ a continuous surjective homomorphism respecting augmentations to $\mathbb{C}$. Then, f is an isomorphism (necessarily a topological isomorphism) iff the induced map between Lie algebras is a bijection.

Applying the above result to $\hat{i}$ and using Assertions I and II, we get Assertion III.

Assertion IV: For a complete local $\mathbb{C}$-algebra B with residue field $\mathbb{C}$, the canonical map $i: \mathscr{G}^{\text{red}} \to \mathscr{G}$ induces a bijection $i_B: \mathscr{G}^{\text{red}}(B) \simeq \mathscr{G}(B)$.

Let B be any (not necessarily noetherian) $\mathbb{C}$-algebra. By the definition of a morphism to an ind-scheme, any morphism $\varphi\colon$ Spec $B \to \mathscr{G}$ lands inside $\mathscr{G}_n$ (for some $n \geq 1$). From this, we see that $i_B: \mathscr{G}^{\text{red}}(B) \to \mathscr{G}(B)$ is injective (for any B). So, it suffices to prove that for any B as in Assertion IV, i_B is surjective.

Take $g \in \mathscr{G}(B)$. Then, it is represented by an algebra homomorphism $\bar{g}_n: A_n \to B$, for some $n \geq 1$. Since B is a complete local $\mathbb{C}$-algebra with residue field $\mathbb{C}$, $\bar{g}_n$ induces a continuous homomorphism $\hat{g}_n: \hat{A}_n(x) \to B$ and hence a continuous homomorphism $\hat{g}\colon \hat{A}(x) \to B$, where $\hat{A}_n(x)$ denotes the completion of A_n with respect to some $\mathbb{C}$-point x (not necessarily e) of $\mathscr{G}_n$ and $\hat{A}(x)$ is the inverse limit of $\{\hat{A}_n(x)\}_n$. But, by Assertion III, $\hat{i}\colon \hat{A} \to \hat{A}^{\text{red}}$ is an isomorphism (and hence so is $\hat{i}(x)\colon \hat{A}(x) \simeq \hat{A}^{\text{red}}(x)$ by translation using $\mathscr{G}(\mathbb{C}) = \mathscr{G}^{\text{red}}(\mathbb{C})$). Thus, we get a continuous homomorphism

$$\hat{g}^{\text{red}} := \hat{g} \circ (\hat{i}(x))^{-1}\colon \hat{A}^{\text{red}}(x) \to B.$$

Composing $\hat{g}^{\text{red}}$ with the canonical $\mathbb{C}$-algebra homomorphism $A^{\text{red}} \to \hat{A}^{\text{red}}(x)$, we get a $\mathbb{C}$-algebra homomorphism $A^{\text{red}} \to B$. This provides the desired lift of g in $\mathscr{G}^{\text{red}}(B)$. Thus, $\mathscr{G}^{\text{red}}(B) = \mathscr{G}(B)$, proving Assertion IV.

Assertion V: For any local noetherian $\mathbb{C}$-algebra B with residue field $\mathbb{C}$, the canonical map $i: \mathscr{G}^{\text{red}} \to \mathscr{G}$ induces a bijection $i_B: \mathscr{G}^{\text{red}}(B) \simeq \mathscr{G}(B)$.

As observed in the proof of Assertion IV, it suffices to prove that $\mathscr{G}^{\text{red}}(B) \to \mathscr{G}(B)$ is surjective. Since B is noetherian, $B \to \hat{B}$ is injective, where $\hat{B}$ is the completion of B with respect to its unique maximal ideal. Take $g \in \mathscr{G}(B)$ and represent it as $\bar{g}_n\colon A_n \to B$ (for some n). By Assertion IV, there exists $N \geq n$ such that $\bar{g}_N: A_N \to B \subset \hat{B}$ (obtained from the composition of $\bar{g}_n$ with the canonical map $A_N \to A_n$) descends to $\hat{g}_N\colon A_N^{\text{red}} \to \hat{B}$. But, since $A_N \to A_N^{\text{red}}$ is surjective, we get that $\hat{g}_N(A_N^{\text{red}}) \subset B$. This proves Assertion V.

With these preparations, we finally come to the proof of the theorem. We need to show that, for any $\mathbb{C}$-algebra B, the canonical map $i: \mathscr{G}^{\text{red}} \to \mathscr{G}$

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induces a bijection $i_B : \mathscr{G}^{\text{red}}(B) \simeq \mathscr{G}(B)$. As observed earlier, i_B is injective. So, we only need to prove the surjectivity of i_B.

Take $g \in \mathscr{G}(B)$. Then, as shown earlier, $g \in \mathscr{G}_n(B)$ for some $n \geq 1$. Since $\mathscr{G}_n$ is a scheme of finite type over $\mathbb{C}$ (by assumption), we can assume that B is a $\mathbb{C}$-algebra of finite type over $\mathbb{C}$. Let $\mathfrak{m}$ be a maximal ideal of B (so $B/\mathfrak{m} = \mathbb{C}$) and let $B_\mathfrak{m}$ be the localization. Then, by Assertion V, we can find $n(\mathfrak{m}) \geq 1$ and

$$g'_\mathfrak{m} \in \mathscr{G}^{\text{red}}_{n(\mathfrak{m})}(B_\mathfrak{m}) = \text{Mor}\left(\mathbb{C}[\mathscr{G}^{\text{red}}_{n(\mathfrak{m})}], B_\mathfrak{m}\right) \simeq \text{Mor}\left(\text{Spec}(B_\mathfrak{m}), \mathscr{G}^{\text{red}}_{n(\mathfrak{m})}\right)$$

such that

$$i_{B_\mathfrak{m}}(g'_\mathfrak{m}) = g_\mathfrak{m}, \tag{3}$$

where $g_\mathfrak{m}$ is the element of $\mathscr{G}_{n(\mathfrak{m})}(B_\mathfrak{m})$ corresponding to g. Since $i_{B_\mathfrak{m}}$ is injective, $g'_\mathfrak{m}$ is unique, satisfying (3). Further, since $\mathscr{G}^{\text{red}}_{n(\mathfrak{m})}$ is of finite type over $\mathbb{C}$, there exists an affine open set $U_\mathfrak{m} \subset \text{Spec}(B)$ containing the point $\mathfrak{m}$ such that $g'_\mathfrak{m}$ spreads out to

$$g'_{U_\mathfrak{m}} \in \mathscr{G}^{\text{red}}_{n(\mathfrak{m})}(\mathbb{C}[U_\mathfrak{m}]).$$

By the injectivity of $i_{\mathbb{C}[U_\mathfrak{m}]}$, we get the following analogue of (3) (possibly after suitably shrinking $U_\mathfrak{m}$ around $\mathfrak{m}$):

$$i_{\mathbb{C}[U_\mathfrak{m}]}(g'_{U_\mathfrak{m}}) = g_{U_\mathfrak{m}}, \tag{4}$$

where $g_{U_\mathfrak{m}}$ is the element of $\mathscr{G}_{n(\mathfrak{m})}(\mathbb{C}[U_\mathfrak{m}])$ corresponding to g. As $\mathfrak{m}$ runs over the maximal ideals of B, $\{U_\mathfrak{m}\}$ clearly covers $\text{Spec}(B)$. Choose a finite subcover $\{U_{\mathfrak{m}_i}\}_{1 \leq i \leq k}$ of $\text{Spec}(B)$ and let $N := \max_i \{n(\mathfrak{m}_i)\}$. From the uniqueness of $g'_{U_{\mathfrak{m}_i}}$ satisfying (4), we get that $g'_{U_{\mathfrak{m}_i}} = g'_{U_{\mathfrak{m}_j}}$ on $U_{\mathfrak{m}_i} \cap U_{\mathfrak{m}_j}$. Thus, we get the element $g' \in \mathscr{G}^{\text{red}}_N(B)$ such that $g'_{|U_{\mathfrak{m}_i}} = g'_{U_{\mathfrak{m}_i}}$ on $U_{\mathfrak{m}_i}$. From this and (3), we get $i_B(g') = g$. This proves the surjectivity of i_B and hence the theorem is proved. □

As a consequence of the above theorem, we deduce the following result.

Theorem 1.3.23 *Let G be a connected semisimple algebraic group. Then the ind-affine group scheme $\bar{G}[t^{-1}]$ is reduced and hence so is $\bar{G}[t^{-1}]^-$.*

Thus, the infinite Grassmannian $\bar{X}_G$ is a reduced ind-scheme.

Proof We first show that $\mathscr{G} := \bar{G}[t^{-1}]$ is reduced. By Theorem 1.3.22, it suffices to show that

$$\text{Lie}(\mathscr{G}^{\text{red}}) = \text{Lie}(\mathscr{G}). \tag{1}$$

Take an embedding $G \hookrightarrow \text{SL}_N \subset M_N$. This gives rise to a closed embedding of groups: $\mathscr{G} \subset \overline{\text{SL}}_N[t^{-1}]$ (cf. Lemma 1.3.2).

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In particular,

$$\mathrm{Lie}(\mathscr{G}) \subset \mathrm{Lie}(\overline{\mathrm{SL}}_N[t^{-1}]) = sl_N \otimes \mathbb{C}[t^{-1}],$$

where for the last equality, see Exercise 1.3.E.12. Considering the evaluation homomorphisms for any $\alpha \in \mathbb{P}^1(\mathbb{C})\backslash\{0\}$ (induced from the $\mathbb{C}$-algebra homomorphisms $R[t^{-1}] \to R$, $t \mapsto \alpha$), $\epsilon(\alpha)\colon \mathscr{G} \to G$ and $\overline{\epsilon}(\alpha)\colon \overline{\mathrm{SL}}_N[t^{-1}] \to \mathrm{SL}_N$, it is easy to see (using Exercise 1.3.E.12 again) that

$$\mathrm{Lie}(\mathscr{G}) \subset \mathfrak{g} \otimes \mathbb{C}[t^{-1}], \quad \text{where} \quad \mathfrak{g} := \mathrm{Lie}\, G. \tag{2}$$

For any root $\beta \in \Delta$ of $\mathfrak{g}$, consider the root subgroup $U_\beta \subset G$ with Lie algebra the root space $\mathfrak{g}_\beta$ (cf. (Jantzen, 2003, Part II, §1.2)). This gives rise to a closed embedding of ind-affine group schemes $j\colon \bar{U}_\beta[t^{-1}] \hookrightarrow \mathscr{G}$. Since $U_\beta \simeq \mathbb{A}^1$, clearly, $\bar{U}_\beta[t^{-1}]$ is reduced. In particular, the embedding $j\colon \bar{U}_\beta[t^{-1}] \hookrightarrow \mathscr{G}$ factors through $\bar{U}_\beta[t^{-1}] \hookrightarrow \mathscr{G}^{\mathrm{red}}$. Further, under the differential $\dot{j}$, similar to (2), we get

$$\mathrm{Lie}(\bar{U}_\beta[t^{-1}]) \subset \mathfrak{g}_\beta \otimes \mathbb{C}[t^{-1}]. \tag{3}$$

In fact, identifying the group U_β with the additive group $G_a \simeq \mathbb{C}$, it is easy to see that

$$\mathrm{Lie}(\bar{U}_\beta[t^{-1}]) = \mathfrak{g}_\beta \otimes \mathbb{C}[t^{-1}]. \tag{4}$$

Thus,

$$\mathrm{Lie}(\mathscr{G}^{\mathrm{red}}) \supset \sum_{\beta \in \Delta} (\mathfrak{g}_\beta \otimes \mathbb{C}[t^{-1}]).$$

But, since $\mathrm{Lie}(\mathscr{G}^{\mathrm{red}})$ is a Lie algebra which is a Lie subalgebra of $sl_N \otimes \mathbb{C}[t^{-1}]$ under the standard bracket as in Exercise 1.3.E.12 and $\sum_{\beta\in\Delta} \mathfrak{g}_\beta$ generates the Lie algebra $\mathfrak{g}$ (this is where we have used the assumption that G is semisimple), we get that

$$\mathrm{Lie}(\mathscr{G}^{\mathrm{red}}) \supset \mathfrak{g} \otimes \mathbb{C}[t^{-1}]. \tag{5}$$

Combining (2) and (5), we get

$$\mathrm{Lie}(\mathscr{G}^{\mathrm{red}}) = \mathrm{Lie}(\mathscr{G}) = \mathfrak{g} \otimes \mathbb{C}[t^{-1}].$$

This proves (1) and hence $\mathscr{G}$ is reduced by Theorem 1.3.22.

The evaluation $\epsilon(\infty)\colon \mathscr{G} \to G$ admits a group splitting obtained from the inclusion $G \to \mathscr{G}$ (which is induced from the $\mathbb{C}$-algebra homomorphism $R \hookrightarrow R[t^{-1}]$). This gives rise to an isomorphism of ind-schemes (cf. Corollary 1.3.3(a)):

$$\mathscr{G} \simeq \mathscr{G}^- \times G, \quad \text{where} \quad \mathscr{G}^- := \bar{G}[t^{-1}]^-. \tag{6}$$

Now, since $\mathscr{G}$ is reduced, so is $\mathscr{G}^-$.

Finally, by Proposition 1.3.18, the infinite Grassmannian $\bar{X}_G$ has an open cover isomorphic with the ind-scheme $\mathscr{G}^-$. Thus, $\bar{X}_G$ is a reduced ind-scheme. □

Recall the ind-projective variety X^r_G (in particular, reduced) with closed points X_G from Kumar (2002, §§13.2.12, 13.2.13 and 13.2.15). Also, recall that the structure X^r_G coincides with the ind-variety structure obtained via the representation theory (cf. (Kumar, 2002, Proposition 13.2.18)).

Proposition 1.3.24 *Let G be a connected, simply-connected simple algebraic group. Then the ind-scheme $\bar{X}_G$ as in Proposition 1.3.18 coincides with the ind-projective variety X^r_G.*

Proof We first prove the proposition for $G = \mathrm{SL}_N$. In this case, following the notation as in Theorem 1.3.8, by definition $\bar{X}_{\mathrm{SL}_N}$ is the ind-scheme given by the filtration $(\bar{H}_n)_{n\geq 0}$. By Theorem 1.3.23 and Lemma 1.3.21, since $\bar{X}_{\mathrm{SL}_N}$ is reduced, $(\bar{H}_n^{\mathrm{red}})_{n\geq 0}$ gives an equivalent filtration with $\bar{H}_n(\mathbb{C}) = \bar{H}_n^{\mathrm{red}}(\mathbb{C}) = Q_n^N$. Hence, $\bar{X}_{\mathrm{SL}_N}$ is an ind-variety. By Theorem 1.3.8 and Kumar (2002, §13.2.13), we get that $\bar{X}_{\mathrm{SL}_N}$ coincides with $X^r_{\mathrm{SL}_N}$.

We now come to the general G. Fix an embedding $G \hookrightarrow \mathrm{SL}_N$. This induces closed embeddings (cf. Corollary 1.3.19 for $\bar{X}_G$ and Kumar (2002, §13.2.15) for X^r_G):

$$\bar{X}_G \hookrightarrow \bar{X}_{\mathrm{SL}_N} \quad \text{and} \quad X^r_G \hookrightarrow X^r_{\mathrm{SL}_N}.$$

Now, since both $\bar{X}_G$ and X^r_G are reduced, by Kumar (2002, Lemma 4.1.2), we see that the identity map $\mathrm{Id}\colon \bar{X}_G \to X^r_G$ is an isomorphism of ind-varieties. □

Unlike noetherian group schemes over $\mathbb{C}$ (which are always reduced by a result due to Cartier), ind-affine group schemes over $\mathbb{C}$ are, in general, not reduced.

Example 1.3.25 The ind-affine group scheme $\mathscr{H} := \bar{H}[t]$ is *not* reduced for $H = \mathbb{C}^*$.

Consider the embedding

$$\mathbb{C}^* \hookrightarrow \mathrm{SL}_2,\ z \mapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}.$$

Then the set of $\mathbb{C}$-points of $\mathscr{H}$ is given by

$$\left\{ \begin{pmatrix} P(t) & 0 \\ 0 & Q(t) \end{pmatrix} : P(t), Q(t) \in \mathbb{C}[t] \quad \text{and} \quad PQ = 1 \right\} \simeq \mathbb{C}^*.$$

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Since the set of $\mathbb{C}$-points of $\mathscr{H}$ coincides with that of $\mathscr{H}^{\text{red}}$, we see that the ind-variety

$$\mathscr{H}^{\text{red}} \simeq \mathbb{C}^*.$$

In particular, for any $R \in \mathbf{Alg}$,

$$\begin{aligned} \operatorname{Mor}(\operatorname{Spec} R, \mathscr{H}^{\text{red}}) &= \mathbb{C}^*(R) \\ &\simeq \text{set of invertible elements in } R. \end{aligned} \tag{1}$$

On the other hand, by Lemma 1.3.2,

$$\begin{aligned} \operatorname{Mor}(\operatorname{Spec} R, \mathscr{H}) &\simeq \operatorname{Hom}_{\text{alg}}(\mathbb{C}[x, x^{-1}], R[t]) \\ &\simeq \text{set of invertible elements in } R[t]. \end{aligned} \tag{2}$$

If R has a nonzero nilpotent element a, then $1 - at \in R[t]$ is invertible. Thus, by comparing (1) and (2), we get that

$$\operatorname{Mor}(\operatorname{Spec} R, \mathscr{H}^{\text{red}}) \subsetneq \operatorname{Mor}(\operatorname{Spec} R, \mathscr{H}).$$

This shows that $\mathscr{H}$ is not reduced. Thus, the infinite Grassmannian $\bar{X}_{\mathbb{C}^*}$ is not reduced (cf. Proposition 1.3.18).

Remark 1.3.26 (a) Similar to the above example, one can see that for any algebraic group H with a surjective algebraic group homomorphism $H \to \mathbb{C}^*$, $\bar{H}[t]$ is *not* reduced.

(b) Any (not necessarily noetherian) affine group scheme $\mathscr{G}$ over a field of characteristic 0 is reduced (extension of Cartier's result to non-noetherian group schemes). We refer to (Oort, 1966) in combination with (Waterhouse, 1979, §3.3)[2] for a short proof.

In particular, for any affine algebraic group H, the affine group scheme $\bar{H}[[t]]$ (cf. Lemma 1.3.2) is reduced.

Thus, combining Theorem 1.3.23 with Lemma 1.3.16, we get that for any connected semisimple group G, the ind-affine group scheme $\bar{G}((t))$ is reduced.

1.3.E Exercises

(1) Let $R \in \mathbf{Alg}$. For any positive integer N, let $L_o^R = L_o^R(N) := R[[t]] \otimes_{\mathbb{C}} V$ be as in Definition 1.3.6, where $V = \mathbb{C}^N$. Let L^R be an $R[[t]]$-submodule of $R((t)) \otimes_{\mathbb{C}} V$ satisfying

$$t^n L_o^R \subset L^R \subset t^{-n} L_o^R, \quad \text{for some } n \geq 0.$$

[2] We thank B. Conrad for providing this reference.

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Then show that L^R is an $R[[t]]$-projective module if and only if $\tilde{L}^R := L^R/t^n L_o^R$ is an R-module direct summand of the R-module $t^{-n}L_o^R/t^n L_o^R$.

(2) Let $R \in \mathbf{Alg}$ and let L^R be a projective $R[[t]]$-submodule of $R((t)) \otimes_{\mathbb{C}} V$ satisfying

$$t^n L_o^R \subset L^R \subset t^{-n} L_o^R, \quad \text{for some } n \geq 0.$$

Then show that for any $\mathbb{C}$-algebra homomorphism $R \to R'$, the above inclusions induce the inclusions

$$t^n L_o^{R'} \subset L^{R'} \subset t^{-n} L_o^{R'}, \;\text{ where } L^{R'} := R'[[t]] \otimes_{R[[t]]} L^R.$$

Show further that if $L^R/t^n L_o^R$ is a direct summand of $t^{-n}L_o^R/t^n L_o^R$ as R-modules, then

$$R' \otimes_R \frac{L^R}{t^n L_o^R} \xrightarrow{\sim} \frac{L^{R'}}{t^n L_o^{R'}}.$$

(3) Let X be a scheme and let f be an automorphism of X. Then show that the fixed subscheme X^f (which is defined as the scheme-theoretic inverse image of the diagonal under the morphism $\theta_f : X \to X \times X$, $x \mapsto (x, f(x))$) represents the functor $\mathscr{X}^f : \mathbf{Alg} \to \mathbf{Set}$ defined by

$$\mathscr{X}^f(R) = X(R)^{f_R},$$

where f_R is the induced automorphism of $X(R) := \mathrm{Mor}(\mathrm{Spec}\, R, X)$.

(4) Let S be a noetherian algebra over $\mathbb{C}$ and let P be a finitely generated projective $S[[t]]$-module. Show that there exists an affine open cover $\{\mathrm{Spec}(S_i)\}_i$ of the scheme $\mathrm{Spec}(S)$ such that $S_i[[t]] \otimes_{S[[t]]} P$ is a free $S_i[[t]]$-module, for each i.

Hint (due to N. Mohan Kumar): Consider the projective S-module $P_o := P/tP$. Then show that $P \simeq \hat{P}_o$, where $\hat{P}_o := P_o \otimes_S S[[t]]$. To show this, using the projectivity of P and $\hat{P}_o$ as $S[[t]]$-modules, get an $S[[t]]$-module lift $\theta : \hat{P}_o \to P$ of the S-module isomorphism $\hat{P}_o/t\hat{P}_o \simeq P/tP$. Prove that θ is an isomorphism by observing that a finitely generated module over $S[[t]]$ is complete with respect to t and is zero if it is zero mod t.

(5) Show that the special lattice functor $\mathscr{Q} = \mathscr{Q}^N$ as in Definition 1.3.6 is the sheafification (cf. Lemma B.2) of the functor $R \rightsquigarrow \mathrm{SL}_N(R((t)))/\mathrm{SL}_N(R[[t]])$.

Hint: Use the fact proved in Proposition 1.3.14 that for $g \in \mathrm{SL}_N(R((t)))$ and $L^R \in \mathscr{Q}^N(R)$, $gL^R \in \mathscr{Q}^N(R)$, i.e., gL^R satisfies properties (a) and (b) of Definition 1.3.6 for some $n \geq 0$.

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(6) Let $f: X \to Z$ and $g: Y \to Z$ be two morphisms of ind-schemes. Then their fiber product $X \underset{Z}{\times} Y$ represents the functor

$$R \rightsquigarrow X(R) \underset{Z(R)}{\times} Y(R),$$

where

$$X(R) \underset{Z(R)}{\times} Y(R) := \{(x, y) \in X(R) \times Y(R) : \; f_R(x) = g_R(y)\}$$

and $f_R: X(R) \to Z(R)$ is the map induced from f.

(7) Show that the functor $h_{\bar{G}[t^{-1}]^-}$ as in the proof of Proposition 1.3.18 is an open subfunctor of $\mathscr{X}_G$.

Moreover, show that $\mathscr{X}_G(k) = \mathscr{X}^o_G(k)$ for any field $k \supset \mathbb{C}$, where the functors $\mathscr{X}_G$ and $\mathscr{X}^o_G$ are defined in Definition 1.3.5.

(8) Let R be a $\mathbb{C}$-algebra generated (as a $\mathbb{C}$-algebra) by countably many elements. Then show that for any maximal ideal $\mathfrak{m}$ of R, $R/\mathfrak{m} \simeq \mathbb{C}$ (as $\mathbb{C}$-algebras).

In particular, any closed ind-subscheme of $\bar{G}((t))$ (for any affine algebraic group G) has nonempty set of $\mathbb{C}$-points.

Hint: We can assume that $R = \mathbb{C}[x_1, x_2, x_3, \dots]$. Now, $R/\mathfrak{m}$ is a field extension k of $\mathbb{C}$. In particular, $\mathbb{C}$ being algebraically closed, $k \supset \mathbb{C}(x)$, where $\mathbb{C}(x)$ is the quotient field of the polynomial ring $\mathbb{C}[x]$. Show that $\mathbb{C}(x)$ as a vector space over $\mathbb{C}$ is of uncountable dimension, whereas clearly R and hence $R/\mathfrak{m}$ is of countable dimension over $\mathbb{C}$.

(9) Show that for any G as H in Remark 1.3.17, the canonical map $\bar{X}_G \hookrightarrow \bar{X}_{\mathrm{SL}_N}$ (induced by an embedding $G \hookrightarrow \mathrm{SL}_N$) is a closed embedding.

(10) For any integer $n \geq 1$, consider the covariant group functor $\mathscr{F}_n$ from **Alg** to **Set** defined by

$$\mathscr{F}_n(R) = G\left(\frac{R[[t]]}{\langle t^n \rangle}\right),$$

where G is an affine algebraic group. Then show that $\mathscr{F}_n$ is a representable functor represented by an affine group scheme of finite type over $\mathbb{C}$ (i.e., an affine algebraic group) $\bar{G}\left(\frac{\mathbb{C}[[t]]}{\langle t^n \rangle}\right)$ with $\mathbb{C}$-points $G\left(\frac{\mathbb{C}[[t]]}{\langle t^n \rangle}\right)$. Since it is a variety, we denote it by $G\left(\frac{\mathbb{C}[[t]]}{\langle t^n \rangle}\right)$ itself.

Hint: Follow the proof of Lemma 1.3.2.

(11) Let G be a connected reductive group and let $P \subset G$ be a parabolic subgroup. Define the (parahoric) closed subgroup scheme $\mathcal{P} \subset \bar{G}((t))$ by $\mathcal{P} := ev_0^{-1}(P)$, under the evaluation map $ev_0: \bar{G}[[t]] \to G$ at $t = 0$

equipped with the scheme-theoretic inverse image structure. Then $\mathcal{P}$ is reduced (cf. Remark 1.3.26(b)). Moreover, by Exercise 6 and Lemma 1.3.2, $\mathcal{P}$ represents the functor $R \rightsquigarrow \tilde{\mathcal{P}}(R)$, where $\tilde{\mathcal{P}}(R) := (ev_0^R)^{-1}(P(R))$, under the evaluation map $ev_0^R : G(R[[t]]) \to G(R)$ at $t = 0$.

Consider the functor

$$\mathbf{Alg} \to \mathbf{Set}, \quad R \rightsquigarrow G(R((t)))/\tilde{\mathcal{P}}(R).$$

Show that its sheafification (cf. Lemma B.2) is a representable functor represented by an ind-projective scheme denoted $\bar{X}_G(P)$ (with $\mathbb{C}$-points $G((t))/\tilde{\mathcal{P}}(\mathbb{C})$). Moreover, show that this ind-scheme is, in fact, an ind-variety if G is semisimple.

Show further that for any connected reductive group G,

$$\bar{G}((t)) \to \bar{X}_G(P)$$

is a locally trivial principal $\mathcal{P}$-bundle and

$$\bar{X}_G(P) \to \bar{X}_G$$

is a locally trivial G/P-fibration, where the ind-scheme $\bar{X}_G$ is as in Proposition 1.3.18.

(12) Show that the Lie algebra Lie $(\overline{\mathrm{SL}}_N[t^{-1}])$ of the ind-affine group scheme $\overline{\mathrm{SL}}_N[t^{-1}]$ is isomorphic with the Lie algebra $sl_N \otimes \mathbb{C}[t^{-1}]$ under the bracket $[x \otimes P, y \otimes Q] = [x, y] \otimes PQ$, for $x, y \in sl_N$ and P, $Q \in \mathbb{C}[t^{-1}]$.

Moreover, the evaluation homomorphism $\epsilon(\alpha) : \overline{\mathrm{SL}}_N[t^{-1}] \to \mathrm{SL}_N$, induced from $t \mapsto \alpha$ for any $\alpha \in \mathbb{P}^1(\mathbb{C})\backslash\{0\}$, induces the Lie algebra homomorphism (cf. Lemma B.19):

$$\dot{\epsilon}(\alpha)_1 : sl_N \otimes \mathbb{C}[t^{-1}] \to sl_N,\ x \otimes P \mapsto P(\alpha)x,$$
$$\text{for } x \in sl_N \text{ and } P \in \mathbb{C}[t^{-1}].$$

(13) For any algebra $R \in \mathbf{Alg}$, any maximal ideal of $R[[t]]$ contains $t\ R[[t]]$.

1.4 Central Extension of Loop Groups

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and let G be the connected, simply-connected complex algebraic group with Lie algebra $\mathfrak{g}$. For $\lambda_c \in \hat{D}$, let $\mathscr{H}(\lambda_c)$ be the integrable highest-weight $\hat{\mathfrak{g}}$-module with highest weight λ_c (cf. Theorem 1.2.10). Recall the definition of $\mathbb{C}$-space and $\mathbb{C}$-group functors from Definition B.1.

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1.4.1

Consider the $\mathbb{C}$-group functor $\mathscr{L}_G(R) := G(R((t)))$, which is represented by the ind-affine group scheme $\bar{G}((t))$ (cf. Lemma 1.3.2). In particular, it satisfies the property (E) (cf. Exercise B.E.4). Recall that $\mathbf{PGL}_{\mathscr{H}(\lambda_c)}$ is the projective linear group functor (cf. Example B.4(2)) with the tangent space at 1 given by $\mathrm{End}_R(\mathscr{H}(\lambda_c)_R)/R \cdot \mathrm{Id}_{\mathscr{H}(\lambda_c)_R}$, where $\mathscr{H}(\lambda_c)_R := \mathscr{H}(\lambda_c) \otimes R$ (cf. Lemma B.13). By Lemma B.13, **PGL** satisfies the condition (E). Of course, since G satisfies the condition (L) (cf. Exercise B.E.5), thinking of $\mathscr{L}_G(R)$ as $R((t))$-points of G,

$$T_1(\mathscr{L}_G)_R = \mathfrak{g} \otimes R((t)), \quad \text{for any } R \in \mathbf{Alg}.$$

Moreover, the Lie algebra bracket in $T_1(\mathscr{L}_G)_R$ coincides with the standard Lie algebra bracket in $\mathfrak{g} \otimes_{\mathbb{C}} R((t))$, as can easily be seen from Definition B.17(c). The functor $\mathscr{L}_G$ satisfies the condition (L) finitely (cf. Definition B.15(b)). Also, by Exercise B.E.6, $\mathbf{PGL}_{\mathscr{H}(\lambda_c)}$ satisfies the condition (L) finitely.

Definition 1.4.2 (Adjoint action of $\bar{G}((t))$) Define the R-linear *adjoint action* of the group functor $\mathscr{L}_G(R)$ on the Lie-algebra functor $\hat{\mathfrak{g}}(R) := \mathfrak{g} \otimes R((t)) \oplus R.C$ (where the R-linear bracket in $\hat{\mathfrak{g}}(R)$ is defined by the same formula as (4) of Definition 1.2.1) by

$$(\mathbf{Ad}_C\gamma)(x[P] \oplus sC) = \gamma x[P]\gamma^{-1} + \left(s + \operatorname*{Res}_{t=0}\langle \gamma^{-1} d\gamma, x[P]\rangle\right) C,$$

for $\gamma \in \mathscr{L}_G(R)$, $x \in \mathfrak{g}$, $P \in R((t))$ and $s \in R$, where $\langle , \rangle$ is the $R((t))$-bilinear extension of the normalized invariant form on $\mathfrak{g}$ and taking an embedding $i : G \hookrightarrow GL_N$ we view $G(R((t)))$ as a subgroup of $N \times N$ invertible matrices over the ring $R((t))$. Observe that for the group functor $GL_N(R((t)))$, the adjoint action (defined in Definition B.17) is given by

$$(\mathbf{Ad}\gamma) \cdot M = \gamma M \gamma^{-1}, \quad \text{for } \gamma \in GL_N(R((t))) \text{ and } M \in M_N(R((t))).$$

From the functoriality of **Ad** (cf. (1) of Definition B.17), $\gamma x[P]\gamma^{-1} \in \mathfrak{g} \otimes R((t))$ (for $\gamma \in \mathscr{L}_G(R)$ and $x[P] \in \mathfrak{g} \otimes R((t))$) and it does not depend upon the choice of the embedding i. A similar remark applies to $\gamma^{-1}d\gamma$. Here $d\gamma$ for $\gamma = (\gamma_{i,j}) \in M_N(R((t)))$ denotes $d\gamma := (d\gamma_{i,j}/dt)$.

It is easy to check that for any $\gamma \in \mathscr{L}_G(R)$, $\mathbf{Ad}_C\gamma : \hat{\mathfrak{g}}(R) \to \hat{\mathfrak{g}}(R)$ is an R-linear Lie algebra homomorphism. Moreover,

$$\mathbf{Ad}_C(\gamma_1\gamma_2) = \mathbf{Ad}_C(\gamma_1) \circ \mathbf{Ad}_C(\gamma_2). \tag{1}$$

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Using Lemma B.18, one easily sees that for any finite-dimensional $\mathbb{C}$-algebra R and $x \in \mathfrak{g} \otimes R((t))$,

$$\dot{\mathbf{Ad}}_C(x)(y) = [x, y], \quad \text{for any } y \in \hat{\mathfrak{g}}(R). \tag{2}$$

It is easy to see that the representation $\mathscr{H}(\lambda_c)$ of $\hat{\mathfrak{g}}$ extends R-linearly to a representation $\bar{\rho}_R$ in $\mathscr{H}(\lambda_c)_R := \mathscr{H}(\lambda_c) \underset{\mathbb{C}}{\otimes} R$ of $\hat{\mathfrak{g}}(R)$.

A proof of the following result due to Faltings can be found in Beauville and Laszlo (1994, Lemma A.3) for $G = \mathrm{SL}_n$. The proof for general G is identical.

Proposition 1.4.3 *With the notation as above, for any $R \in \mathbf{Alg}$ and $\gamma \in \mathscr{L}_G(R)$, locally over Spec R, there exists an R-linear automorphism $\hat{\rho}(\gamma)$ of $\mathscr{H}(\lambda_c)_R$ uniquely determined up to an invertible element of R satisfying*

$$\hat{\rho}(\gamma)\bar{\rho}_R(x)\hat{\rho}(\gamma)^{-1} = \bar{\rho}_R(\mathbf{Ad}_C(\gamma) \cdot x), \quad \textit{for any } x \in \hat{\mathfrak{g}}(R), \tag{1}$$

where the adjoint representation of $\mathscr{L}_G(R)$ on $\hat{\mathfrak{g}}(R)$ is defined in the previous Definition 1.4.2.

As a corollary of the above Proposition 1.4.3, we get the following.

Theorem 1.4.4 *With the notation and assumptions as at the beginning of this section, there exists a homomorphism $\rho \colon \mathscr{L}_G \to \mathbf{PGL}_{\mathscr{H}(\lambda_c)}$ of group functors such that*

$$\dot{\rho} = \dot{\rho}(\mathbb{C}) \colon T_1(\mathscr{L}_G)_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}((t)) \to End_{\mathbb{C}}(\mathscr{H}(\lambda_c))/\mathbb{C} \cdot Id_{\mathscr{H}(\lambda_c)} \tag{1}$$

coincides with the projective representation $\mathscr{H}(\lambda_c)$ of $\mathfrak{g} \otimes \mathbb{C}((t))$ (cf. Lemmas B.13 and B.14).

By Exercise 1.4.E.1, in fact, $\dot{\rho}_R \colon \mathfrak{g} \otimes R((t)) \to End_R(\mathscr{H}(\lambda_c)_R)/ R \cdot Id_{\mathscr{H}(\lambda_c)_R}$ coincides with the projective representation $\mathscr{H}(\lambda_c)_R$ of $\mathfrak{g} \otimes R((t))$.

Proof Fix $\gamma \in \mathscr{L}_G(R)$. As guaranteed by the existence of an R-linear automorphism $\hat{\rho}(\gamma)$ of $\mathscr{H}(\lambda_c)_R$ locally in Spec R and its uniqueness up to an invertible element of R, we get an fppf R-algebra S_γ (depending upon γ) (cf. (Stacks, 2019, Tag 021N)) and a unique element (obtained by glueing locally obtained $\hat{\rho}(\gamma)$) $\bar{\rho}_{S_\gamma}(\gamma) \in \mathrm{PGL}_{S_\gamma}(\mathscr{H}(\lambda_c)_{S_\gamma}) := \mathrm{Aut}_{S_\gamma}(\mathscr{H}(\lambda_c)_{S_\gamma})/S_\gamma^* I$, where S_γ^* denotes the set of invertible elements in S_γ. Consider the exact sequence (cf. Definition B.1):

$$\mathscr{L}_G(R) \xrightarrow{i_{S_\gamma}} \mathscr{L}_G(S_\gamma) \rightrightarrows \mathscr{L}_G\left(S_\gamma \underset{R}{\otimes} S_\gamma\right).$$

Since $i_{S_\gamma}(\gamma)$ goes to the same element in $\mathscr{L}_G(S_\gamma \underset{R}{\otimes} S_\gamma)$ under the above two homomorphisms $\mathscr{L}_G(S_\gamma) \rightrightarrows \mathscr{L}_G(S_\gamma \underset{R}{\otimes} S_\gamma)$, we get that $\bar{\rho}_{S_\gamma}(\gamma) \in \mathrm{PGL}_{S_\gamma}(\mathscr{H}(\lambda_c)_{S_\gamma})$ goes to the same element under the two maps

$$\mathrm{PGL}_{S_\gamma}(\mathscr{H}(\lambda_c)_{S_\gamma}) \rightrightarrows \mathrm{PGL}_{S_\gamma \underset{R}{\otimes} S_\gamma}\left(\mathscr{H}(\lambda_c)_{S_\gamma \otimes_R S_\gamma}\right).$$

Hence, $\bar{\rho}_{S_\gamma}(\gamma) \in K_R(S_\gamma)$ for the functor $\mathbf{PGL}_{\mathscr{H}(\lambda_c)}$ (cf. proof of Lemma B.2 for the notation $K_R(S_\gamma)$). Finally, define $\rho(\gamma) \in \mathbf{PGL}_{\mathscr{H}(\lambda_c)}(R)$ as the image of $\bar{\rho}_{S_\gamma}(\gamma)$ under the canonical map $K_R(S_\gamma) \to \mathbf{PGL}_{\mathscr{H}(\lambda_c)}(R)$. From the uniqueness of R-linear automorphisms $\hat{\rho}(\gamma)$ locally in Spec R up to an invertible element in R, we get that $\rho(\gamma)$ is well defined (i.e., it does not depend upon the choice of S_γ).

Again using the uniqueness of $\hat{\rho}(\gamma)$ (up to invertible elements in R locally) satisfying (1) of Proposition 1.4.3 and using (1) of Definition 1.4.2 and Exercise 1.4.E.4, we get that ρ is a group homomorphism and, in fact, it is a morphism from the group functor $\mathscr{L}_G$ to the group functor $\mathbf{PGL}_{\mathscr{H}(\lambda_c)}$.

We now prove (1). Take $x \in \mathfrak{g}((t))$ and $y \in \hat{\mathfrak{g}}$. Then, by Proposition 1.4.3 applied to $R = \mathbb{C}(\epsilon)$, we get

$$\hat{\rho}(e^{\epsilon x})\bar{\rho}_R(y)\hat{\rho}(e^{-\epsilon x}) = \bar{\rho}_R(\mathbf{Ad}_C(e^{\epsilon x}) \cdot y), \tag{2}$$

where for the notation $e^{\epsilon x} \in \mathscr{L}_G(R)$ see Definition B.15(a). By Lemma B.18 applied to the representation $\mathbf{Ad}_C$ of $\mathscr{L}_G$, and identity (2) of Definition 1.4.2,

$$\bar{\rho}_R(\mathbf{Ad}_C(e^{\epsilon x})y) = \bar{\rho}_R(y + \epsilon[x, y]). \tag{3}$$

Similarly, fixing a lift of $\dot{\rho}(x)$ in $\mathrm{End}_{\mathbb{C}}(\mathscr{H}(\lambda_c))$, for $v \in \mathscr{H}(\lambda_c)$, by Lemma B.18,

$$\begin{aligned}
&\hat{\rho}(e^{\epsilon x})\bar{\rho}_R(y)\hat{\rho}(e^{-\epsilon x})v \\
&\quad = \hat{\rho}(e^{\epsilon x})\bar{\rho}_R(y)\left(v - \epsilon\dot{\rho}(x)v - \epsilon\lambda_x v\right), \quad \text{for some } \lambda_x \in \mathbb{C} \\
&\quad = \hat{\rho}(e^{\epsilon x})\left(\bar{\rho}(y)v - \epsilon\bar{\rho}(y)\dot{\rho}(x)v - \epsilon\lambda_x\bar{\rho}(y)v\right) \\
&\quad = \bar{\rho}(y)v - \epsilon\bar{\rho}(y)\dot{\rho}(x)v - \epsilon\lambda_x\bar{\rho}(y)v + \epsilon\dot{\rho}(x)\bar{\rho}(y)v + \epsilon\lambda_x\bar{\rho}(y)v.
\end{aligned} \tag{4}$$

Combining the equations (2)–(4), we get

$$\bar{\rho}[x, y] = \left[\dot{\rho}(x), \bar{\rho}(y)\right],$$

i.e.,

$$\left[\bar{\rho}(x) - \dot{\rho}(x), \bar{\rho}(y)\right] = 0, \quad \text{for all } x \in \mathfrak{g}((t)) \text{ and } y \in \hat{\mathfrak{g}}.$$

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Thus, by Exercise 1.2.E.5,

$$\bar{\rho}(x) - \dot{\rho}(x) = \mu_x \operatorname{Id}_{\mathscr{H}(\lambda_c)}, \text{ for some } \mu_x \in \mathbb{C}.$$

This proves the theorem. □

Definition 1.4.5 (Central extensions of loop groups) Following the notation and assumptions at the beginning of this section, take any $\lambda_c \in \hat{D}$. By Theorem 1.4.4, we have a homomorphism of group functors:

$$\rho : \mathscr{L}_G \to \mathbf{PGL}_{\mathscr{H}(\lambda_c)}.$$

Also, there is a canonical homomorphism of group functors (cf. Example B.4(2)):

$$\pi : \mathbf{GL}_{\mathscr{H}(\lambda_c)} \to \mathbf{PGL}_{\mathscr{H}(\lambda_c)}.$$

All these $\mathbb{C}$-group functors $\mathscr{L}_G$, $\mathbf{GL}_{\mathscr{H}(\lambda_c)}$ and $\mathbf{PGL}_{\mathscr{H}(\lambda_c)}$ satisfy the condition (L) finitely (cf. §1.4.1 for $\mathscr{L}_G$ and Exercise B.E.6 for **GL** and **PGL**). Thus, by Exercise B.E.7, we get the fiber product group functor $\hat{\mathscr{G}}_{\lambda_c}$ satisfying the condition (L) finitely:

$$\hat{\mathscr{G}}_{\lambda_c} := \mathscr{L}_G \underset{\mathbf{PGL}_{\mathscr{H}(\lambda_c)}}{\times} \mathbf{GL}_{\mathscr{H}(\lambda_c)}.$$

By the definition, we get homomorphisms of group functors

$$p : \hat{\mathscr{G}}_{\lambda_c} \to \mathscr{L}_G \quad \text{and} \quad \hat{\rho} : \hat{\mathscr{G}}_{\lambda_c} \to \mathbf{GL}_{\mathscr{H}(\lambda_c)}$$

making the following diagram commutative:

$$\begin{array}{ccc} \hat{\mathscr{G}}_{\lambda_c} & \xrightarrow{\hat{\rho}} & \mathbf{GL}_{\mathscr{H}(\lambda_c)} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathscr{L}_G & \xrightarrow[\rho]{} & \mathbf{PGL}_{\mathscr{H}(\lambda_c)}. \end{array}$$

By Exercise B.E.7, the Lie algebra $\operatorname{Lie} \hat{\mathscr{G}}_{\lambda_c}(R) := T_1(\hat{\mathscr{G}}_{\lambda_c})_R$ is identified with the fiber product Lie algebra (cf. §1.4.1, Example B.12 and Lemma B.13)

$$\hat{\mathfrak{G}}_{\lambda_c}(R) = \mathfrak{g} \otimes R((t)) \underset{\operatorname{End}_R(\mathscr{H}(\lambda_c)_R)/R.\operatorname{Id}}{\times} \operatorname{End}_R(\mathscr{H}(\lambda_c)_R),$$

for any finite-dimensional $\mathbb{C}$-algebra R.

Lemma 1.4.6 *The Lie algebra $\hat{\mathfrak{g}}_{\lambda_c} := \operatorname{Lie} \hat{\mathscr{G}}_{\lambda_c}(\mathbb{C})$ can canonically be identified with the affine Lie algebra $\hat{\mathfrak{g}}$.*

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Proof Let $\bar{\rho}\colon \hat{\mathfrak{g}} \to \mathrm{End}_{\mathbb{C}}(\mathscr{H}(\lambda_c))$ denote the representation. Define

$$\psi\colon \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}_{\lambda_c}, \quad x[P] + zC \mapsto (x[P], \bar{\rho}(x[P]) + zc\,\mathrm{Id}),$$
$$\text{for } x \in \mathfrak{g}, P \in K \text{ and } z \in \mathbb{C}.$$

From the definition of the bracket in $\hat{\mathfrak{g}}$ and Theorem 1.4.4, ψ is an isomorphism of Lie algebras. □

Combining Theorem 1.4.4, Definition 1.4.5 and Lemma 1.4.6, we get the following.

Corollary 1.4.7 *For any $\lambda_c \in \hat{D}$, we have a homomorphism of group functors*

$$\hat{\rho}\colon \hat{\mathscr{G}}_{\lambda_c} \to \mathbf{GL}_{\mathscr{H}(\lambda_c)}$$

such that its derivative for $R = \mathbb{C}$

$$\dot{\hat{\rho}}\colon \hat{\mathfrak{g}}_{\lambda_c} \to \mathrm{End}_{\mathbb{C}}(\mathscr{H}(\lambda_c))$$

under the identification of Lemma 1.4.6 coincides with the Lie algebra representation

$$\bar{\rho}\colon \hat{\mathfrak{g}} \to \mathrm{End}_{\mathbb{C}}(\mathscr{H}(\lambda_c)).$$

Moreover, for any $\mathbb{C}$-algebra R, $\hat{\gamma} \in \hat{\mathscr{G}}_{\lambda_c}(R)$ and $x \in \hat{\mathfrak{g}}(R)$,

$$\hat{\rho}(\hat{\gamma})\bar{\rho}_R(x)\hat{\rho}(\hat{\gamma})^{-1} = \bar{\rho}_R(\mathbf{Ad}_C(p(\hat{\gamma}))x), \ \ \textit{as operators on } \mathscr{H}(\lambda_c)_R. \quad (1)$$

The following lemma is trivial to verify.

Lemma 1.4.8 *Let V be a vector space over $\mathbb{C}$ and let $v_+ \in V$ be a nonzero vector and $V' \subset V$ a subspace such that $V = \mathbb{C}v_+ \oplus V'$. Then the following subgroup functors of $\mathbf{GL}_V$:*

$$\mathbf{GL}_V^{\circ}(R) = \{T \in \mathrm{GL}_R(V_R) : Tv_+ = v_+\} \quad \textit{and}$$
$$\mathbf{GL}'_V(R) = \left\{T \in \mathrm{GL}_R(V_R) : Tv_+ - v_+ \in V'_R \ \textit{and}\ T(V'_R) \subset V'_R\right\}$$

are $\mathbb{C}$-group functors, i.e., they satisfy the sheaf condition for the fppf topology.

Moreover, the projection homomorphism $\pi : \mathbf{GL}_V \to \mathbf{PGL}_V$ is an isomorphism of group functors restricted to either of $\mathbf{GL}_V^{\circ}$ or $\mathbf{GL}'_V$ onto their images.

Lemma 1.4.9 *Let Y be a connected variety (over $\mathbb{C}$). Then any morphism $f : Y \to \mathbb{C}^*$, which is null-homotopic in the topological category with the analytic topology Y^{an} on Y, is constant.*

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Observe that if the singular cohomology $H^1(Y^{an}, \mathbb{Z}) = 0$, then any continuous map $Y^{an} \to \mathbb{C}^*$ is null-homotopic since $\mathbb{C}^*$ is a $K(\mathbb{Z}, 1)$-space (cf. (Spanier, 1966, Chap. 8, §1, Theorem 8)).

Proof Assume, if possible, that there exists a null-homotopic nonconstant morphism $f: Y \to \mathbb{C}^*$. Since f is a morphism, there exists $N_o > 0$ such that the number of irreducible components of $f^{-1}(z) \leq N_o$, for any $z \in \mathbb{C}^*$. Now take the N-sheeted covering $\pi_N: \mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^N$, for any $N > N_o$. Since f is null-homotopic, there exists a lift as a morphism $\tilde{f}: Y \to \mathbb{C}^*$ (cf. (Serre, 1958, Proposition 20)) making the following diagram commutative:

$$\begin{array}{ccc} & & \mathbb{C}^* \\ & \overset{\tilde{f}}{\nearrow} & \big\downarrow \pi_N \\ Y & \xrightarrow[f]{} & \mathbb{C}^* . \end{array}$$

Since $\tilde{f}$ is a morphism and nonconstant, by Chevalley's Theorem (cf. (Hartshorne, 1977, Chap. II, Exercise 3.19)) $\operatorname{Im} \tilde{f}$ (being a constructible and connected set) misses only finitely many points of $\mathbb{C}^*$. In particular, there exists $z_o \in \mathbb{C}^*$ (in fact, an open dense set of points) such that $\pi_N^{-1}(z_o) \subset \operatorname{Im} \tilde{f}$. But then the number of irreducible components of $f^{-1}(z_o) = \tilde{f}^{-1}\pi_N^{-1}(z_o) \geq N > N_o$, which is a contradiction to the choice of N. This proves the lemma. □

1.4.10

Recall the homomorphism of group functors $p: \hat{\mathscr{G}}_{\lambda_c} \to \mathscr{L}_G$ (for any $\lambda_c \in \hat{D}$) from Definition 1.4.5. Since

$$\mathbf{GL}_{\mathscr{H}(\lambda_c)}(R) \to \mathbf{PGL}_{\mathscr{H}(\lambda_c)}(R)$$

has kernel R^* for any $R \in \mathbf{Alg}$, we get an exact sequence of group functors (i.e., for any R, the following sequence specialized at R is exact):

$$1 \to \mathbb{G}_m \xrightarrow{i} \hat{\mathscr{G}}_{\lambda_c} \xrightarrow{p} \mathscr{L}_G, \qquad (1)$$

where i is given by $r \mapsto (1, r\,\mathrm{Id})$, for any $r \in R^*$. Of course, $\mathbb{G}_m$ is central in $\hat{\mathscr{G}}_{\lambda_c}$ (for any $R \in \mathbf{Alg}$). By Exercise 1.4.E.2, $\hat{\mathscr{G}}_{\lambda_c}(R) \to \mathscr{L}_G(R)$ is surjective for any field $R \supset \mathbb{C}$. Combining Lemma 1.4.8 and Proposition 1.4.3, we get the following.

Theorem 1.4.11 *Take any $\lambda_c \in \hat{D}$. Then the homomorphism of group functors $p\colon \hat{\mathscr{G}}_{\lambda_c} \to \mathscr{L}_G$ splits over the subgroup functors*

$$\mathscr{L}_G^+(R) := G(R[[t]])_e,\ \ \mathscr{L}_G^-(R) := G(R[t^{-1}])^- \ \text{ and } \mathscr{G}(R) := G(R);$$

where $G(R[[t]])_e$ and $G(R[t^{-1}])^-$ are defined in Corollary 1.3.3 with $H = (e)$.

Thus, p also splits over the group functor $\mathscr{F}_1\colon R \rightsquigarrow G(R[[t]])$ (cf. Lemma 1.3.2).

Similarly, p also splits over the group functor $\mathscr{F}_3\colon R \rightsquigarrow G(R[t^{-1}])$.

Proof We first prove the theorem for $\mathscr{L}_G^+$. Fix $v_+ \in \mathscr{H}(\lambda_c)$ a highest-weight vector (which is unique up to a scalar multiple). We claim that for any $\gamma \in \mathscr{L}_G^+(R)$,

$$\hat{\rho}(\gamma_S)v_+ \in S^* v_+, \tag{1}$$

for any fppf R-algebra $S = S_\gamma$ such that $\rho(\gamma_S)$ lifts to an element $\hat{\rho}(\gamma_S)$ of $\mathrm{Aut}_S(\mathscr{H}(\lambda_c)_S)$, where γ_S is the image of γ in $\mathscr{L}_G^+(S)$. By (1) of Proposition 1.4.3,

$$\bar{\rho}_S(x)\hat{\rho}(\gamma_S)v_+ = \hat{\rho}(\gamma_S)\bar{\rho}_S(\mathbf{Ad}_C(\gamma_S^{-1})x)v_+,\\ \text{for any } x \in \hat{\mathfrak{u}}_S := \mathfrak{g} \otimes tS[[t]] \oplus (S \otimes \mathfrak{u}), \tag{2}$$

where $\mathfrak{u}$ is the nil-radical of $\mathfrak{b}$. But, by definition,

$$\mathbf{Ad}_C(\gamma_S^{-1})(\hat{\mathfrak{u}}_S) \subset \hat{\mathfrak{u}}_S. \tag{3}$$

Moreover,

$$\bar{\rho}_S(\hat{\mathfrak{u}}_S) \cdot v_+ = 0, \ \text{ since } v_+ \text{ is a highest-weight vector.} \tag{4}$$

Thus, combining (2)–(4), we get

$$\bar{\rho}_S(x)(\hat{\rho}(\gamma_S)v_+) = 0, \quad \text{for all } x \in \hat{\mathfrak{u}}_S. \tag{5}$$

By Exercise 1.4.E.4, we get that

$$\hat{\rho}(\gamma_S)v_+ \in Sv_+, \quad \text{and hence} \quad \hat{\rho}(\gamma_S)v_+ \in S^* \cdot v_+$$

($\hat{\rho}(\gamma_S)$ being represented by an invertible S-linear map). This proves (1).

Thus, by Exercise B.E.9,

$$\rho\left(\mathscr{L}_G^+(R)\right) \subset \pi\left(\mathbf{GL}^0_{\mathscr{H}(\lambda_c)}(R)\right),$$

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where π is the canonical morphism $\mathbf{GL}_{\mathscr{H}(\lambda_c)} \to \mathbf{PGL}_{\mathscr{H}(\lambda_c)}$. Therefore, by Lemma 1.4.8, we get the splitting of p over the subgroup functor $\mathscr{L}_G^+$.

We next come to the case of $\mathscr{L}_G^-$. We claim that for any $\gamma \in \mathscr{L}_G^-(R)$,

$$[\hat{\rho}(\gamma_S)v_+]_+ \subset S^*, \tag{6}$$

for any fppf R-algebra $S = S_\gamma$ such that $\rho(\gamma_S)$ lifts to an element $\hat{\rho}(\gamma_S)$ of $\mathrm{Aut}_S(\mathscr{H}(\lambda_c)_S)$, where $[\hat{\rho}(\gamma_S)v_+]_+$ is the coefficient of v_+ in the component of $\hat{\rho}(\gamma_S)v_+$ under the decomposition

$$\mathscr{H}(\lambda_c) \otimes S = Sv_+ \oplus (\mathscr{H}'(\lambda_c) \otimes S),$$

where $\mathscr{H}'(\lambda_c)$ is the sum of weight spaces of $\mathscr{H}(\lambda_c)$ of weights $< \lambda_c$. Applying (1) of Proposition 1.4.3 to $\hat{\rho}(\gamma_S)v_+$, we get

$$\hat{\rho}(\gamma_S)\,(\bar{\rho}_S(x_1)\dots\bar{\rho}_S(x_n)v_+) = \bar{\rho}_S(\mathbf{Ad}_C(\gamma_S)x_1)\dots\bar{\rho}_S(\mathbf{Ad}_C(\gamma_S)x_n)\hat{\rho}(\gamma_S)v_+, \tag{7}$$

for any $x_i \in \hat{\mathfrak{u}}_S^- := (\mathfrak{g} \otimes t^{-1}S[t^{-1}]) \oplus (S \otimes \mathfrak{u}^-)$, where $\mathfrak{u}^-$ is the nil-radical of the opposite Borel $\mathfrak{b}^-$. From the definition of $\mathbf{Ad}_C$,

$$\mathbf{Ad}_C(\gamma_S) \cdot \left(\hat{\mathfrak{u}}_S^-\right) \subset \hat{\mathfrak{u}}_S^-.$$

Thus, from (7) we get (since $\hat{\mathfrak{u}}_{\mathbb{C}}^- \cdot \mathscr{H}(\lambda_c) \subset \mathscr{H}'(\lambda_c)$)

$$\hat{\rho}(\gamma_S)\,(\bar{\rho}_S(x_1)\dots\bar{\rho}_S(x_n)v_+) \in \mathscr{H}'(\lambda_c) \otimes S, \ \text{ for any } n \geq 1. \tag{8}$$

But, since $\mathscr{H}(\lambda_c)$ is an irreducible $\hat{\mathfrak{g}}$-module and $\hat{\mathfrak{u}}_S$ annihilates v_+, the span of $\bar{\rho}_S(x_1)\dots\bar{\rho}_S(x_n)v_+$, as x_i run over $\hat{\mathfrak{u}}_S^-$, is equal to $\mathscr{H}'(\lambda_c) \otimes S$. Thus, from (8), we get

$$\hat{\rho}(\gamma_S)(\mathscr{H}'(\lambda_c) \otimes S) \subset \mathscr{H}'(\lambda_c) \otimes S. \tag{9}$$

From this we immediately obtain (6) by applying $\hat{\rho}(\gamma_S^{-1})$ to (9). Thus, by (6) and (9), and Exercise B.E.9,

$$\rho\left(\mathscr{L}_G^-(R)\right) \subset \pi\left(\mathbf{GL}'_{\mathscr{H}(\lambda_c)}(R)\right)$$

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under the decomposition $\mathscr{H}(\lambda_c) = \mathbb{C}v_+ \oplus \mathscr{H}'(\lambda_c)$. Again applying Lemma 1.4.8, we get the splitting of p over the subgroup functor $\mathscr{L}_G^-$.

Since the action of $\mathfrak{g}$ on $\mathscr{H}(\lambda_c)$ decomposes as a direct sum of finite-dimensional $\mathfrak{g}$-submodules V_i:

$$\mathscr{H}(\lambda_c) = \bigoplus_i V_i$$

and G is simply connected, the action of $\mathfrak{g}$ on any V_i integrates to give an action of G on V_i. This gives a representation of G in $\mathrm{GL}_{\mathbb{C}}(\mathscr{H}(\lambda_c))$. From this we get a homomorphism of group functors

$$\mathscr{G} \to \mathbf{GL}_{\mathscr{H}(\lambda_c)},$$

which provides a splitting of p over $\mathscr{G}$.

We finally prove that p splits over the subgroup functor $\mathscr{F}_1(R) := G(R[[t]])$ (cf. Lemma 1.3.2) of $\mathscr{L}_G$.

First of all, from the definition of $\mathscr{L}_G^+(R)$ as the kernel of

$$\epsilon_R(0)\colon G(R[[t]]) \to G(R),\ t \mapsto 0 \quad \text{(cf. Corollary 1.3.3)},$$

and the splitting of $\epsilon_R(0)$ obtained from the embedding $G(R) \hookrightarrow G(R[[t]])$, induced from the embedding $R \hookrightarrow R[[t]]$, we see that there is a semidirect product decomposition of the group functor

$$\mathscr{F}_1 = \mathscr{L}_G^+ \ltimes \mathscr{G}. \tag{10}$$

Take splittings σ_+ and σ_0 of p over $\mathscr{L}_G^+$ and $\mathscr{G}$, respectively. Now define (for any $R \in \mathbf{Alg}$ and $g \in \mathscr{F}_1(R)$ uniquely written as $g = g_+g_0$, with $g_+ \in \mathscr{L}_G^+(R)$ and $g_0 \in \mathscr{G}(R)$)

$$\sigma(g) = \sigma_+(g_+) \cdot \sigma_0(g_0). \tag{11}$$

It is clear that σ is a $\mathbb{C}$-space functor section of p. We now prove that σ, in fact, is a homomorphism of group functors. To prove this, since $\mathscr{G}$ normalizes $\mathscr{L}_G^+$, it suffices to show

$$\sigma_+(g_0g_+g_0^{-1}) = \sigma_0(g_0)\sigma_+(g_+)\sigma_0(g_0)^{-1}, \quad \text{for } g_0 \in \mathscr{G}(R) \text{ and } g_+ \in \mathscr{L}_G^+(R). \tag{12}$$

Consider the morphism of $\mathbb{C}$-space functors

$$\psi : \mathscr{G} \times \mathscr{L}_G^+ \to \hat{\mathscr{G}}_{\lambda_c},\ (g_0, g_+) \mapsto \sigma_+(g_0g_+g_0^{-1})\sigma_0(g_0)\sigma_+(g_+)^{-1}\sigma_0(g_0)^{-1}.$$

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Its image clearly lands in $\mathbb{G}_m$ under the sequence (1) of §1.4.10. Thus, the morphism ψ gives rise to a morphism of $\mathbb{C}$-space functors:

$$\bar{\psi} : \mathscr{G} \times \bar{\mathscr{L}}_G^+ \to \mathbb{G}_m \text{ such that } \bar{\psi}(1, g_+) = 1, \text{ for any } g_+ \in \bar{\mathscr{L}}_G^+.$$

Since $\bar{\mathscr{L}}_G^+$ is a representable functor (cf. Corollary 1.3.3) represented by a scheme denoted $\bar{L}_G^+$ with $\mathbb{C}$-points L_G^+ (the kernel of $G[[t]] \to G, t \mapsto 0$), $\bar{\psi}$ is induced from a morphism of schemes:

$$\bar{\psi}_o : G \times \bar{L}_G^+ \to \mathbb{G}_m, \text{ such that } \bar{\psi}_o(1, g_+) = 1, \text{ for any } g_+ \in \bar{L}_G^+.$$

But any morphism $f : G \to \mathbb{G}_m$ is a constant (cf. Lemma 1.4.9). Thus, $\bar{\psi}_o \equiv 1$ and hence so is ψ. This proves (12) and hence we obtain a splitting of p over $\mathscr{F}_1$.

The proof for $\mathscr{F}_3$ is identical to that of $\mathscr{F}_1$. This proves the theorem. □

Proposition 1.4.12 *For $\lambda_c \in \hat{D}$, the group functor $\hat{\mathscr{G}}_{\lambda_c}$ is represented by a reduced ind-affine group scheme denoted $\bar{\hat{G}}_{\lambda_c}$ (with $\mathbb{C}$-points $\hat{G}_{\lambda_c} = \hat{\mathscr{G}}_{\lambda_c}(\mathbb{C})$). This gives rise to an exact sequence of ind-group schemes:*

$$1 \to \mathbb{G}_m \to \bar{\hat{G}}_{\lambda_c} \xrightarrow{\bar{p}} \bar{G}((t)) \to 1. \tag{1}$$

Moreover, $\bar{p}$ admits a regular section over $N := \bar{G}[t^{-1}]^- \times \bar{G}[[t]]$ (cf. Lemma 1.3.16).

Thus, $\bar{\hat{G}}_{\lambda_c} \to \bar{G}((t))$ is a Zariski locally trivial principal $\mathbb{G}_m$-bundle.

Proof We first show that the group functor $\hat{\mathscr{G}}_{\lambda_c}$ is represented by an ind-scheme. Consider the open cover of $\bar{G}((t))$:

$$\bar{G}((t)) = \bigcup_{g \in G((t))} gN. \tag{2}$$

Then, the subfunctors $\{h_{gN}\}_{g \in G((t))}$ are an open covering of $h_{\bar{G}((t))}$ (cf. Definition B.5). To prove the above equality, observe that

$$\bar{G}((t))(\mathbb{C}) = G((t)) = \cup_{g \in G((t))} gN(\mathbb{C}).$$

(Now, use the fact that any closed ind-subscheme of $\bar{G}((t))$ has nonempty set of $\mathbb{C}$-points as observed in Exercise 1.3.E.8.) Thus, by Exercise B.E.8, $\{p^{-1}h_{gN}\}_{g \in G((t))}$ is an open cover consisting of subfunctors of $\hat{\mathscr{G}}_{\lambda_c}$. (Observe that $p^{-1}h_{gN}$ indeed satisfies the sheaf condition by using condition (1) of Exercise B.E.8 and using the fact that Spec $R' \to$ Spec R is surjective for any faithfully flat homomorphism $R \to R'$, cf. (Matsumura, 1989, Theorem 7.3).)

Recall from Theorem 1.4.11 that the homomorphism of group functors $p\colon \hat{\mathscr{G}}_{\lambda_c} \to \mathscr{L}_G$ admits splittings over the subgroup functors $G(R[[t]])$ and $G(R[t^{-1}])^-$. Combining them we get a morphism s from the $\mathbb{C}$-space functor $N(R) = G(R[[t]]) \times G(R[t^{-1}])^-$ to $\hat{\mathscr{G}}_{\lambda_c}$ such that $p \circ s = \mathrm{Id}$, i.e., a functorial section s of p over $N(R)$. For any $g \in G((t))$, choosing $\hat{g} \in \hat{G}_{\lambda_c}$ over g (which is possible by the surjectivity of $\hat{G}_{\lambda_c} \to G((t))$, cf. §1.4.10), we get a functorial section $\hat{g}s$ of p over h_{gN}. This gives rise to an isomorphism of $\mathbb{C}$-space functors

$$f\colon h_{gN} \times \mathbb{G}_m \simeq p^{-1}(h_{gN})$$

given by (for any $R \in \mathbf{Alg}$)

$$h_{gN}(R) \times R^* \xrightarrow{\sim} \left(p^{-1}h_{gN}\right)(R), \quad (\theta, r) \mapsto \hat{g}s(\theta)\cdot r,$$
$$\text{for } \theta \in h_{gN}(R) \text{ and } r \in R^*, \tag{3}$$

making the following diagram commutative:

$$\begin{array}{ccc} h_{gN}(R) \times R^* & \xrightarrow{\sim} & (p^{-1}h_{gN})(R) \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle p} & \\ & h_{gN}(R). & \end{array}$$

In particular, this implies that the open subfunctor $p^{-1}h_{gN}$ of $\hat{\mathscr{G}}_{\lambda_c}$ (for any $g \in G((t))$) over h_{gN} is represented by the ind-scheme $gN \times \mathbb{G}_m \to gN$. We now show that $\hat{\mathscr{G}}_{\lambda_c}$ is a representable functor with $\mathbb{C}$-points $\hat{G}_{\lambda_c}$.

Since $\{p^{-1}h_{gN}\}_{g \in G((t))}$ is an open cover consisting of subfunctors of $\hat{\mathscr{G}}_{\lambda_c}$ and $p^{-1}h_{gN}$ is represented by the ind-scheme $gN \times \mathbb{G}_m$ (by the isomorphism (3)), we get that the functor $\hat{\mathscr{G}}_{\lambda_c}$ is represented by an ind-scheme denoted $\bar{\hat{G}}_{\lambda_c}$ (cf. the proof of Proposition 1.3.18(a) using (Eisenbud and Harris, 2000, Theorem VI-14 and Exercise VI-11); since a Zariski cover is an fppf cover by Stacks (2019, Tag 021N)). Since $\hat{\mathscr{G}}_{\lambda_c}$ is a group functor, we get that $\bar{\hat{G}}_{\lambda_c}$ is an ind-group scheme giving rise to the exact sequence (1) of ind-group schemes. Moreover, the morphism $\bar{p}\colon \bar{\hat{G}}_{\lambda_c} \to \bar{G}((t))$ admits regular sections over gN (by (3)) for any $g \in G((t))$.

We now show that $\bar{\hat{G}}_{\lambda_c}$ is a reduced ind-affine group scheme. Let $\{(\bar{G}((t)))^n\}_{n \geq 0}$ be an increasing filtration of $\bar{G}((t))$ by reduced closed affine subschemes (cf. Remark 1.3.26(b) and Lemma 1.3.21), giving an ind-affine group scheme structure. Then, under the inverse image ind-scheme structure,

$$(\bar{\hat{G}}_{\lambda_c})^n := \bar{p}^{-1}\left((\bar{G}((t)))^n\right)$$

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is a closed subset of $\bar{\hat{G}}_{\lambda_c}$ acquiring a reduced scheme structure by virtue of the isomorphism (3). Moreover, $(\bar{\hat{G}}_{\lambda_c})^n \hookrightarrow (\bar{\hat{G}}_{\lambda_c})^{n+1}$ is a closed embedding since closed embedding is preserved under base change (cf. (Hartshorne, 1977, Chap. II, Exercise 3.11(a))). Thus, $\bar{\hat{G}}_{\lambda_c}$ is a reduced ind group scheme.

Since a (Zariski locally trivial) principal $\mathbb{G}_m$-bundle over an affine scheme is affine (being an affine morphism), from the affineness of $(\bar{G}((t)))^n$ and (2), we get that $\bar{\hat{G}}_{\lambda_c}$ is ind-affine. This completes the proof of the proposition. □

Remark 1.4.13 As proved later (cf. Corollary 8.2.3), the (group) splittings of $\bar{p}\colon \bar{\hat{G}}_{\lambda_c} \to \bar{G}((t))$ over either of $\bar{G}[[t]]$ or $\bar{G}[t^{-1}]^-$ are unique (cf. Theorem 1.4.11).

Moreover, for any two regular sections s_1, s_2 of $\bar{p}$ over $N = \bar{G}[t^{-1}]^- \times \bar{G}[[t]]$ (cf. Proposition 1.4.12 and Corollary 8.2.3),

$$s_1 = s_2 z, \quad \text{for a fixed } z \in \mathbb{G}_m.$$

1.4.E Exercises

(1) With the notation as in Theorem 1.4.4, show that

$$\dot{\rho}_R\colon \mathfrak{g} \otimes R((t)) \to \mathrm{End}_R(\mathscr{H}(\lambda_c)_R)/R.\,\mathrm{Id}_{\mathscr{H}(\lambda_c)_R}$$

coincides with the projective representation of $\mathfrak{g} \otimes R((t))$ in $\mathscr{H}(\lambda_c)_R$, for any $R \in \mathbf{Alg}$.

(2) For any $\mathbb{C}$-algebra R which is a field, show that $\mathbf{GL}_V(R) \to \mathbf{PGL}_V(R)$ is surjective for any (not necessarily finite dimensional) $\mathbb{C}$-vector space V.

Hint: Let R be any noetherian $\mathbb{C}$-algebra. Let S be an fppf R-algebra. In particular, S is a noetherian $\mathbb{C}$-algebra. By Stacks (2019, Tag 0311), there is an embedding of rings

$$S \hookrightarrow \Pi_{i=1}^N S_{\mathfrak{p}_i},$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_N$ are the associated prime ideals of S.

(a) Show that for any injective $\mathbb{C}$-algebra homomorphism $T \hookrightarrow T'$,

$$\mathrm{PGL}_V(T) \subset \mathrm{PGL}_V(T'), \quad \text{where } \mathrm{PGL}_V(T) := \mathrm{GL}_T(V_T)/T^* \cdot \mathrm{Id}.$$

Moreover, if $R \subset T \hookrightarrow T'$ and $K_R(T') = \mathrm{PGL}_V(R)$ for the functor $\mathbf{PGL}_V$ (cf. proof of Lemma B.2 for the notation $K_R(T)$), show that $K_R(T) = \mathrm{PGL}_V(R)$.

(b) Let $R \subset T$, where T is a local ring such that R is an R-module direct summand of T and T is R-flat. Show that $K_R(T) = \mathrm{PGL}_V(R)$.

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(c) Let $R \subset T_1$ and $R \subset T_2$ be two $\mathbb{C}$-algebras such that T_2 is flat over R. Assume further that

$$K_R(T_1) = K_R(T_2) = \mathrm{PGL}_V(R).$$

Show that $K_R(T_1 \times T_2) = \mathrm{PGL}_V(R)$. Combining (a)–(c), the exercise follows.

(3) Show that, for any $\lambda_c \in \hat{D}$, there is an isomorphism of ind-group schemes

$$\bar{\hat{G}}_{\lambda_c} \xrightarrow{\psi} \bar{\hat{G}}_{0_c}$$

making the following diagram commutative:

$$\begin{array}{ccc} \bar{\hat{G}}_{\lambda_c} & \xrightarrow{\quad\psi\quad} & \bar{\hat{G}}_{0_C} \\ & {\scriptstyle \bar{p}}\searrow \quad \swarrow{\scriptstyle \bar{p}} & \\ & \bar{G}((t)). & \end{array}$$

Hint: To prove this, show first that the $\mathbb{G}_m$-bundles $\bar{\hat{G}}_{\lambda_c}$ and $\bar{\hat{G}}_{0_C}$ are isomorphic. Then show that a $\mathbb{G}_m$-bundle isomorphism which takes 1 to 1 over $1 \in \bar{G}((t))$ is automatically a group homomorphism.

(4) For any $\lambda_c \in \hat{D}$ and any $\mathbb{C}$-algebra R, show that $Rv_+ \subset \mathscr{H}(\lambda_c)_R$ is the unique line annihilated by $\hat{\mathfrak{u}}(R) := (\mathfrak{g} \otimes tR[[t]]) \oplus (R \otimes \mathfrak{u})$, where $\mathfrak{u}$ is the nil-radical of $\mathfrak{b}$.

Hence, show that any $\hat{\mathfrak{g}}(R)$-module endomorphism of $\mathscr{H}(\lambda_c)_R$ is the identity map up to a scalar multiple.

Hint: Use Exercise 1.2.E.5.

1.C Comments

The content of Section 1.2 is fairly standard (cf. (Kac, 1990, Chaps. 7 and 12 and Lemma 9.10) and (Kumar, 2002, Chaps. 1 and 13)).

The content of Section 1.3 is also fairly standard by now. Lemma 1.3.2, Theorem 1.3.8, Corollary 1.3.15 for $G = \mathrm{SL}_n$, GL_N; Corollary 1.3.19 for $G = \mathrm{SL}_N$; and Lemma 1.3.21 are proved in Beauville and Laszlo (1994). The proof of Theorem 1.3.8 is an elaboration of Beauville and Laszlo (1994, Proof of Proposition 2.4) (with some help from P. Belkale). The approach we have taken in this section is largely derived from Faltings (2003) (though we have supplied here significantly more details). For example, Proposition 1.3.14, Corollary 1.3.15 and Lemma 1.3.16 are taken from Faltings (2003, §2).

As mentioned before, the detailed proof of Theorem 1.3.22 given here was provided by B. Conrad (and a brief outline was given earlier by G. Faltings in a private communication). Theorem 1.3.23 and Proposition 1.3.24 are given in Laszlo and Sorger (1997, Propositions 4.6 and 4.7). However, the proof of Theorem 1.3.23 outlined in Laszlo and Sorger (1997, Proposition 4.6) is incorrect (since it wrongly uses an incorrect theorm of Shafarevich). For a different representation-theoretic approach to many of the results in this Section 1.3 and the next, see Kumar (2002, Chap. 13.2).

Lemma 1.4.9 is taken from Kumar, Narasimhan and Ramanathan (1994, Lemma 2.5). Theorem 1.4.11 is taken from Laszlo and Sorger (1997). There is an alternative construction of the central extension of $\mathrm{SL}_N((t))$ via the Fredholm group in Beauville and Laszlo (1994, §4) (also see Kumar (2002, Theorem 13.2.8)).