8

Microlocal Aspects, Surjectivity of I_0^*

This chapter provides the key microlocal input of the monograph. We will prove that on a simple manifold M the normal operator $I_0^* I_0$ is an elliptic pseudodifferential operator of order -1 in the interior of M, thus establishing an analogue of Theorem 1.3.16 for the Radon transform in the plane. Combining this result with the injectivity of I_0, we will prove a surjectivity result for the adjoint I_0^*. This surjectivity result may be rephrased as an existence result for first integrals of the geodesic flow with prescribed zero Fourier modes, and it will play a prominent role in subsequent chapters. At the end of this chapter we shall extend these properties to include matrix weights and attenuations.

8.1 The Normal Operator

Let (M, g) be a compact non-trapping manifold with strictly convex boundary, and let I_0 be the geodesic X-ray transform acting on $C^\infty(M)$. By (4.1), I_0 is a bounded operator $L^2(M) \to L^2_\mu(\partial_+ SM)$, and Lemma 4.1.4 states that the adjoint of this operator is given by

$$(I_0^* h)(x) = \int_{S_x M} h^\sharp(x, v)\, dS_x(v).$$

We will consider the *normal operator*

$$\mathcal{N} := I_0^* I_0 \colon L^2(M) \to L^2(M).$$

The following result is an analogue of the fact proved in Theorem 1.3.16 that the normal operator of the Radon transform in the plane is an elliptic pseudodifferential operator (ΨDO) of order -1. For our geometric setting this can be traced back to Guillemin and Sternberg (1977, section 6.3) and Stefanov and Uhlmann (2004). The references Guillemin and Sternberg (1977) and Guillemin (1985) state the property under the so-called *Bolker condition*,

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which is seen to be equivalent in our case to the absence of conjugate points. The references Stefanov and Uhlmann (2004) and Pestov and Uhlmann (2005, Lemma 3.1) provide a more recent version of this result fitting with our presentational aims.

Theorem 8.1.1 (The normal operator is elliptic) *Let (M, g) be a simple manifold. Then $\mathcal{N} = I_0^* I_0$ is a classical elliptic ΨDO on M^{int} of order -1 with principal symbol*

$$\sigma_{\text{pr}}(\mathcal{N}) = c_n |\xi|_g^{-1}.$$

We discussed ΨDOs in $\mathbb{R}^n$ in Section 1.3. ΨDOs on manifolds can be defined in terms of local coordinates. See Hörmander (1983–1985, Section 18.1) for the following facts.

Definition 8.1.2 (ΨDOs on manifolds) Let Z be a smooth manifold without boundary and let $A \colon C_c^\infty(Z) \to C^\infty(Z)$ be a linear operator. We say that A is a ΨDO of order m, written $A \in \Psi^m(Z)$, if for any local coordinate chart $\kappa \colon U \to \tilde{U}$, where $U \subset Z$ and $\tilde{U} \subset \mathbb{R}^n$ are open sets, the operator

$$A_\kappa \colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n), \;\; A_\kappa f = (\psi A(\phi(f \circ \kappa))) \circ \kappa^{-1}$$

is in $\Psi^m(\mathbb{R}^n)$ whenever $\phi, \psi \in C_c^\infty(U)$. We say that A is a classical ΨDO, denoted by $A \in \Psi_{\text{cl}}^m(Z)$, if each A_κ is in $\Psi_{\text{cl}}^m(\mathbb{R}^n)$.

We also need the notion of ellipticity. For the case of $\Psi^m(\mathbb{R}^n)$ we gave a definition involving the *full symbol*. On manifolds we need to deal with the fact that the full symbol is not invariant under changes of coordinates. However, for classical ΨDOs the *principal symbol* can be invariantly defined as a smooth function on T^*Z that is homogeneous in ξ.

Proposition 8.1.3 (Principal symbol) *For any $m \in \mathbb{R}$, there is a linear map*

$$\sigma_{\text{pr}} \colon \Psi_{\text{cl}}^m(Z) \to C^\infty(T^*Z \setminus \{0\})$$

such that $\sigma_{\text{pr}}(A)$ is homogeneous of degree m in ξ and $\sigma_{\text{pr}}(A) = 0$ if and only if $A \in \Psi_{\text{cl}}^{m-1}(Z)$. Moreover, if $A \in \Psi_{\text{cl}}^m(Z)$ and $B \in \Psi_{\text{cl}}^{m'}(Z)$, then $AB \in \Psi_{\text{cl}}^{m+m'}(Z)$ and

$$\sigma_{\text{pr}}(AB) = \sigma_{\text{pr}}(A)\sigma_{\text{pr}}(B).$$

Definition 8.1.4 (Ellipticity) An operator $A \in \Psi_{\text{cl}}^m(Z)$ is *elliptic* if its principal symbol $\sigma_{\text{pr}}(A)$ is non-vanishing on $T^*Z \setminus \{0\}$.

To motivate the proof of Theorem 8.1.1, note that from the Schwartz kernel theorem we know that the bounded operator $\mathcal{N}\colon L^2(M) \to L^2(M)$ must have a Schwartz kernel $K(x, y)$ so that

$$(\mathcal{N} f)(x) = \int_M K(x, y) f(y)\, dV^n(y). \tag{8.1}$$

For general operators K could be very singular, in general it is just a distribution on $M^{\mathrm{int}} \times M^{\mathrm{int}}$, but ΨDOs are characterized by having kernels of a very special type, namely K is what is called a *conormal distribution* with respect to the diagonal of $M^{\mathrm{int}} \times M^{\mathrm{int}}$. This means that it is smooth off the diagonal and at the diagonal, it has a singularity of a special type. We refer to Hörmander (1983–1985, Section 18.2) for further details.

Our first task is then to find out what the Schwartz kernel K of $\mathcal{N}$ looks like. We begin by deriving an integral expression for $\mathcal{N}$.

Lemma 8.1.5 (First expression for $\mathcal{N}$) *Let (M, g) be a compact non-trapping manifold with strictly convex boundary. Then*

$$(\mathcal{N} f)(x) = 2 \int_{S_x M} \int_0^{\tau(x,v)} f\left(\gamma_{x,v}(t)\right)\, dt\, dS_x(v). \tag{8.2}$$

Proof From the definitions we have

$$\int_{S_x M} (I_0 f)^\sharp(x, v)\, dS_x(v) = \int_{S_x M} \int_{-\tau(x,-v)}^{\tau(x,v)} f(\gamma_{x,v}(t))\, dt\, dS_x(v).$$

Thus

$$\begin{aligned}(\mathcal{N} f)(x) = &\int_{S_x M} \int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\, dt\, dS_x(v) \\ &+ \int_{S_x M} \int_{-\tau(x,-v)}^{0} f(\gamma_{x,v}(t))\, dt\, dS_x(v).\end{aligned}$$

The result follows after performing the change of variables $(t, v) \mapsto (-t, -v)$ in the second integral. □

The next example determines the Schwartz kernel K when M is a Euclidean domain.

Example 8.1.6 ($\mathcal{N}$ in the Euclidean case) Let $M = \overline{\Omega}$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with strictly convex smooth boundary, and let $g = e$ be the Euclidean metric. Extend f by zero to $\mathbb{R}^n$. Then the formula (8.2) becomes

$$(\mathcal{N} f)(x) = 2\int_0^\infty \int_{S^{n-1}} f(x+tv)\, dS(v)\, dt.$$

Let x be fixed. It is natural to change to *polar coordinates*, i.e. consider $y = x + tv$, where $t \geq 0$ and $v \in S^{n-1}$. This requires that we introduce the Jacobian t^{n-1} as follows:

$$(\mathcal{N} f)(x) = 2\int_0^\infty \int_{S^{n-1}} \frac{f(x+tv)}{t^{n-1}} t^{n-1}\, dS(v)\, dt = 2\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}}\, dy.$$

We have proved that the Schwartz kernel of $\mathcal{N}$ has the simple form

$$K(x,y) = \frac{2}{|x-y|^{n-1}}.$$

We would like to determine $K(x,y)$ in a similar way for more general manifolds (M,g). First we show that one can always change to polar coordinates in T_xM. Recall from Proposition 3.7.10 the notation

$$D_x = \{tv \in T_xM \,:\, v \in S_xM,\ t \in [0, \tau(x,v)]\}.$$

Also recall that T_xM has metric $g|_x$ whose volume form is denoted by dT_x.

Lemma 8.1.7 (Second expression for $\mathcal{N}$) *Let (M,g) be a compact non-trapping manifold with strictly convex boundary. Then*

$$(\mathcal{N} f)(x) = 2\int_{D_x} \frac{f(\exp_x(w))}{|w|_g^{n-1}}\, dT_x(w). \tag{8.3}$$

The proof uses the following basic result.

Lemma 8.1.8 (Change of variables) *Let (M,g) and (N,h) be oriented Riemannian manifolds and let $\Phi\colon M \to N$ be a diffeomorphism. Then*

$$\int_N f\, dV_h = \int_M (f\circ\Phi)|\det d\Phi|\, dV_g,$$

where

$$\det d\Phi|_p := \det(\langle f_j, d\Phi|_p e_k\rangle_h),$$

where (e_k) and (f_j) are positively oriented orthonormal bases of T_pM and $T_{\Phi(p)}N$, respectively (the definition of $\det d\Phi$ *is independent of the choice of such bases).*

Exercise 8.1.9 Prove Lemma 8.1.8.

Proof of Lemma 8.1.7 Fix $x \in M^{\text{int}}$. We will change variables in (8.2) from $(t,v) \in \tilde{D}_x := (0, \tau(x,v)] \times S_xM$ to $w = tv \in T_xM$. In fact, define

$$q\colon \tilde{D}_x \to D_x \setminus \{0\},\ \ q(t,v) = tv.$$

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Then q is a diffeomorphism. Noting that the manifold $\tilde{D}_x$ carries the metric $dt^2 + g_x$ and volume form $dt \wedge dS_x$, we can write (8.2) as

$$(\mathcal{N} f)(x) = 2 \int_{\tilde{D}_x} f(\exp_x(q(t,v)))\, dt \wedge dS_x.$$

We wish to use Lemma 8.1.8, which involves the Jacobian $\det dq|_{(t,v)}$. For $v \in S_x M$ let $\{e_1 = v, e_2, \ldots, e_n\}$ be a positive orthonormal basis of $T_x M$. Then $\{\partial_t, e_2, \ldots, e_n\}$ is a positive orthonormal basis of $T_{(t,v)} \tilde{D}_x$. Moreover, $\{e_1, e_2, \ldots, e_n\}$ is a positive orthonormal basis of $T_{tv} D_x \approx T_x M$ with metric g_x and volume form dT_x. Now $dq|_{(t,v)}(\partial_t) = v = e_1$ and $dq|_{(t,v)}(e_j) = te_j$ for $2 \leq j \leq n$. This shows that

$$\det dq|_{(t,v)} = t^{n-1}.$$

We can now change variables using Lemma 8.1.8:

$$\begin{aligned}(\mathcal{N} f)(x) &= 2 \int_{\tilde{D}_x} \frac{f(\exp_x(q(t,v)))}{t^{n-1}} t^{n-1}\, dt \wedge dS_x \\ &= 2 \int_{D_x} \frac{f(\exp_x(w))}{|w|_g^{n-1}}\, dT_x(w).\end{aligned}$$

□

Finally, to determine the Schwartz kernel of $\mathcal{N}$ we would like to make another change of coordinates $y = \exp_x(w)$ in (8.3). This boils down to the property that the exponential map

$$\exp_x \colon D_x \to M$$

should be a diffeomorphism onto M for any fixed $x \in M$. By Proposition 3.8.5 this is always true when (M, g) is a simple manifold.

Lemma 8.1.10 (Schwartz kernel of $\mathcal{N}$) *Let (M, g) be a simple manifold. Then*

$$(\mathcal{N} f)(x) = \int_M \frac{2a(x,y)}{d_g(x,y)^{n-1}} f(y)\, dV^n(y),$$

where the function

$$a(x,y) := \frac{1}{\det(d \exp_x |_{\exp_x^{-1}(y)})}$$

is smooth and positive in $M \times M$ and satisfies $a(x,x) = 1$.

Proof Since $\exp_x \colon D_x \to M$ is a diffeomorphism when (M, g) is simple by Proposition 3.8.5, we can change variables $y = \exp_x(w)$ in (8.3) using Lemma 8.1.8. Since $|w|_g = d_g(x,y)$, we obtain the formula

$$(\mathcal{N} f)(x) = \int_M \frac{2a(x,y)}{d_g(x,y)^{n-1}} f(y)\, dV^n(y),$$

where $a(x, y)$ has the given expression. Now $\exp_x$ is an orientation preserving diffeomorphism, so $\det d \exp_x|_w$ is a smooth positive function of $w \in D_x$ and it also depends smoothly on $x \in M$. Since $d \exp_x|_0 = \mathrm{id}$, we obtain that $a(x, x) = 1$. $\square$

Remark 8.1.11 The function $a(x, y)$ in Lemma 8.1.10 can be studied further by using the fact that $d \exp_x$ can be expressed in terms of Jacobi fields. In fact, let $(e_1 = v, e_2, \dots, e_n)$ be a positive orthonormal basis of $T_x M$. Proposition 3.7.10 implies that

$$\begin{aligned} d \exp_x|_{tv}(e_1) &= \dot{\gamma}_{x,v}(t), \\ d \exp_x|_{tv}(te_k) &= J_k(t) \quad \text{for } 2 \le k \le n, \end{aligned}$$

where $J_k(t)$ is the Jacobi field along $\gamma_{x,v}$ with initial conditions $J_k(0) = 0$ and $D_t J_k(0) = e_k$. Note that $\{e_1(t) = \dot{\gamma}_{x,v}(t), e_2(t), \dots, e_n(t)\}$ is a positive orthonormal basis of $T_{\exp_x(tv)} M$ if we let $e_j(t)$ be the parallel transport of e_j along $\gamma_{x,v}$. Thus we obtain from Lemma 8.1.8 that

$$t^{n-1} \det d \exp_x|_{tv} = \det(\langle e_j(t), J_k(t) \rangle)_{j,k=2}^n =: A_x(v, t).$$

The last expression is an ubiquitous quantity in Riemannian geometry as it dictates how to compute the volume of balls in M of radius r by integrating over $S_x M \times [0, r]$. Note that since M is simple, $\exp_x$ is an orientation-preserving diffeomorphism and therefore $A_x > 0$ for all $(t, v) \in \tilde{D}_x$.

We have now proved that on simple manifolds, the Schwartz kernel of the normal operator $\mathcal{N}$ has a singularity at the diagonal that behaves like $\frac{1}{d_g(x,y)^{n-1}}$. At this point we shall need the following lemma:

Lemma 8.1.12 *In local coordinates, there are smooth functions $G_{jk}(x, y)$ such that $G_{jk}(x, x) = g_{jk}(x)$ and*

$$[d_g(x, y)]^2 = G_{jk}(x, y)(x - y)^j (x - y)^k.$$

Exercise 8.1.13 Prove the lemma. Hint: do a Taylor expansion at x of the function $f(y) = |\exp_x^{-1}(y)|_g^2$.

To show that we have a ΨDO, by Definition 8.1.2 we need to localize matters by considering two cut-off functions $\psi(x)$ and $\phi(y)$ supported in a chart of M^{int} (since M is simple, M^{int} is in fact diffeomorphic to a ball, so one chart will do). Working in local coordinates, if we let

$$\tilde{K}(x, y) := \psi(x) K(x, y) \sqrt{\det g(y)} \phi(y),$$

we need to show that the operator whose Schwartz kernel is $\tilde{K}$ is a ΨDO in $\mathbb{R}^n$. (Recall that in local coordinates $dV^n = \sqrt{\det g(y)}\, dy$.)

By Lemmas 8.1.10 and 8.1.12, one has

$$\tilde{K}(x,y) := \psi(x)\frac{2a(x,y)}{(G_{jk}(x,y)(x-y)^j(x-y)^k)^{\frac{n-1}{2}}}\sqrt{\det g(y)}\phi(y).$$

Since $a(x,y)$ and $G_{jk}(x,y)$ are smooth and ϕ and ψ have compact support, the kernel $k(x,z) := \tilde{K}(x,x-z)$ satisfies estimates of the form

$$\left|\partial_x^\alpha \partial_z^\beta k(x,z)\right| \leq C_{\alpha\beta}|z|^{-n+1-|\beta|}.$$

By the next result (see Stein (1993, VI.4 and VI.7.4)) this implies that the operator with Schwartz kernel $\tilde{K}$ is a ΨDO of order -1.

Proposition 8.1.14 (Schwartz kernel of a ΨDO in $\mathbb{R}^n$) *Let $m < 0$. If $k \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ satisfies*

$$\left|\partial_x^\alpha \partial_z^\beta k(x,z)\right| \leq C_{\alpha\beta N}|z|^{-n-m-|\beta|-N}, \tag{8.4}$$

whenever $n+m+|\beta|+N>0$, then the operator A defined by

$$Af(x) = \int_{\mathbb{R}^n} k(x,x-y)f(y)\,dy$$

belongs to $\Psi^m(\mathbb{R}^n)$ and its full symbol $a \in S^m(\mathbb{R}^n)$ is given by

$$a(x,\xi) = \int_{\mathbb{R}^n} e^{-iz\cdot\xi}k(x,z)\,dz.$$

Conversely, if $A \in \Psi^m(\mathbb{R}^n)$ and if $K(x,y)$ is the Schwartz kernel of A, then $k(x,z) := K(x,x-z)$ satisfies (8.4).

We have now proved that $\mathcal{N} \in \Psi^{-1}(M^{\mathrm{int}})$. The last part of the proof consists in proving ellipticity, which requires that we compute the *principal symbol* of $\mathcal{N}$. We first show that $\mathcal{N} \in \Psi_{\mathrm{cl}}^{-1}(M^{\mathrm{int}})$. It is enough to compute a corresponding expansion in local coordinates. Write

$$\tilde{K}(x,y) = |x-y|^{-(n-1)}\tilde{h}\left(x,|x-y|,\frac{x-y}{|x-y|}\right),$$

where

$$\tilde{h}(x,r,\omega) = \psi(x)\frac{2a(x,x-r\omega)\sqrt{\det g(x-r\omega)}}{(G_{jk}(x,x-r\omega)\omega^j\omega^k)^{\frac{n-1}{2}}}\phi(x-r\omega).$$

Then $\tilde{h}$ is smooth in $\mathbb{R}^n \times [0,\infty) \times S^{n-1}$ (this uses the support properties of ϕ and ψ). Taylor expanding $\tilde{h}$ at $r=0$ leads to the formula

$$\tilde{K}(x,y) = \sum_{j=0}^N \tilde{K}_{-1-j}(x,y) + R_N(x,y),$$

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where

$$\tilde{K}_{-1-j}(x,y) = |x-y|^{-n+j+1} \frac{\partial_r^j h\left(x, 0, \frac{x-y}{|x-y|}\right)}{j!}.$$

By Proposition 8.1.14, $\tilde{K}_{-1-j}$ is the Schwartz kernel of some ΨDO with symbol $\tilde{a}_{-1-j} \in S^{-1-j}(\mathbb{R}^n)$ and R_N corresponds to a symbol in $S^{-N-2}(\mathbb{R}^n)$. This shows that $\mathcal{N}$ is a classical ΨDO, and its principal symbol in local coordinates (computed in the set where $\phi = \psi = 1$) is

$$\begin{aligned}
\tilde{a}_{-1}(x,\xi) &= \int_{\mathbb{R}^n} e^{-iz\cdot\xi} \tilde{K}_{-1}(x, x-z)\, dz \\
&= \int_{\mathbb{R}^n} e^{-iz\cdot\xi} \frac{2\sqrt{\det g(x)}}{(g_{jk}(x) z^j z^k)^{\frac{n-1}{2}}}\, dz \\
&= \int_{\mathbb{R}^n} e^{-iz\cdot g(x)^{-1/2}\xi} \frac{2}{|z|^{n-1}}\, dz \\
&= c_n |\xi|_g^{-1}.
\end{aligned}$$

Here we used the change of variables $z \mapsto g(x)^{-1/2} z$ and the fact that the Fourier transform of $z \mapsto 2|z|^{1-n}$ is $c_n|\xi|^{-1}$. Thus the principal symbol of $\mathcal{N}$ is $c_n|\xi|_g^{-1}$ and $\mathcal{N}$ is elliptic. This concludes the proof of Theorem 8.1.1.

8.2 Surjectivity of I_0^*

Let (M, g) be a compact simple manifold. In this section we prove a fundamental surjectivity result for I_0^* that underpins the successful solution of many geometric inverse problems in two dimensions. Recall from Theorem 5.1.1 the space

$$C_\alpha^\infty(\partial_+ SM) = \{h \in C^\infty(\partial_+ SM) \colon h^\sharp \in C^\infty(SM)\}.$$

Recall the notation ℓ_0 in Exercise 4.1.5. Since

$$(I_0^* h)(x) = \int_{S_x M} h^\sharp(x, v)\, dS_x(v) = (\ell_0^* h^\sharp)(x),$$

we see that I_0^* maps $C_\alpha^\infty(\partial_+ SM)$ to $C^\infty(M)$.

Theorem 8.2.1 *Let (M, g) be a simple manifold. Then the operator*

$$I_0^* \colon C_\alpha^\infty(\partial_+ SM) \to C^\infty(M)$$

is surjective.

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We can reformulate the result in another very useful form. Recall from Lemma 6.1.3 that $\ell_0^* w = \sigma_{n-1} w_0$, where w_0 is the zeroth Fourier mode of $w \in C^\infty(SM)$, and σ_{n-1} is the volume of the $(n-1)$-sphere.

Theorem 8.2.2 (Invariant functions with prescribed zeroth Fourier mode) *Let (M, g) be a manifold with I_0^* surjective. Given any $f \in C^\infty(M)$, there is $w \in C^\infty(SM)$ so that*

$$Xw = 0 \text{ in } SM, \qquad \ell_0^* w = f.$$

Proof Given $f \in C^\infty(M)$, use surjectivity of I_0^* to find $h \in C_\alpha^\infty(\partial_+ SM)$ with $I_0^* h = f$. Writing $w = h^\sharp$, we have $w \in C^\infty(SM)$ since $h \in C_\alpha^\infty(\partial_+ SM)$. Clearly $Xw = 0$, and $\ell_0^* w = \ell_0^* h^\sharp = I_0^* h = f$. □

The proof of Theorem 8.2.1 is based on the following two facts:

- I_0 is injective.
- $I_0^* I_0$ is an elliptic ΨDO.

Here I_0 is a linear operator between infinite-dimensional spaces, and in general surjectivity of the adjoint I_0^* would follow from injectivity of I_0 combined with a suitable closed range condition for I_0. The ellipticity of the normal operator ensures the closed range condition. In the argument below it is convenient to extend $I_0^* I_0$ to an elliptic operator P in a closed manifold and use the fact that P has closed range.

As usual, we consider (M, g) isometrically embedded into a closed manifold (N, g). Since M is simple, by Proposition 3.8.7 there is an open neighbourhood U_1 of M in N such that its closure $M_1 := \overline{U}_1$ is a compact simple manifold. Let $I_{0,1}$ denote the geodesic ray transform associated to (M_1, g) and let $\mathcal{N}_1 = I_{0,1}^* I_{0,1}$.

As in Pestov and Uhlmann (2005) we may cover (N, g) with finitely many simple open sets U_k with $M \subset U_1$, $M \cap \overline{U}_j = \emptyset$ for $j \geq 2$, and consider a partition of unity $\{\varphi_k\}$ subordinate to $\{U_k\}$ so that $\varphi_k \geq 0$, $\operatorname{supp} \varphi_k \subset U_k$ and $\sum \varphi_k^2 = 1$. We pick φ_1 such that $\varphi_1 \equiv 1$ on a neighbourhood of M and compactly supported in U_1. Hence, for $I_{0,k}$, the ray transform associated to $(\overline{U}_k, g)$, we can define

$$Pf := \sum_k \varphi_k (I_{0,k}^* I_{0,k})(\varphi_k f), \qquad f \in C^\infty(N). \tag{8.5}$$

Lemma 8.2.3 *P is an elliptic ΨDO of order -1 in N.*

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Proof Each operator $\mathcal{N}_k := I_{0,k}^* I_{0,k} \colon C_c^\infty(U_k) \to C^\infty(U_k)$ is an elliptic ΨDO of order -1 with principal symbol $c_n|\xi|^{-1}$. By Proposition 8.1.3, the operator P has the principal symbol

$$\sigma_{\mathrm{pr}}(P) = \sum_k \varphi_k \sigma_{\mathrm{pr}}(I_{0,k}^* I_{0,k}) \varphi_k = c_n|\xi|^{-1} \sum_k \varphi_k^2 = c_n|\xi|^{-1}.$$

Thus also P is elliptic. □

Having P defined on a closed manifold is convenient, since one can use standard mapping properties for ΨDOs without having to worry about boundary behaviour. For instance for P defined by (8.5) we have

$$P \colon H^s(N) \to H^{s+1}(N) \qquad \text{for all } s \in \mathbb{R},$$

where $H^s(N)$ denotes the standard L^2 Sobolev space of the closed manifold N.

Remark 8.2.4 There are other natural ways of producing an ambient operator P with the desired properties. Let ψ be a smooth function on N with support contained in U_1 and such that it is equal to 1 near M. Let Δ_g denote the Laplacian of (N, g). Define

$$P := \psi \mathcal{N}_1 \psi + (1-\psi)(1-\Delta_g)^{-1/2}(1-\psi).$$

As we have already mentioned, $\mathcal{N}_1$ is an elliptic ΨDO of order -1 on U_1, and thus P is also an elliptic ΨDO of order -1 in N. Instead of $(1-\Delta_g)^{-1/2}$ we could have used any other invertible self-adjoint elliptic ΨDO of order -1.

Lemma 8.2.5 *The operator P is injective. Moreover, $P \colon C^\infty(N) \to C^\infty(N)$ is a bijection.*

The proof follows from the injectivity of I_0 (Theorem 4.4.1) together with basic properties of elliptic ΨDOs that we recall next. Part (a) gives the existence of a *parametrix* (approximate inverse), part (b) is elliptic regularity, and parts (c) and (d) are related to Fredholm properties.

Proposition 8.2.6 *Let N be a closed manifold, and let $A \in \Psi_{\mathrm{cl}}^m(N)$ be elliptic.*

(a) There is an elliptic $B \in \Psi_{\mathrm{cl}}^{-m}(N)$ so that

$$AB = \mathrm{Id} + R_1,$$
$$BA = \mathrm{Id} + R_2,$$

where R_j are smoothing operators, i.e. they have C^∞ integral kernels and map $H^s(N)$ to $H^t(N)$ boundedly for any $s, t \in \mathbb{R}$.

(b) If $Au = f$ and $f \in H^s(N)$, then $u \in H^{s+m}(N)$.

(c) $\mathrm{Ker}(A) = \{u \in C^\infty(N) \colon Au = 0\}$ *is finite dimensional.*

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(d) Given $f \in C^\infty(N)$, the equation

$$Au = f$$

has a solution $u \in C^\infty(N)$ if and only if $(f, w)_{L^2(N)} = 0$ for all $w \in$ $\mathrm{Ker}(A^*)$.

Proof Part (a) is a standard parametrix construction for elliptic ΨDOs (Hörmander, 1983–1985, Section 18.1). Let us show how the other parts follow from this.

To prove (b), note that if $Au = f$, then by (a),

$$Bf = BAu = u + R_2 u.$$

Thus $u = Bf - R_2 u$, where $Bf \in H^{s+m}(N)$ and $R_2 u \in C^\infty(N)$, so $u \in H^{s+m}(N)$.

To prove (c), note that if $Au = 0$, then by (a),

$$0 = BAu = (\mathrm{Id} + R_2)u.$$

Now R_2 is compact on $L^2(N)$ (it is bounded $L^2(N) \to H^1(N)$ and the embedding $H^1(N) \to L^2(N)$ is compact). Thus the kernel of $\mathrm{Id} + R_2$ on $L^2(N)$ is finite dimensional, and hence so is $\mathrm{Ker}(A)$.

Finally, to prove (d), consider the operator A acting between the spaces

$$A\colon H^m(N) \to Y := \{f \in L^2(N) \colon (f, w)_{L^2(N)} = 0 \quad \text{for all } w \in \mathrm{Ker}(A^*)\}.$$

Equip Y with the $L^2(N)$ norm. If $u \in H^m(N)$ then Au is indeed in Y, since $(Au, w)_{L^2(N)} = (u, A^* w)_{L^2(N)} = 0$ for any $w \in \mathrm{Ker}(A^*)$. We wish to prove that A is surjective.

- A has dense range: if $f \in Y$ satisfies $(Au, f)_{L^2(N)} = 0$ for all $u \in H^m(N)$, then $(u, A^* f)_{L^2(N)} = 0$ for $u \in H^m(N)$ that yields $A^* f = 0$. Thus $f \in \mathrm{Ker}(A^*)$, and by the definition of Y one has $(f, f)_{L^2(N)} = 0$, showing that $f = 0$.
- A has closed range: if $u_j \in H^m(N)$ and $Au_j \to f$ in Y, then by (a) one has $u_j + R_2 u_j \to Bf$ in $H^m(N)$. Since R_2 is compact on $H^m(N)$, some subsequence $(R_2 u_{j_k})$ converges in $H^m(N)$. Then (u_{j_k}) converges in $H^m(N)$ to some $u \in H^m(N)$. It follows that $f = Au$.

By the above two points $A\colon H^m(N) \to Y$ is surjective. Part (d) follows from this and part (b). □

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Proof of Lemma 8.2.5 Since P is elliptic, any element in the kernel of P must be smooth. Let f be such that $Pf = 0$, and write

$$0 = (Pf, f)_{L^2(N)} = \sum_k \left(I_{0,k}^* I_{0,k}(\varphi_k f), \varphi_k f\right)_{L^2(\overline{U}_k)}$$
$$= \sum_k \|I_{0,k}(\varphi_k f)\|^2_{L^2_\mu(\partial_+ S\overline{U}_k)}.$$

Hence $I_{0,k}(\varphi_k f) = 0$ for each k. Using injectivity of I_0 on simple manifolds it follows that $\varphi_k f = 0$ for each k and thus $f = 0$.

We have proved that P is injective. Since P is self-adjoint, P^* is also injective. Then surjectivity follows from Proposition 8.2.6(d). □

Exercise 8.2.7 Prove that $P\colon H^s(N) \to H^{s+1}(N)$ is a homeomorphism for all $s \in \mathbb{R}$.

We are now ready to prove the main result of this section.

Proof of Theorem 8.2.1 Let $h \in C^\infty(M)$ be given, and extend it smoothly to a smooth function in N, still denoted by h. By Lemma 8.2.5 there is a unique $f \in C^\infty(N)$ such that $Pf = h$. Let $w_1 := I_{0,1}(\varphi_1 f)$. Clearly $w_1^\sharp|_{SM} \in C^\infty(SM)$, and we let $w := w_1^\sharp|_{\partial_+ SM}$. We must have

$$w^\sharp = w_1^\sharp|_{SM},$$

since both functions are constant along geodesics and they agree on $\partial_+ SM$. Hence $w \in C_\alpha^\infty(\partial_+ SM)$. To complete the proof we must check that $I_0^* w = h$. To this end, we write for $x \in M$,

$$\begin{aligned}
(I_0^* w)(x) &= \int_{S_x M} w^\sharp(x, v)\, dS_x(v) \\
&= \int_{S_x M} w_1^\sharp(x, v)\, dS_x(v) \\
&= (I_{0,1}^* w_1)(x) \\
&= I_{0,1}^* I_{0,1}(\varphi_1 f)(x) \\
&= Pf(x) \\
&= h(x),
\end{aligned}$$

where in the penultimate line we used (8.5) and that $x \in M$. □

Remark 8.2.8 It turns out that it is possible to give a proof of Theorem 8.2.1 without the need to extend the normal operator to a larger closed manifold N. In order to do this, one requires finer mapping properties for $\mathcal{N}$. Let ρ denote

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a positive boundary defining function; it was shown in Monard et al. (2019, Theorem 4.4) that

$$\mathcal{N}\colon \rho^{-1/2}C^\infty(M) \to C^\infty(M)$$

is a *bijection*. This can be combined with an additional mapping property for I established in Monard et al. (2021b) for any non-trapping manifold with strictly convex boundary, namely

$$I\colon \rho^{-1/2}C^\infty(SM) \to C^\infty_\alpha(\partial_+ SM).$$

These two assertions show that given $h \in C^\infty(M)$, the function

$$w := I_0\mathcal{N}^{-1}h \in C^\infty_\alpha(\partial_+ SM)$$

and satisfies $I_0^* w = h$. Knowing the precise mapping properties of $\mathcal{N}$ and when it can be inverted is of fundamental importance when addressing statistical questions about inversion. We refer to Monard et al. (2019, 2021b) for more details. For the purposes of this text the proof of Theorem 8.2.1 as presented is more than sufficient.

8.3 Stability Estimates Based on the Normal Operator

In this section we will explain how we can derive stability estimates for the normal operator $\mathcal{N}$ using some of the tools developed, in particular, the existence of a parametrix as in Proposition 8.2.6. We will keep the notation and set up from the previous section, so that (M, g) is a compact simple manifold and U_1 is an open neighbourhood of M in the closed manifold N whose closure $\overline{U}_1$ is a compact simple manifold.

We start by noticing that a forward estimate for $\mathcal{N}$ follows easily from the mapping properties of the ΨDO P. Indeed, let $r_M\colon L^2(N) \to L^2(M)$ denote restriction to M and let $e_M\colon L^2(M) \to L^2(N)$ denote extension by zero. Both operators are bounded and dual to each other. From (8.2) one easily obtains the following truncation formula

$$\mathcal{N} = r_M P e_M \quad \text{in } L^2(M). \tag{8.6}$$

Exercise 8.3.1 Prove (8.6)

Since $P\colon L^2(N) \to H^1(N)$ and $r_M\colon H^1(N) \to H^1(M)$, this gives immediately the mapping property

$$\mathcal{N}\colon L^2(M) \to H^1(M),$$

and hence a forward estimate $\|\mathcal{N} f\|_{H^1(M)} \le C\|f\|_{L^2(M)}$.

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In order to derive the stability estimate for the normal operator there is a small price to pay: we shall measure the L^2-norm of f on M, but we shall consider the H^1-norm of the normal operator $\mathcal{N}_1$ defined on the slightly larger manifold U_1. This is to avoid the boundary effects as described in Remark 8.2.8 and the need to use Hörmander spaces adapted to the appropriate transmission condition (cf. Monard et al. (2019)). We will prove:

Theorem 8.3.2 (Stefanov and Uhlmann, 2004) *There is a constant $C > 0$ such that for any function $f \in L^2(M)$,*

$$C^{-1}\|f\|_{L^2(M)} \leq \|\mathcal{N}_1 f\|_{H^1(U_1)} \leq C\|f\|_{L^2(M)}.$$

Here we regard $\mathcal{N}_1 \colon L^2(M) \to H^1(U_1)$ simply extending f by zero to U_1.

Proof We have already proved the inequality on the right, so we now focus on the stability estimate on the left. The injectivity of I_0 implies that $P \colon H^s(N) \to H^{s+1}(N)$ is a homeomorphism simply by extending the proof of Lemma 8.2.5 to Sobolev spaces, cf. Exercise 8.2.7. Thus

$$\|f\|_{L^2(M)} \lesssim \|Pf\|_{H^1(N)}.$$

But from the definition of P in (8.5) we see that

$$Pf = \varphi_1 \mathcal{N}_1 f,$$

where φ_1 is such that $\varphi_1 \equiv 1$ on a neighbourhood of M and compactly supported in U_1 (with f extended by zero). It follows that

$$\|Pf\|_{H^1(N)} \lesssim \|\mathcal{N}_1 f\|_{H^1(U_1)},$$

and the theorem is proved. □

It was shown in Stefanov and Uhlmann (2004) and Sharafutdinov et al. (2005) that for a simple manifold s-injectivity of I_m implies stability estimates for the normal operator. As before, this is based on the fact that $\mathcal{N}^m := I_m^* I_m$ is an elliptic pseudodifferential operator acting on solenoidal tensor fields. We shall not prove these results here; instead we give a brief account of them. Since I_1 is always s-injective for simple manifolds we have:

Theorem 8.3.3 *Let (M, g) be simple. There is a constant $C > 0$ such that for any 1-form f in $L^2(S^1(T^*M))$, we have*

$$C^{-1}\left\|f^s\right\|_{L^2(S^1(T^*M))} \leq \|\mathcal{N}_1^1 f\|_{H^1(U_1)} \leq C\left\|f^s\right\|_{L^2(S^1(T^*M))}.$$

A sharp stability estimate for $\mathcal{N}^2$, assuming that I_2 is known to be s-injective, was proved in Stefanov (2008):

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Theorem 8.3.4 *Let (M, g) be simple and assume that I_2 is s-injective. There is a constant $C > 0$ such that for any symmetric 2-tensor field f in $L^2(S^2(T^*M))$,*

$$C^{-1}\|f^s\|_{L^2(S^2(T^*M))} \leq \|\mathcal{N}_1^2 f\|_{H^1(U_1)} \leq C\|f^s\|_{L^2(S^2(T^*M))}.$$

We refer to Assylbekov and Stefanov (2020) for recent sharp stability estimates for I_m using these results.

Remark 8.3.5 One can also consider the normal operator and stability on compact non-trapping surfaces with strictly convex boundary, but when conjugate points are present. This situation is studied in detail in Monard et al. (2015). It turns out that I_0 is a Fourier integral operator of order $-1/2$, but if there is a pair of interior conjugate points then $I_0^* I_0$ is not a pseudodifferential operator anymore. Moreover, I_0 has an infinite-dimensional microlocal kernel, and some singularities of functions f in the microlocal kernel cannot be recovered from the knowledge of $I_0 f$. This implies that even if I_0 were injective (like it is for radial sound speeds satisfying the Herglotz condition by Theorem 2.4.1), the recovery of f from $I_0 f$ will be highly unstable if conjugate points are present. The instability issue is also discussed in Koch et al. (2021).

8.4 The Normal Operator with a Matrix Weight

Virtually everything that we have done in this chapter so far can be upgraded to include an invertible matrix weight. Let (M, g) be a compact non-trapping manifold with strictly convex boundary and let $\mathbb{W}\colon SM \to GL(m, \mathbb{C})$ be a smooth invertible matrix function, called a *weight*.

Recall from Definition 5.4.5 that the geodesic X-ray transform with matrix weight $\mathbb{W}$ is the operator $I_{\mathbb{W}}\colon C^\infty(SM, \mathbb{C}^m) \to C^\infty(\partial_+ SM, \mathbb{C}^m)$ defined by

$$I_{\mathbb{W}} f(x, v) = \int_0^{\tau(x,v)} (\mathbb{W} f)(\varphi_t(x, v))\, dt, \qquad (x, v) \in \partial_+ SM.$$

By Remark 5.4.7, $I_{\mathbb{W}}$ is bounded $L^2(SM, \mathbb{C}^m) \to L^2(\partial_+ SM, \mathbb{C}^m)$. To compute the adjoint we use the L^2_μ space: the adjoint of

$$I_{\mathbb{W}}\colon L^2(SM, \mathbb{C}^m) \to L^2_\mu(\partial_+ SM, \mathbb{C}^m)$$

is the bounded operator $I_{\mathbb{W}}^*\colon L^2_\mu(\partial_+ SM, \mathbb{C}^m) \to L^2(SM, \mathbb{C}^m)$ given by (see Remark 5.4.7)

$$I_{\mathbb{W}}^* h = \mathbb{W}^* h^\sharp.$$

We will be interested in the weighted transform $I_{\mathbb{W},0}$ acting on 0-tensors.

Definition 8.4.1 The matrix weighted X-ray transform on 0-tensors is the operator

$$I_{\mathbb{W},0}\colon C^\infty(M,\mathbb{C}^m) \to C^\infty(\partial_+ SM,\mathbb{C}^m), \quad I_{\mathbb{W},0} := I_{\mathbb{W}} \circ \ell_0.$$

As in Lemma 4.1.4 one has

$$(I_{\mathbb{W},0}^* h)(x) = \int_{S_x M} \mathbb{W}^* h^\sharp(x,v)\, dS_x(v).$$

The *normal operator*

$$\mathcal{N}_{\mathbb{W}} := I_{\mathbb{W},0}^* I_{\mathbb{W},0}\colon L^2(M,\mathbb{C}^m) \to L^2(M,\mathbb{C}^m)$$

is now an elliptic ΨDO.

Theorem 8.4.2 ($\mathcal{N}_{\mathbb{W}}$ is an elliptic ΨDO) *Let (M,g) be a simple manifold and let $\mathbb{W} \in C^\infty(SM, GL(m,\mathbb{C}))$. Then $\mathcal{N}_{\mathbb{W}} = I_{\mathbb{W},0}^* I_{\mathbb{W},0}$ is a classical elliptic ΨDO on M^{int} of order -1.*

Proof We follow the argument in Section 8.1. From the definitions

$$\begin{aligned}
\mathcal{N}_{\mathbb{W}} f(x) &= \int_{S_x M} \mathbb{W}^*(x,v)(I_{\mathbb{W},0} f)^\sharp(x,v)\, dS_x(v)\\
&= \int_{S_x M} \mathbb{W}^*(x,v) \int_{-\tau(x,-v)}^{\tau(x,v)} \mathbb{W}(\varphi_t(x,v)) f(\gamma_{x,v}(t))\, dt\, dS_x(v)\\
&= \int_{S_x M} \int_0^{\tau(x,v)} \mathbb{W}^*(x,v)\mathbb{W}(\varphi_t(x,v)) f(\gamma_{x,v}(t))\, dt\, dS_x(v)\\
&\quad + \int_{S_x M} \int_0^{\tau(x,v)} \mathbb{W}^*(x,-v)\mathbb{W}(\varphi_{-t}(x,-v)) f(\gamma_{x,v}(t))\, dt\, dS_x(v).
\end{aligned}$$

Following the arguments in Lemmas 8.1.7 and 8.1.10, we have

$$\begin{aligned}
\mathcal{N}_{\mathbb{W}} f(x) &= \int_{D_x} \frac{\mathbb{W}^*\left(x, \frac{w}{|w|}\right) \mathbb{W}\left(\varphi_{|w|}\left(x, \frac{w}{|w|}\right)\right) f(\exp_x(w))}{|w|^{n-1}}\, dT_x(w)\\
&\quad + \int_{D_x} \frac{\mathbb{W}^*\left(x, -\frac{w}{|w|}\right) \mathbb{W}\left(\varphi_{-|w|}\left(x, -\frac{w}{|w|}\right)\right) f(\exp_x(w))}{|w|^{n-1}}\, dT_x(w)\\
&= \int_M \frac{A_{\mathbb{W}}(x, v(x,y), y, w(x,y))}{d_g(x,y)^{n-1}} f(y)\, dV^n(y),
\end{aligned}$$

where $A_{\mathbb{W}}(x,v,y,w)$ (with $v \in S_x M$ and $w \in S_y M$) is the matrix function

$$A_{\mathbb{W}}(x,v,y,w) := \frac{\mathbb{W}^*(x,v)\mathbb{W}(y,w) + \mathbb{W}^*(x,-v)\mathbb{W}(y,-w)}{\det(d\exp_x|_{\exp_x^{-1}(y)})},$$

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and

$$v(x,y) := \frac{\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|}, \qquad w(x,y) := \nabla_y d_g(x,y).$$

Here $A_{\mathbb{W}} \in C^\infty(SM \times SM)$, which shows that $A_{\mathbb{W}}(x, v(x,y), y, w(x,y))$ is bounded in $M^{\text{int}} \times M^{\text{int}}$ and smooth away from the diagonal.

Having computed the Schwartz kernel of $\mathcal{N}_{\mathbb{W}}$, we move to local coordinates and choose cut-off functions $\phi, \psi \in C_c^\infty(M^{\text{int}})$. After multiplying by cutoffs, the Schwartz kernel of $\mathcal{N}_{\mathbb{W}}$ has the expression

$$\begin{aligned}\tilde{K}_{\mathbb{W}}(x,y) &= \frac{\psi(x) A_{\mathbb{W}}(x, v(x,y), y, w(x,y))\sqrt{\det g(y)}\phi(y)}{(G_{jk}(x,y)(x-y)^j(x-y)^k)^{\frac{n-1}{2}}} \\ &= |x-y|^{-(n-1)}\tilde{h}_{\mathbb{W}}\left(x, |x-y|, \frac{x-y}{|x-y|}\right),\end{aligned}$$

where

$$\begin{aligned}&\tilde{h}_{\mathbb{W}}(x,r,\omega) = \psi(x) \\ &\quad \times \frac{A_{\mathbb{W}}(x, v(x, x-r\omega), x-r\omega, w(x, x-r\omega))\sqrt{\det g(x-r\omega)}}{(G_{jk}(x, x-r\omega)\omega^j\omega^k)^{\frac{n-1}{2}}}\phi(x-r\omega).\end{aligned}$$

We claim that $\tilde{h}_{\mathbb{W}}$ is smooth in $\mathbb{R}^n \times [0,\infty) \times S^{n-1}$. To prove this, it is enough to show that the functions $\tilde{v}(x,r,\omega) = v(x, x-r\omega)$ and $\tilde{w}(x,r,\omega) = w(x, x-r\omega)$ are smooth up to $r = 0$.

Let $U \subset \mathbb{R}^n$ be the open subset where the local coordinates are defined, and let g also denote the Riemannian metric on U. Fix $x \in U$; we are interested in the behaviour of $y = \exp_x(t\hat{v}) = \gamma_{x,\hat{v}}(t)$ for small $|t|$, where $\hat{v} \in S_x U$. Note that the map $(t,\hat{v}) \mapsto y$ is smooth. Hence, the function $m(t,\hat{v};x) := (\gamma_{x,\hat{v}}(t) - x)/t$ with $m(0,\hat{v};x) = \hat{v}$ is also smooth. We may introduce new variables $(r,\omega) \in \mathbb{R} \times S^{n-1}$ such that

$$r = t|m(t,\hat{v};x)| \ \text{ and } \ \omega = -\frac{m(t,\hat{v};x)}{|m(t,\hat{v};x)|}.$$

Then $x - r\omega = \gamma_{x,\hat{v}}(t)$. It is straightforward to check that the Jacobian of the change of coordinates $(t,\hat{v}) \mapsto (r,\omega)$ is non-zero for $t = 0$ and thus by the inverse function theorem and the fact that $(0,\hat{v}) \mapsto (0,\omega)$ is injective (cf. Lemma 11.2.6 for a related formulation) there is δ small enough such that this change of coordinates is a diffeomorphism from $(-\delta,\delta) \times S_x U$ onto its image. Thus we have smooth inverse functions $t(r,\omega)$ and $\hat{v}(r,\omega)$ for r small enough and $\omega \in S^{n-1}$.

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To complete the proof that $\tilde{h}_{\mathbb{W}}$ is smooth, observe that $\tilde{v}(x,r,\omega) = \hat{v}(r,\omega)$ and

$$\tilde{w}(x,r,\omega) = d\exp_x|_{t(r,\omega)\hat{v}(r,\omega)}(\hat{v}(r,\omega)),$$

and thus both are smooth as functions of (r,ω) as desired. Now the same argument as in the end of Section 8.1 implies that $\mathcal{N}_{\mathbb{W}} \in \Psi_{\mathrm{cl}}^{-1}(M^{\mathrm{int}})$. Ellipticity follows from Exercise 8.4.3 below. □

Exercise 8.4.3 Show that the principal symbol of $\mathcal{N}_{\mathbb{W}}$ in local coordinates as above is given by

$$\begin{aligned}\sigma_{\mathrm{pr}}(\mathcal{N}_{\mathbb{W}})(x,\xi) = \int_{\mathbb{R}^n} e^{-iz\cdot\xi}\frac{\sqrt{\det g(x)}}{|z|_g^{n-1}} \times (\mathbb{W}^*(x,z/|z|_g)\mathbb{W}(x,z/|z|_g)\\ + \mathbb{W}^*(x,-z/|z|_g)\mathbb{W}(x,-z/|z|_g))\,dz.\end{aligned}$$

Using that $\mathbb{W}$ is invertible, conclude that $\mathcal{N}_{\mathbb{W}}$ is elliptic. What happens if $\mathbb{W}$ is not invertible? Show that if $\mathbb{W}$ takes values in the unitary group, the principal symbol is $c_n|\xi|_g^{-1}\mathrm{Id}$.

With this result in hand, Theorem 8.2.1 can be upgraded to the following.

Theorem 8.4.4 *Let (M,g) be a simple manifold. Then $I_{\mathbb{W},0}$ is injective on $L^2(M,\mathbb{C}^m)$ if and only if*

$$I_{\mathbb{W},0}^*\colon C_\alpha^\infty(\partial_+SM,\mathbb{C}^m) \to C^\infty(M,\mathbb{C}^m)$$

is surjective.

Proof Let $f \in L^2(M,\mathbb{C}^m)$ be such that $I_{\mathbb{W},0}f = 0$. Consider a slightly larger simple manifold $\widetilde{M}$ engulfing M and extend $\mathbb{W}$ smoothly to it. Extending f by zero to $\widetilde{M}$ we see that

$$I_{\widetilde{\mathbb{W}},0}f = 0,$$

and thus $\mathcal{N}_{\widetilde{\mathbb{W}}}f = 0$. By Theorem 8.4.2, $\mathcal{N}_{\widetilde{\mathbb{W}}}$ is elliptic and hence f is smooth in the interior of $\widetilde{M}$ and hence on M. Assume now that $I_{\mathbb{W},0}^*$ is surjective. Then there exists $h \in C_\alpha^\infty(\partial_+SM,\mathbb{C}^m)$ such that $I_{\mathbb{W},0}^*h = f$. Now write

$$0 = (I_{\mathbb{W},0}f,h) = \left(f, I_{\mathbb{W},0}^*h\right) = (f,f),$$

and thus $f = 0$.

Assume now that $I_{\mathbb{W},0}$ is injective. We wish to show that $I_{\mathbb{W},0}^*$ is surjective. This part of the proof proceeds exactly as the proof of Theorem 8.2.1. We construct an elliptic operator $P\colon C^\infty(N,\mathbb{C}^m) \to C^\infty(N,\mathbb{C}^m)$, and we show it is a bijection by showing first that it has trivial kernel. The surjectivity of P implies the surjectivity of $I_{\mathbb{W},0}^*$ exactly as in the proof of Theorem 8.2.1. □

Exercise 8.4.5 Fill in the details of the proof of Theorem 8.4.4.

Let us state explicitly the following rephrasing of Theorem 8.4.4 that will be useful later on.

Corollary 8.4.6 *Let (M, g) be a simple manifold with $I_{\mathbb{W},0}$ injective. Given $f \in C^\infty(M, \mathbb{C}^m)$ there exists $u \in C^\infty(SM, \mathbb{C}^m)$ such that*

$$\begin{cases} Xu + \mathcal{A}u = 0, \\ \qquad \ell_0^* u = f \end{cases}$$

where $\mathcal{A} = -X(\mathbb{W}^*)(\mathbb{W}^*)^{-1}$ and $\ell_0^* u = \int_{S_x M} u(x, v)\, dS_x(v)$.

Proof By Theorem 8.4.4 there is $h \in C_\alpha^\infty(\partial_+ SM, \mathbb{C}^m)$ such that $\ell_0^* \mathbb{W}^* h^\sharp = f$. We let $u := \mathbb{W}^* h^\sharp \in C^\infty(SM, \mathbb{C}^m)$. Since $Xh^\sharp = 0$, the function u satisfies

$$Xu = X(\mathbb{W}^*)h^\sharp = -\mathcal{A}u,$$

and the corollary follows. $\square$