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ARENS REGULARITY AND AMENABILITY OF LAU PRODUCT OF BANACH ALGEBRAS DEFINED BY A BANACH ALGEBRA MORPHISM

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Abstract

Given a morphism T from a Banach algebra $\mathcal{B}$ to a commutative Banach algebra $\mathcal{A}$, a multiplication is defined on the Cartesian product space $\mathcal{A} \times \mathcal{B}$ perturbing the coordinatewise product resulting in a new Banach algebra $\mathcal{A} \times_T \mathcal{B}$. The Arens regularity as well as amenability (together with its various avatars) of $\mathcal{A} \times_T \mathcal{B}$ are shown to be stable with respect to T.

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1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras, let $T : \mathcal{B} \to \mathcal{A}$ be an algebra homomorphism, and let $\mathcal{A}$ be assumed commutative throughout. (It suffices to assume that the range of T is contained in the centre of $\mathcal{A}$.) We define a product on $\mathcal{A} \times \mathcal{B}$ as follows:

$$(a, b)(a', b') = (aa' + T(b)a' + T(b')a, bb') \quad ((a, b),\ (a', b') \in \mathcal{A} \times \mathcal{B}).$$

Then the Cartesian product space $\mathcal{A} \times \mathcal{B}$ is an associative, not necessarily commutative, algebra with this product. We denote $\mathcal{A} \times \mathcal{B}$ with this product by $\mathcal{A} \times_T \mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras and if $\|T\| \le 1$, then $\mathcal{A} \times_T \mathcal{B}$ is a Banach algebra with the norm

$$\|(a, b)\| = \|a\| + \|b\| \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

We note that $\mathcal{A}$ is a closed ideal of $\mathcal{A} \times_T \mathcal{B}$ and $(\mathcal{A} \times_T \mathcal{B})/\mathcal{A}$ is isometrically isomorphic to $\mathcal{B}$. When $T = 0$, this gives the coordinatewise product. Thus $\times_T$ is the perturbation of the coordinatewise product induced by T. Besides giving a new

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method of constructing Banach algebras, the product $\times_T$ has relevance in at least the following two situations.

(a) Let $\mathcal{A}$ be unital, and let $\theta : \mathcal{B} \to \mathbb{C}$ be a multiplicative linear functional. Define $T : \mathcal{B} \to \mathcal{A}$ as $T(x) = \theta(x)e$ $(x \in \mathcal{B})$. Then the above product coincides with the product investigated by Lau [11]. This product is of relevance in Lau algebras arising in harmonic analysis and providing an abstract setting for several Banach algebras of harmonic analysis. A *Lau algebra* (called an F-algebra in [11]) is a pair $(\mathcal{A}, M)$, where $\mathcal{A}$ is a complex Banach algebra; M is a W^*-algebra and $\mathcal{A}$ is the predual of M such that the identity e of M is a multiplicative linear functional on $\mathcal{A}$. Examples of Lau algebras include the group algebra $L^1(G)$, the measure algebra $M(G)$ and the Fourier algebra $A(G)$ of a locally compact group G [11]; the Fourier–Stieltjes algebra of a topological group [12]; the measure algebra $M(S)$ of a locally compact semigroup or a hypergroup; as well as the predual algebra of a Hopf von Neumann algebra [16]. The Banach algebra $M \times_T \mathcal{A}$, $T : \mathcal{A} \to M$ being $T(x) = e(x)e$, encodes a Lau algebra $(\mathcal{A}, M)$.

(b) In the framework of Brown–Douglas–Fillmore theory [4], given an extension of compact operators $\mathcal{K}(H)$ on a separable Hilbert space H by the abelian C^*-algebra $C(X)$ of continuous functions on a compact space X manifested by a short exact sequence $0 \longrightarrow \mathcal{K}(H) \xrightarrow{i} \mathcal{A} \xrightarrow{\varphi} C(X) \longrightarrow 0$ of C^*-algebras, the above product produces a Banach $*$-algebra $C(X) \times_\varphi \mathcal{A}$ encoding the extension. This aspect of the extension is yet to be explored.

The purpose of the present paper is to determine the Gel'fand space of $\mathcal{A} \times_T \mathcal{B}$ which turns out to be nontrivial even though $\mathcal{A} \times_T \mathcal{B}$ need not be commutative and to discuss the Arens regularity as well as the amenability of $\mathcal{A} \times_T \mathcal{B}$. These topics are central to the general theory of Banach algebras [3]; and are of current relevance [3, 5, 6]. Arens [1, 2] showed that the given product on a Banach algebra $\mathcal{A}$ induces two canonical products on the second dual $\mathcal{A}''$ of $\mathcal{A}$; and $\mathcal{A}$ is *Arens regular* if these two products coincide. We prove in Theorem 3.1 that for an Arens regular commutative Banach algebra $\mathcal{A}$, $\mathcal{A} \times_T \mathcal{B}$ is Arens regular if and only if $\mathcal{B}$ is Arens regular, thereby showing Arens regularity to be independent of T. A Banach algebra is *amenable* if any bounded derivation of $\mathcal{A}$ into the dual E' of a Banach $\mathcal{A}$-module E is *inner* in the sense that it is of the form δ_x, $\delta_x(a) = a \cdot x - x \cdot a$ $(a \in \mathcal{A})$, for some $x \in E'$. Replacing E' by the dual $\mathcal{A}'$ (which is an $\mathcal{A}$-module) results in *weak amenability*. There are closely related notions like approximate amenability, weak approximate amenability, and cyclic amenability. It is shown in Theorem 4.1 that $\mathcal{A} \times_T \mathcal{B}$ is amenable (weakly amenable, approximately amenable, weak approximately amenable, cyclic amenable, approximate cyclic amenable, respectively) if and only if $\mathcal{A}$ and $\mathcal{B}$ also are. A localised version is also discussed in Theorem 4.2. These results provide analogues for the perturbed product $\times_T$ of the results for the Lau product proved recently in [13]. In fact, the present paper and the arguments herein are inspired by [13]. The message of the paper is that the notions of Arens regularity and amenability are fairly stable with respect to Cartesian product.

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2. Gel'fand space and geometrisation of $\times_T$

THEOREM 2.1. *Let $\mathcal{A}$ be a commutative Banach algebra with the Gel'fand space $\Delta(\mathcal{A})$, let $\mathcal{B}$ be a Banach algebra, and let $T : \mathcal{B} \to \mathcal{A}$ be a homomorphism with $\|T\| \le 1$. Then $\Delta(\mathcal{A} \times_T \mathcal{B}) = \{(\varphi, \varphi \circ T) : \varphi \in \Delta(\mathcal{A})\} \,\dot{\cup}\, \{(0, \psi) : \psi \in \Delta(\mathcal{B})\}$, a disjoint union, and $E := \{(\varphi, \varphi \circ T) : \varphi \in \Delta(\mathcal{A})\}$ and $F := \{(0, \psi) : \psi \in \Delta(\mathcal{B})\}$ are closed in $\Delta(\mathcal{A} \times_T \mathcal{B})$.*

PROOF. It is easy to see that

$$\{(\varphi, \varphi \circ T) : \varphi \in \Delta(\mathcal{A})\} \cup \{(0, \psi) : \psi \in \Delta(\mathcal{B})\} \subset \Delta(\mathcal{A} \times_T \mathcal{B}).$$

Conversely, let $(\varphi, \psi) \in \Delta(\mathcal{A} \times_T \mathcal{B})$. Then

$$(\varphi, \psi)[(a, b)(a', b')] = (\varphi, \psi)(a, b)(\varphi, \psi)(a', b')$$

gives

$$\varphi(aa' + T(b')a + T(b)a') + \psi(bb') = \varphi(a)\varphi(a') + \varphi(a)\psi(b') + \varphi(a')\psi(b) + \psi(b)\psi(b').$$

Taking $b = b' = 0$, we get $\varphi(aa') = \varphi(a)\varphi(a')$, and taking $a = a' = 0$, we get $\psi(bb') = \psi(b)\psi(b')$.

First let $\varphi \ne 0$. Then

$$\varphi(T(b))\varphi(a') + \varphi(T(b'))\varphi(a) = \varphi(a')\psi(b) + \varphi(a)\psi(b').$$

Taking $a = a'$ and $b = b'$, we get $\varphi(T(b)) = \psi(b)$. Hence $\psi = \varphi \circ T$.

Suppose that $\varphi = 0$. Then $(0, \psi) \in \Delta(\mathcal{A} \times \mathcal{B})$.

Thus $E \,\dot{\cup}\, F = \Delta(\mathcal{A} \times_T \mathcal{B})$. Let $(\varphi_0, \varphi_0 \circ T) \in E$. Then there exists $a \in \mathcal{A}$ such that $\varphi_0(a) \ne 0$. Let $\epsilon = |\varphi_0(a)|/2$, and $U = U((\varphi_0, \varphi_0 \circ T), \epsilon, (a, 0))$. Then

$$\begin{aligned} U &= \{(\varphi, \psi) \in \Delta(\mathcal{A} \times_T \mathcal{B}) : |(\varphi, \psi)(a, 0) - (\varphi_0, \varphi_0 \circ T)(a, 0)| < \epsilon\} \\ &= \{(\varphi, \psi) \in \Delta(\mathcal{A} \times_T \mathcal{B}) : |\varphi(a) - \varphi_0(a)| < \epsilon\}. \end{aligned}$$

If $(0, \psi) \in U$, then $|\varphi_0(a)| < \epsilon$, which is not possible. Hence $U \subset E$. This shows that E is open in $\Delta(\mathcal{A} \times_T \mathcal{B})$ and hence F is closed in $\Delta(\mathcal{A} \times_T \mathcal{B})$.

Let $(0, \psi) \in \Delta(\mathcal{A} \times_T \mathcal{B})$ be in the closure of E. Then there is a net $((\varphi_\alpha, \varphi_\alpha \circ T)) \subset E$ converging to $(0, \psi)$, that is,

$$\varphi_\alpha(a) + \varphi_\alpha \circ T(b) \to \psi(b) \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

In particular, taking $b = 0$, $\varphi_\alpha(a) \to 0$ $(a \in \mathcal{A})$. Taking $a = 0$, $\varphi_\alpha \circ T(b) \to \psi(b)$ $(b \in \mathcal{B})$. Since $\varphi_\alpha \to 0$, we have $\varphi_\alpha \circ T \to 0$, that is, $\psi = 0$. This is a contradiction. Hence E is closed in $\Delta(\mathcal{A} \times_T \mathcal{B})$. □

COROLLARY 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras, and let $T : \mathcal{B} \to \mathcal{A}$ be an algebra homomorphism with $\|T\| \le 1$. Then $\mathcal{A} \times_T \mathcal{B}$ is semisimple if and only if both $\mathcal{A}$ and $\mathcal{B}$ are semisimple.*

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Proof. Let $\mathcal{A} \times_T \mathcal{B}$ be semisimple. Let $a \in \mathcal{A}$ be such that $\varphi(a) = 0$ $(\varphi \in \Delta(\mathcal{A}))$. Then $(\varphi, \varphi \circ T)(a, 0) = 0$ $(\varphi \in \Delta(\mathcal{A}))$ and $(0, \psi)(a, 0) = 0$ $(\psi \in \Delta(\mathcal{B}))$. Since $\mathcal{A} \times_T \mathcal{B}$ is semisimple, $a = 0$. Therefore $\mathcal{A}$ is semisimple. It also follows by an analogous argument that $\mathcal{B}$ is semisimple.

Let $(a, b) \in \mathcal{A} \times_T \mathcal{B}$ be such that $(\varphi, \psi)(a, b) = 0$ $((\varphi, \psi) \in \Delta(\mathcal{A} \times_T \mathcal{B}))$. In particular, $\psi(b) = (0, \psi)(a, b) = 0$ $(\psi \in \Delta(\mathcal{B}))$. Since $\mathcal{B}$ is semisimple, it follows that $b = 0$. Since $b = 0$, we have $\varphi(a) = 0$ $(\varphi \in \Delta(\mathcal{A}))$. Since $\mathcal{A}$ is semisimple, $a = 0$. Hence $\mathcal{A} \times_T \mathcal{B}$ is semisimple. □

Since $\mathcal{A}$ is a closed ideal of $\mathcal{A} \times_T \mathcal{B}$ and $(\mathcal{A} \times_T \mathcal{B})/\mathcal{A}$ is isometrically isomorphic to $\mathcal{B}$, it follows from Theorems 4.2.6 and 4.3.8 in [8] that $\mathcal{A} \times_T \mathcal{B}$ is regular if and only if both $\mathcal{A}$ and $\mathcal{B}$ are regular.

Theorem 2.1 shows that given locally compact Hausdorff spaces X and Y and a proper continuous map $h : X \to Y$ (that is, a continuous map for which the preimage under h of any compact set in Y is compact in X), the product $X \times_h Y$ defined below gives a geometrisation of the perturbed product $\times_T$:

$$X \times_h Y := \{(x, h(x)) : x \in X\} \cup \{(0, y) : y \in Y\}.$$

3. Arens regularity

Let $\mathcal{A}$ be a Banach algebra. Let $\mathcal{A}'$ and $\mathcal{A}''$ be the dual and second dual Banach spaces, respectively. Let $a \in \mathcal{A}$, $\lambda \in \mathcal{A}'$ and $\Phi, \Psi \in \mathcal{A}''$. Then $\lambda \cdot a$ and $a \cdot \lambda$ are defined as $\lambda \cdot a(x) = \lambda(ax)$ $(x \in \mathcal{A})$ and $a \cdot \lambda(x) = \lambda(xa)$ $(x \in \mathcal{A})$, making $\mathcal{A}'$ an $\mathcal{A}$-module. Now $\mathcal{A}''$ is an $\mathcal{A}'$-module by

$$\langle a, \Phi \cdot \lambda \rangle = \langle \Phi, \lambda \cdot a \rangle, \quad \langle a, \lambda \cdot \Phi \rangle = \langle \Phi, a \cdot \lambda \rangle.$$

This defines two Arens products $\Box$ and $\Diamond$ on $\mathcal{A}''$ as

$$\langle \Phi \Box \Psi, \lambda \rangle = \langle \Phi, \Psi \cdot \lambda \rangle, \quad \langle \Phi \Diamond \Psi, \lambda \rangle = \langle \Psi, \lambda \cdot \Phi \rangle,$$

making $\mathcal{A}''$ a Banach algebra with each. For each $\Psi \in \mathcal{A}''$, the maps $L_\Psi : \Phi \mapsto \Psi \Diamond \Phi$ and $R_\Psi : \Phi \mapsto \Phi \Box \Psi$ are continuous on $(\mathcal{A}'', \sigma)$, where $\sigma \equiv \sigma(\mathcal{A}'', \mathcal{A}')$ denotes the weak*-topology on $\mathcal{A}''$ by the duality $\langle \mathcal{A}', \mathcal{A}'' \rangle$. The products $\Box$ and $\Diamond$ are respectively the *first* and *second Arens products* on $\mathcal{A}''$. The algebra $\mathcal{A}$ is *Arens regular* if these products coincide on $\mathcal{A}''$.

The *left* and *right topological centres* of $\mathcal{A}''$ are defined by

$$\mathfrak{Z}_t^{(\ell)}(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \Phi \Box \Psi = \Phi \Diamond \Psi \ (\Psi \in \mathcal{A}'')\},$$
$$\mathfrak{Z}_t^{(r)}(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \Psi \Box \Phi = \Psi \Diamond \Phi \ (\Psi \in \mathcal{A}'')\}.$$

The Banach algebra $\mathcal{A}$ is *left strongly Arens irregular* if $\mathfrak{Z}_t^{(\ell)}(\mathcal{A}'') = \mathcal{A}$, *right strongly Arens irregular* if $\mathfrak{Z}_t^{(r)}(\mathcal{A}'') = \mathcal{A}$, and *strongly Arens irregular* if it is both left and right strongly Arens irregular.

Let $T : \mathcal{A} \to \mathcal{B}$ be a continuous algebra homomorphism. Define $T' : \mathcal{B}' \to \mathcal{A}'$ as $T'(\lambda) = \lambda \circ T$, and $T'' : \mathcal{A}'' \to \mathcal{B}''$ as $T''(F) = F \circ T'$. Then by [3, p. 251] both $T'' : (\mathcal{A}'', \square) \to (\mathcal{B}'', \square)$ and $T'' : (\mathcal{A}'', \diamond) \to (\mathcal{B}'', \diamond)$ are continuous homomorphisms. If $\|T\| \le 1$, then $\|T''\| \le 1$ in both the cases.

Let $\mathcal{A}$ be a commutative Banach algebra. Then $\mathcal{A}$ is Arens regular if and only if $(\mathcal{A}'', \square)$ is commutative [3].

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, and let $\mathcal{A}$ be commutative and Arens regular. Let $T : \mathcal{B} \to \mathcal{A}$ be an algebra homomorphism with norm at most* 1.

(1) *Suppose that $\mathcal{A}''$, $\mathcal{B}''$, and $(\mathcal{A} \times_T \mathcal{B})''$ are equipped with their first (respectively, second) Arens products. Then*

$$(\mathcal{A} \times_T \mathcal{B})'' \cong \mathcal{A}'' \times_{T''} \mathcal{B}'' \quad \textit{(isometric isomorphism).}$$

(2) *Let $\mathfrak{Z}_t$ be either a left or a right topological centre of $\mathcal{A}''$. Then $\mathfrak{Z}_t((\mathcal{A} \times_T \mathcal{B})'') = \mathcal{A}'' \times_{T''} \mathfrak{Z}_t(\mathcal{B}'')$. In particular, $\mathcal{A} \times_T \mathcal{B}$ is Arens regular if and only if $\mathcal{B}$ is Arens regular.*

Proof. (1) Since $\mathcal{A}$ is commutative and Arens regular, $(\mathcal{A}'', \square)$ is commutative. The first Arens product on $\mathcal{A}'' \times_{T''} \mathcal{B}''$ is given as follows. Let $(\Phi, \Psi), (\Phi', \Psi') \in \mathcal{A}'' \times_{T''} \mathcal{B}''$. Then

$$(\Phi, \Psi)(\Phi', \Psi') := (\Phi \,\square\, \Phi' + T''(\Psi') \,\square\, \Phi + T''(\Psi) \,\square\, \Phi', \Psi \,\square\, \Psi'). \tag{3.1}$$

We compute the Arens product $\square$ on $(\mathcal{A} \times_T \mathcal{B})''$. For this purpose let $(a, b) \in \mathcal{A} \times_T \mathcal{B}$, $(\phi, \psi) \in \mathcal{A}' \times \mathcal{B}'$, and (Φ, Ψ), $(\Phi', \Psi') \in \mathcal{A}'' \times \mathcal{B}''$. Let $(a', b') \in \mathcal{A} \times_T \mathcal{B}$. Then

$$\begin{aligned}
((\phi, \psi) \cdot (a, b))(a', b') &= (\phi, \psi)((a, b)(a', b')) \\
&= (\phi, \psi)(aa' + T(b)a' + T(b')a, bb') \\
&= \phi(aa' + T(b)a' + T(b')a) + \phi(bb') \\
&= (\phi \cdot a + \phi \cdot T(b))(a') + (T'(\phi \cdot a) + \psi \cdot b)(b') \\
&= (\phi \cdot a + \phi \cdot T(b), T'(\phi \cdot a) + \psi \cdot b)(a', b').
\end{aligned}$$

Therefore

$$(\phi, \psi) \cdot (a, b) = (\phi \cdot a + \phi \cdot T(b), T'(\phi \cdot a) + \psi \cdot b).$$

Also

$$\begin{aligned}
((\Phi, \Psi) \cdot (\phi, \psi))(a, b) &= (\Phi, \Psi)((\phi, \psi) \cdot (a, b)) \\
&= (\Phi, \Psi)(\phi \cdot a + \phi \cdot T(b), T'(\phi \cdot a) + \psi \cdot b) \\
&= \Phi(\phi \cdot a + \phi \cdot T(b)) + \Psi(T'(\phi \cdot a) + \psi \cdot b) \\
&= \Phi(\phi \cdot a + \phi \cdot T(b)) + \Psi \circ T'(\phi \cdot a) + \Psi(\psi \cdot b) \\
&= \Phi(\phi \cdot a + \phi \cdot T(b)) + T''(\Psi)(\phi \cdot a) + \Psi(\psi \cdot b) \\
&= (\Phi \cdot \phi + T''(\Psi) \cdot \phi)(a) + (T'(\Phi \cdot \phi) + \Psi \cdot \psi)(b) \\
&= (\Phi \cdot \phi + T''(\Psi) \cdot \phi, T'(\Phi \cdot \phi) + \Psi \cdot \psi)(a, b).
\end{aligned}$$

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Therefore

$$(\Phi, \Psi) \cdot (\phi, \psi) = (\Phi \cdot \phi + T''(\Psi) \cdot \phi, T'(\Phi \cdot \phi) + \Psi \cdot \psi).$$

Now

$$\begin{aligned}((\Phi, \Psi) \square (\Phi', \Psi'))(\phi, \psi) &= (\Phi, \Psi)((\Phi', \Psi') \cdot (\phi, \psi)) \\ &= (\Phi, \Psi)(\Phi' \cdot \phi + T''(\Psi') \cdot \phi, T'(\Phi' \cdot \phi) + \Psi' \cdot \psi) \\ &= \Phi(\Phi' \cdot \phi + T''(\Psi') \cdot \phi) + \Psi(T'(\Phi' \cdot \phi) + \Psi' \cdot \psi) \\ &= \Phi(\Phi' \cdot \phi + T''(\Psi') \cdot \phi) + T''(\Psi)(\Phi' \cdot \phi) + (\Psi' \cdot \psi) \\ &= (\Phi \square \Phi' + \Phi \square T''(\Psi') + \Phi' \square T''(\Psi))(\phi) + (\Psi \square \Psi')(\psi) \\ &= (\Phi \square \Phi' + \Phi \square T''(\Psi') + \Phi' \square T''(\Psi), \Psi \square \Psi')(\phi, \psi).\end{aligned}$$

Therefore

$$(\Phi, \Psi) \square (\Phi', \Psi') = (\Phi \square \Phi' + \Phi \square T''(\Psi') + \Phi' \square T''(\Psi), \Psi \square \Psi'),$$

which is the same as (3.1). Calculations for the second Arens product are analogous.

(2) Since $\mathcal{A}$ is commutative and Arens regular, $\mathcal{A}'' = \mathfrak{Z}^{(\ell)}(\mathcal{A}'') = \mathfrak{Z}^{(r)}(\mathcal{A}'')$. Let $(\Phi, \Psi) \in \mathfrak{Z}^{(\ell)}((\mathcal{A} \times_T \mathcal{B})'') = \mathfrak{Z}^{(\ell)}(\mathcal{A}'' \times_{T''} \mathcal{B}'')$. Then for any $(\Phi', \Psi') \in \mathcal{A}'' \times_{T''} \mathcal{B}''$, $(\Phi, \Psi) \square (\Phi', \Psi') = (\Phi, \Psi) \diamond (\Phi', \Psi')$, that is,

$$\begin{aligned}&(\Phi \square \Phi' + \Phi \square T''(\Psi') + \Phi' \square T''(\Psi), \Psi \square \Psi') \\ &\quad = (\Phi \diamond \Phi' + \Phi \diamond T''(\Psi') + \Phi' \diamond T''(\Psi), \Psi \diamond \Psi').\end{aligned}$$

In particular, $\Psi \square \Psi' = \Psi \diamond \Psi'$ for every $\Psi' \in \mathcal{B}''$. Therefore

$$\mathfrak{Z}^{(\ell)}(\mathcal{A}'' \times_{T''} \mathcal{B}'') \subset \mathcal{A}'' \times_{T''} \mathfrak{Z}^{(\ell)}(\mathcal{B}'').$$

Conversely, assume that $(\Phi, \Psi) \in \mathcal{A}'' \times_{T''} \mathfrak{Z}^{(\ell)}(\mathcal{B}'')$. Since $\mathcal{A}'' = \mathfrak{Z}^{(\ell)}(\mathcal{A}'')$, it follows that $(\Phi, \Psi) \square (\Phi', \Psi') = (\Phi, \Psi) \diamond (\Phi', \Psi')$ for every $(\Phi', \Psi') \in \mathcal{A}'' \times_{T''} \mathcal{B}''$, that is, $(\Phi, \Psi) \in \mathfrak{Z}^{(\ell)}(\mathcal{A}'' \times_{T''} \mathcal{B}'')$. Therefore $\mathcal{A}'' \times_{T''} \mathfrak{Z}^{(\ell)}(\mathcal{B}'') \subset \mathfrak{Z}^{(\ell)}(\mathcal{A}'' \times_{T''} \mathcal{B}'')$. This means that $\mathfrak{Z}^{(\ell)}((\mathcal{A} \times_T \mathcal{B})'') = \mathcal{A}'' \times_{T''} \mathfrak{Z}^{(\ell)}(\mathcal{B}'')$. It also follows that $\mathcal{A} \times_T \mathcal{B}$ is Arens regular if and only if $\mathcal{B}$ is regular. □

4. Amenability

Let $\mathcal{A}$ be a Banach algebra, and let E be a Banach $\mathcal{A}$-module. A *bounded E-derivation* is a bounded linear map $D : \mathcal{A} \to E$ such that

$$D(ab) = (Da) \cdot b + a \cdot (Db) \quad (a, b \in \mathcal{A}).$$

The set of all bounded E-derivations on $\mathcal{A}$ is denoted by $\mathcal{Z}^1(\mathcal{A}, E)$.

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Given $x \in E$, let $\delta_x : \mathcal{A} \to E$ be given by $\delta_x(a) = a \cdot x - x \cdot a$ $(a \in \mathcal{A})$. Then $\delta_x \in \mathcal{Z}^1(\mathcal{A}, E)$. The derivation δ_x is called an *inner E-derivation*. Let $\mathcal{B}^1(\mathcal{A}, E)$ be the set of all inner E-derivations. Let

$$\mathcal{H}^1(\mathcal{A}, E) = \mathcal{Z}^1(\mathcal{A}, E)/\mathcal{B}^1(\mathcal{A}, E).$$

Then $\mathcal{H}^1(\mathcal{A}, E)$ is the *first cohomology group of $\mathcal{A}$ with coefficients in E*.

A Banach algebra $\mathcal{A}$ is *amenable* if $\mathcal{H}^1(\mathcal{A}, E') = \{0\}$ for every Banach $\mathcal{A}$-module E and it is *weakly amenable* if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}') = \{0\}$. By [3, Proposition 2.8.59], a Banach algebra $\mathcal{A}$ is amenable if and only if for each Banach $\mathcal{A}$-module E and each $D \in \mathcal{Z}^1(\mathcal{A}, E)$, there exists a bounded net (x_α) in E such that

$$D(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in \mathcal{A}).$$

If I is a closed ideal of $\mathcal{A}$, then, by [3, Proposition 2.8.66], $\mathcal{A}$ is amenable if I and $\mathcal{A}/I$ are amenable and $\mathcal{A}$ is weakly amenable if I and $\mathcal{A}/I$ are weakly amenable.

A derivation $D : \mathcal{A} \to E$ is *approximately inner* if there exists a net $(x_\alpha) \subset E$ such that $D(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a)$ $(a \in \mathcal{A})$. The algebra $\mathcal{A}$ is *approximately amenable* if for each Banach $\mathcal{A}$-module E every bounded derivation $D : \mathcal{A} \to E$ is approximately inner and $\mathcal{A}$ is *approximately weakly amenable* if every bounded derivation $D : \mathcal{A} \to \mathcal{A}'$ is approximately inner. By [6, Proposition 2.2 (iii)], if I is a closed ideal in a Banach algebra $\mathcal{A}$, I is weakly amenable and $\mathcal{A}/I$ is approximately weakly amenable, then $\mathcal{A}$ is approximately weakly amenable.

A derivation $D : \mathcal{A} \to \mathcal{A}'$ is *cyclic* if $D(a)(b) + D(b)(a) = 0$ $(a, b \in \mathcal{A})$. Every inner derivation from $\mathcal{A}$ to $\mathcal{A}'$ is cyclic. A Banach algebra $\mathcal{A}$ is *cyclic amenable* if every cyclic derivation is inner.

A Banach algebra $\mathcal{A}$ is *approximately cyclic amenable* if every cyclic derivation $D : \mathcal{A} \to \mathcal{A}'$ is approximately inner.

The following theorem exhibits the stability of amenability with respect to the product $\times_T$. Looking to the elementary nature of product $\times_T$, the proof is quite elementary, avoiding deeper results in amenability such as its functorial properties [15].

THEOREM 4.1. *Let $\mathcal{A}$ be a commutative Banach algebra, let $\mathcal{B}$ be a Banach algebra, and let $T : \mathcal{B} \to \mathcal{A}$ be an algebra homomorphism with $\|T\| \leq 1$. Then:*

(1) *$\mathcal{A} \times_T \mathcal{B}$ is amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are amenable;*
(2) *$\mathcal{A} \times_T \mathcal{B}$ is weakly amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are weakly amenable;*
(3) *$\mathcal{A} \times_T \mathcal{B}$ is approximately weakly amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are approximately weakly amenable;*
(4) *$\mathcal{A} \times_T \mathcal{B}$ is cyclic amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are cyclic amenable;*
(5) *$\mathcal{A} \times_T \mathcal{B}$ is approximately cyclic amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are approximately cyclic amenable.*

PROOF. (1) Assume that both $\mathcal{A}$ and $\mathcal{B}$ are amenable. Since I is a closed ideal in $\mathcal{A} \times_T \mathcal{B}$ and $(\mathcal{A} \times_T \mathcal{B})/\mathcal{A} \cong \mathcal{B}$, $\mathcal{A} \times_T \mathcal{B}$ is amenable.

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Conversely, assume that $\mathcal{A} \times_T \mathcal{B}$ is amenable. Let E be a Banach $\mathcal{A}$-module, and let $d : \mathcal{A} \to E$ be a bounded derivation. We may consider this map as $d : \mathcal{A} \times \{0\} \to E \times \{0\}$. We note that $E \times \{0\}$ is a Banach $\mathcal{A} \times_T \mathcal{B}$-module. Let $P : \mathcal{A} \times_T \mathcal{B}$ be defined as

$$P(a, b) = (a + T(b), 0) \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

Then $D = d \circ P : \mathcal{A} \times_T \mathcal{B} \to E \times \{0\}$ is a bounded derivation on $\mathcal{A} \times_T \mathcal{B}$. Since $\mathcal{A} \times_T \mathcal{B}$ is amenable, there exists a bounded net $((x_\alpha, 0))$ in $E \times \{0\}$ such that

$$D(a, b) = \lim_\alpha((a, b) \cdot (x_\alpha, 0) - (x_\alpha, 0) \cdot (a, b)) \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

In particular,

$$\begin{aligned} d(a) = d(a, 0) = D(a, 0) &= \lim_\alpha((a, 0) \cdot (x_\alpha, 0) - (x_\alpha, 0) \cdot (a, 0)) \\ &= \lim_\alpha(a \cdot x_\alpha - x_\alpha \cdot a). \end{aligned}$$

Hence $\mathcal{A}$ is amenable.

Similarly, by taking $P : \mathcal{A} \times_T \mathcal{B} \to \{0\} \times E$ as $P(a, b) = (0, b)$ it follows that $\mathcal{B}$ is amenable.

(2) Assume that both $\mathcal{A}$ and $\mathcal{B}$ are weakly amenable. Since I is a closed ideal in $\mathcal{A} \times_T \mathcal{B}$ and $(\mathcal{A} \times_T \mathcal{B})/\mathcal{A} \cong \mathcal{B}$, $\mathcal{A} \times_T \mathcal{B}$ is weakly amenable.

Conversely, let $\mathcal{A} \times_T \mathcal{B}$ be weakly amenable. Let $d : \mathcal{A} \to \mathcal{A}'$ be a continuous derivation. Let $P : \mathcal{A} \times_T \mathcal{B} \to \mathcal{A}$ be defined as

$$P(a, b) = a + T(b) \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

Let $D = P' \circ d \circ P$. Then D is a derivation of $\mathcal{A} \times_T \mathcal{B}$ to $\mathcal{A}' \times \mathcal{B}'$. Since $\mathcal{A} \times_T \mathcal{B}$ is weakly amenable, there exists (φ, ψ) in $\mathcal{A}' \times \mathcal{B}'$ such that $D = \delta_{(\varphi,\psi)}$.

Let $a, a' \in \mathcal{A}$. Then

$$\begin{aligned} \langle d(a), a' \rangle &= \langle P' \circ dP(a, 0), (a', 0) \rangle = \langle (a, 0) \cdot (\varphi, \psi) - (\varphi, \psi) \cdot (a, 0), (a', 0) \rangle \\ &= \langle a \cdot \varphi - \varphi \cdot a, a' \rangle = 0 \quad (\text{as } \mathcal{A} \text{ is commutative}). \end{aligned}$$

Hence $\mathcal{Z}^1(\mathcal{A}, \mathcal{A}') = \{0\}$, that is, $\mathcal{A}$ is weakly amenable.

Let $d : \mathcal{B} \to \mathcal{B}'$ be a continuous derivation. Let $P : \mathcal{A} \times_T \mathcal{B} \to \mathcal{B}$ be defined as

$$P(a, b) = b \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

Let $D = P' \circ d \circ P$. Then D is a derivation of $\mathcal{A} \times_T \mathcal{B}$ to $\mathcal{A}' \times \mathcal{B}'$. Since $\mathcal{A} \times \mathcal{B}$ is weakly amenable, there exists $(\varphi, \psi) \in \mathcal{A}' \times \mathcal{B}'$ such that $D = \delta_{(\varphi,\psi)}$.

Let $b, b' \in \mathcal{B}$. Then

$$\begin{aligned} \langle d(b), b' \rangle &= \langle P' \circ d \circ P(0, b), (0, b') \rangle \\ &= \langle (0, b) \cdot (\varphi, \psi) - (\varphi, \psi) \cdot (0, b), (0, b') \rangle \\ &= \langle b \cdot \psi - \psi \cdot b, b' \rangle. \end{aligned}$$

Hence $d = \delta_\psi$. Therefore $\mathcal{B}$ is weakly amenable.

(3) Assume that both $\mathcal{A}$ and $\mathcal{B}$ are approximately weakly amenable. Since $\mathcal{A}$ is commutative, it is weakly amenable. Since $\mathcal{A}$ is a closed ideal in $\mathcal{A} \times_T \mathcal{B}$ and $(\mathcal{A} \times_T \mathcal{B})/\mathcal{A} \cong \mathcal{B}$, $\mathcal{A} \times_T \mathcal{B}$ is approximately weakly amenable.

Conversely, assume that $\mathcal{A} \times_T \mathcal{B}$ is approximately weakly amenable. Let $d : \mathcal{A} \to \mathcal{A}'$ be a continuous derivation. Then $D = P' \circ d \circ P$ is a continuous derivation of $\mathcal{A} \times_T \mathcal{B}$. Since $\mathcal{A} \times_T \mathcal{B}$ is weakly approximately amenable, there exists a net $((\varphi_\alpha, \psi_\alpha)) \subset \mathcal{A}' \times \mathcal{B}'$ such that

$$D(a, b) = \lim_\alpha ((a, b) \cdot (\varphi_\alpha, \psi_\alpha) - (\varphi_\alpha, \psi_\alpha) \cdot (a, b)) \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

Now

$$\begin{aligned} \langle d(a), a' \rangle &= \langle D(a, 0), (a', 0) \rangle \\ &= \langle \lim_\alpha ((a, 0) \cdot (\varphi_\alpha, \psi_\alpha) - (\varphi_\alpha, \psi_\alpha) \cdot (a, 0)), (a', 0) \rangle \\ &= \langle \lim_\alpha (a \cdot \varphi_\alpha - \varphi_\alpha \cdot a), a' \rangle = 0. \end{aligned}$$

Therefore $d = 0$. Hence $\mathcal{A}$ is approximately weakly amenable.

The proof of approximately weak amenability of $\mathcal{B}$ is analogous to the above.

(4) Let $\mathcal{A}$ and $\mathcal{B}$ be cyclic amenable. Let $D : \mathcal{A} \times_T \mathcal{B} \to \mathcal{A}' \times \mathcal{B}'$ be a bounded cyclic derivation. Then $D_{|_\mathcal{A}} : \mathcal{A} \to \mathcal{A}'$ and $D_{|_\mathcal{B}} : \mathcal{B} \to \mathcal{B}'$ are cyclic derivations. Since $\mathcal{A}$ is commutative, $D_{|_\mathcal{A}} = 0$, and since $\mathcal{B}$ is cyclic amenable, there exists $\psi \in \mathcal{B}'$ such that $D_{|_\mathcal{B}}(0, b) = (0, b) \cdot (0, \psi) - (0, \psi) \cdot (0, b)$ $(b \in \mathcal{B})$. Let $(a, b) \in \mathcal{A} \times_T \mathcal{B}$. Then

$$\begin{aligned} D(a, b) &= D(a, 0) + D(0, b) \\ &= b \cdot \psi - \psi \cdot b \\ &= (a, b) \cdot (0, \psi) - (0, \psi) \cdot (a, b). \end{aligned}$$

Therefore $\mathcal{A} \times_T \mathcal{B}$ is cyclic amenable.

Conversely, assume that $\mathcal{A} \times_T \mathcal{B}$ is cyclic amenable. Let $d : \mathcal{A} \to \mathcal{A}'$ be a cyclic derivation. Then d can be considered as a map $d : \mathcal{A} \times \{0\} \to \mathcal{A}' \times \{0\} \subset \mathcal{A}' \times \mathcal{B}'$. Let $P : \mathcal{A} \times_T \mathcal{B} \to \mathcal{A}$ be defined as $P(a, b) = (a + T(b), 0)$ $((a, b) \in \mathcal{A} \times_T \mathcal{B})$. Then $d \circ P : \mathcal{A} \times_T \mathcal{B} \to \mathcal{A}' \times \{0\} \subset \mathcal{A}' \times \mathcal{B}'$ is a cyclic derivation. Since $\mathcal{A} \times_T \mathcal{B}$ is cyclic amenable, there exists $(\varphi, \psi) \in \mathcal{A}' \times \mathcal{B}'$ such that

$$d \circ P(a, b) = (a, b) \cdot (\varphi, \psi) - (\varphi, \psi) \cdot (a, b) \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

Let $a \in \mathcal{A}$. Then

$$\begin{aligned} d(a) &= d \circ P(a, 0) = (a, 0) \cdot (\varphi, \psi) - (\varphi, \psi) \cdot (a, 0) \\ &= a \cdot \varphi - \varphi \cdot a = 0 \quad (\text{as } \mathcal{A} \text{ is commutative}). \end{aligned}$$

Therefore $\mathcal{A}$ is cyclic amenable. Similarly, one can show that $\mathcal{B}$ is cyclic amenable.

(5) Let $\mathcal{A}$ and $\mathcal{B}$ be approximately cyclic amenable. Let $D : \mathcal{A} \times_T \mathcal{B} \to \mathcal{A}' \times \mathcal{B}'$ be a bounded cyclic derivation. Then $D_{|_\mathcal{A}} : \mathcal{A} \to \mathcal{A}'$ and $D_{|_\mathcal{B}} : \mathcal{B} \to \mathcal{B}'$ are cyclic

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derivations. Since $\mathcal{A}$ is commutative, $D_{|\mathcal{A}} = 0$, and since $\mathcal{B}$ is approximately cyclic amenable, there exists a net (ψ_α) in $\mathcal{B}'$ such that $D_{|\mathcal{B}}(0, b) = \lim_\alpha((0, b) \cdot (0, \psi_\alpha) - (0, \psi_\alpha) \cdot (0, b))$ $(b \in \mathcal{B})$. Let $(a, b) \in \mathcal{A} \times_T \mathcal{B}$. Then

$$\begin{aligned} D(a, b) &= D(a, 0) + D(0, b) \\ &= \lim_\alpha (b \cdot \psi_\alpha - \psi_\alpha \cdot b) \\ &= \lim_\alpha ((a, b) \cdot (0, \psi_\alpha) - (0, \psi_\alpha) \cdot (a, b)). \end{aligned}$$

Therefore $\mathcal{A} \times_T \mathcal{B}$ is approximately cyclic amenable.

Conversely, assume that $\mathcal{A} \times_T \mathcal{B}$ is approximately cyclic amenable. Let $d : \mathcal{A} \to \mathcal{A}'$ be a cyclic derivation. Then d can be considered as a map $d : \mathcal{A} \times \{0\} \to \mathcal{A}' \times \{0\} \subset \mathcal{A}' \times \mathcal{B}'$. Let $P : \mathcal{A} \times_T \mathcal{B} \to \mathcal{A}$ be defined as $P(a, b) = (a + T(b), 0)$ $((a, b) \in \mathcal{A} \times_T \mathcal{B})$. Then $d \circ P : \mathcal{A} \times_T \mathcal{B} \to \mathcal{A}' \times \{0\} \subset \mathcal{A}' \times \mathcal{B}'$ is a cyclic derivation. Since $\mathcal{A} \times_T \mathcal{B}$ is cyclic amenable, there exists a net $(\varphi_\alpha, \psi_\alpha)$ in $\mathcal{A}' \times \mathcal{B}'$ such that

$$d \circ P(a, b) = \lim_\alpha ((a, b) \cdot (\varphi_\alpha, \psi_\alpha) - (\varphi_\alpha, \psi_\alpha) \cdot (a, b)) \quad ((a, b) \in \mathcal{A} \times_T \mathcal{B}).$$

Let $a \in \mathcal{A}$. Then

$$\begin{aligned} d(a) = d \circ P(a, 0) &= \lim_\alpha ((a, 0) \cdot (\varphi_\alpha, \psi_\alpha) - (\varphi_\alpha, \psi_\alpha) \cdot (a, 0)) \\ &= \lim_\alpha (a \cdot \varphi_\alpha - \varphi_\alpha \cdot a) = 0 \quad (\text{as } \mathcal{A} \text{ is commutative}). \end{aligned}$$

Therefore $\mathcal{A}$ is approximately cyclic amenable. Similarly, one can show that $\mathcal{B}$ is approximately cyclic amenable. □

There is a localised version of amenability of much recent interest, namely character amenability [7, 9, 10, 14]. Let $\varphi \in \Delta(\mathcal{A})$. Following [5], $\mathcal{A}$ is *φ-inner amenable* if there exists $m \in \mathcal{A}''$ such that $m(\varphi) = 1$ and $m \,\square\, a = a \,\square\, m$ $(a \in \mathcal{A})$. Such an m is called *φ-inner mean*. A Banach algebra $\mathcal{A}$ is *character inner amenable* if $\mathcal{A}$ is φ-inner amenable for all $\varphi \in \Delta(\mathcal{A})$. We note that every commutative Banach algebra is character inner amenable.

Theorem 4.2. *Let $\mathcal{A}$ be a commutative Banach algebra, $\mathcal{B}$ be a Banach algebra, and let $T : \mathcal{B} \to \mathcal{A}$ be an algebra homomorphism with $\|T\| \le 1$.*

(1) *$\mathcal{A} \times_T \mathcal{B}$ is $(\varphi, \varphi \circ T)$-inner amenable for all $\varphi \in \Delta(\mathcal{A})$. If (m, n) is a $(\varphi, \varphi \circ T)$-inner mean and $n(\varphi \circ T) \neq 0$, then $\mathcal{B}$ is $\varphi \circ T$-inner amenable.*

(2) *For all $\psi \in \Delta(\mathcal{B})$, $\mathcal{A} \times_T \mathcal{B}$ is $(0, \psi)$-inner amenable if and only if $\mathcal{B}$ is ψ-inner amenable.*

(3) *$\mathcal{A} \times_T \mathcal{B}$ is character inner amenable if and only if $\mathcal{B}$ is character inner amenable.*

Proof. (1) Let $\varphi \in \Delta(\mathcal{A})$. Since $\mathcal{A}$ is commutative, there exists $m \in \mathcal{A}''$ such that $m(\varphi) = 1$ and $m \,\square\, a = a \,\square\, m$ for all $a \in \mathcal{A}$. Then $(m, 0)(\varphi, \varphi \circ T) = 1$ and since $\mathcal{A}$ is commutative, $(m, 0) \,\square\, (a, b) = (a, b) \,\square\, (m, 0)$ for all $(a, b) \in \mathcal{A} \times_T \mathcal{B}$, that is, $\mathcal{A} \times_T \mathcal{B}$ is $(\varphi, \varphi \circ T)$-inner amenable.

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Let (m, n) be a $(\varphi, \varphi \circ T)$-inner mean and $n(\varphi \circ T) \neq 0$. Then it follows that $(n/n(\varphi \circ T))(\varphi \circ T) = 1$ and $(n/n(\varphi \circ T)) \,\square\, b = b \,\square\, (n/n(\varphi \circ T))$ for all $b \in \mathcal{B}$, that is, $\mathcal{B}$ is $\varphi \circ T$-inner amenable.

(2) The proof is analogous to that of (1).

(3) Assume that $\mathcal{B}$ is character amenable. Let $(\varphi, \psi) \in \Delta(\mathcal{A} \times_T \mathcal{B})$. First let $\varphi = 0$. Since $\mathcal{B}$ is character amenable, there exists $n \in \mathcal{B}''$ such that $n(\psi) = 1$ and $n \,\square\, b = b \,\square\, n$ for all $b \in \mathcal{B}$. Now

$$(0, n)((0, \psi)) = n(\psi) = 1, \qquad (0, n) \,\square\, (a, b) = (T''(n) \,\square\, a, n \,\square\, b)$$

and

$$(a, b) \,\square\, (0, n) = (T''(n) \,\square\, a, b \,\square\, n).$$

Since $\mathcal{A}$ is commutative, $m \,\square\, a = a \,\square\, m$ for every $m \in \mathcal{A}''$ and $a \in \mathcal{A}$. Hence $(0, n)$ is a $(0, \psi)$- inner mean for $\mathcal{A} \times_T \mathcal{B}$.

Second, assume that $\varphi \neq 0$. Then $(\varphi, \psi) = (\varphi, \varphi \circ T)$. Since $\mathcal{A}$ is character inner amenable, there exists $m \in \mathcal{A}''$ such that $m(\varphi) = 1$ and $m \,\square\, a = a \,\square\, m$ for every $m \in \mathcal{A}''$. Now

$$(m, 0)((\varphi, \varphi \circ T)) = m(\varphi) = 1,$$
$$(m, 0) \,\square\, (a, b) = (m \,\square\, a + T(b) \,\square\, m, 0) = (a, b) \,\square\, (m, 0).$$

Hence $\mathcal{A} \times_T \mathcal{B}$ is character inner amenable.

Conversely, assume that $\mathcal{A} \times_T \mathcal{B}$ is character inner amenable. Let $\psi \in \Delta(\mathcal{B})$. Then there exists $(m, n) \in \mathcal{A}'' \times_{T''} \mathcal{B}''$ such that $(m, n)((0, \psi)) = n(\psi) = 1$ and $(m, n) \,\square\, (a, b) = (a, b) \,\square\, (m, n)$ $((a, b) \in \mathcal{A} \times_T \mathcal{B})$. It follows that $n(\psi) = 1$ and $n \,\square\, b = b \,\square\, n$ $(b \in \mathcal{B})$, that is, n is a ψ-inner mean for $\mathcal{B}$. $\square$

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References

[1] R. Arens, 'Operations induced in function classes', *Monatsh. Math.* **55** (1951), 1–19.
[2] R. Arens, 'The adjoint of a bilinear operation', *Proc. Amer. Math. Soc.* **2** (1951), 839–848.
[3] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Mathematical Society Monographs New Series, 24 (Oxford University Press, Oxford, 2000).
[4] K. R. Davidson, *C^*-Algebras by Example*, Text and Readings in Mathematics, 11 (Hindustan Book Agency, New Delhi, 1996).
[5] H. R. Ebrahimi Vishki and A. R. Khoddami, 'Character inner amenability of certain Banach algebras', arXiv:1007.1654v1 [math.FA], 9 July 2010.
[6] G. H. Esslamzadeh and B. Shojaee, 'Approximate weak amenability of Banach algebras', arXiv:0908.3577v2 [math.FA], 18 Jan. 2011.
[7] Z. Hu, M. S. Monfared and T. Traynor, 'On character amenable Banach algebras', *Studia Math.* **193**(1) (2009), 53–78.

[8] E. Kaniuth, *A Course in Commutative Banach Algebras* (Springer, New York, 2009).
[9] E. Kaniuth, A. T.-M. Lau and J. Pym, 'On character amenability of Banach algebras', *J. Math. Anal. Appl.* **344**(2) (2008), 942–955.
[10] E. Kaniuth, A. T.-M. Lau and J. Pym, 'On φ-amenability of Banach algebras', *Math. Proc. Cambridge Philos. Soc.* **144**(1) (2008), 85–96.
[11] A. T.-M. Lau, 'Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups', *Fund. Math.* **118** (1983), 161–175.
[12] A. T.-M. Lau and J. Ludwig, 'Fourier–Stieltjes algebra of topological group', *Adv. Math.* **229**(3) (2012), 2000–2023.
[13] M. S. Monfared, 'On certain products of Banach algebras with application to Harmonic analysis', *Studia Math.* **178**(3) (2007), 277–294.
[14] M. S. Monfared, 'Character amenability of Banach algebras', *Math. Proc. Cambridge Philos. Soc.* **144**(3) (2008), 697–706.
[15] V. Runde, *Lectures on Amenability*, Lectures Notes in Mathematics, 1774 (Springer, Berlin–Heidelberg–New York, 2002).
[16] M. Takesaki, 'Duality and von Neumann algebras', in: *Lectures on Operator Algebras*, Lecture Notes in Mathematics, 247 (Springer, Berlin), pp. 665–779.

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