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ON THE CONDITIONAL RESIDUAL LIFE AND INACTIVITY TIME OF COHERENT SYSTEMS

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Abstract

In this paper we derive mixture representations for the reliability functions of the conditional residual life and inactivity time of a coherent system with n independent and identically distributed components. Based on these mixture representations we carry out stochastic comparisons on the conditional residual life, and the inactivity time of two coherent systems with independent and identical components.

Keywords: Residual lifetime; signature; inactivity time; hazard rate order; likelihood ratio order; reversed hazard rate order

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1. Introduction

In this paper we examine some stochastic and ageing properties of coherent systems. According to Barlow and Proschan (1975), a coherent system is a reliability system such that the structure function of the system is monotone in its components and each component of the system is relevant (a component is irrelevant if it does not matter whether or not it is working with regard to the functioning of the system). Many results on stochastic and ageing properties of coherent systems, and the comparison of coherent systems, are based mainly on the concept of the signature of a system. The concept of the signature of a coherent system was introduced by Samaniego (1985). For a coherent system with n components whose lifetimes $X_1, \dots, X_n$ are independent and identically distributed (i.i.d.) random variables with a continuous distribution function F, suppose the lifetime of the system can be expressed as $T = \tau(X_1, \dots, X_n)$, where τ is a structural function (see Barlow and Proschan (1975)). Then, the signature of the system is defined as a probability vector $\boldsymbol{s} = (s_1, \dots, s_n)$ with

$$s_i = \mathbb{P}\{T = X_{i:n}\}, \qquad i = 1, \dots, n,$$

where $X_{i:n}$ is the ith order statistic among $X_1, \dots, X_n$.

It is known that the signature vector is a distribution-free function (that is, does not depend on the common continuous lifetime distribution F of the components) and that $s_i = \mathbb{P}\{T = X_{i:n}\} = |A_i|/n!$, where A_i is the set of all permutations of the component lifetimes for which the ith ordered component failure is fatal to the system and $|A_i|$ denotes the cardinality of A_i.

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Samaniego (1985) then showed that the distribution function of T can be expressed as

$$F_T(t) = \sum_{i=1}^{n} s_i F_{i:n}(t); \tag{1}$$

that is, the distribution function of T can be represented as a mixture of the distributions $F_{i:n}$ of the ordered component lifetimes $X_{i:n}$, $i = 1, \dots, n$.

Kochar *et al.* (1999) made use of the representation in (1) to carry out stochastic comparisons between different systems. Subsequently, many authors have studied reliability properties of the lifetime, the residual lifetime, and the inactivity time of coherent systems; see, for example, Khaledi and Shaked (2007), Li and Zhao (2006), (2008), Li and Zhang (2008), Poursaeed and Nematollahi (2008), Tavangar and Asadi (2010), Navarro *et al.* (2005), (2008), (2013), Zhang (2010a), (2010b), Zhang and Li (2010), Zhang and Yang (2010), Golifroushani and Asadi (2011), Golifroushani *et al.* (2012), Nama and Asadi (2013), Zhang and Meeker (2013), and the references therein.

In this paper we concentrate on the two cases when the signature of the system has the following specific forms:

(i) $\boldsymbol{s} = (0, \dots, 0, s_i, s_{i+1}, \dots, s_n)$ for $i = 2, \dots, n$,

(ii) $\boldsymbol{s} = (s_1, \dots, s_i, 0, \dots, 0)$ for $i = 1, \dots, n-1$.

Now, suppose the system under consideration is monitored continuously and that the jth failure is observed to occur at time t, i.e. $X_{j:n} = t$. Under such a setting, in the case of (i), we then study the random variables (residual lifetime of the system)

$$(T - t \mid X_{j:n} = t) \quad \text{for } j = 1, \dots, i-1,$$

and in the case of (ii), we study the random variables (inactivity time of the system)

$$(t - T \mid X_{j:n} = t) \quad \text{for } j = i, \dots, n.$$

The following definition introduces some orderings that will be useful in later discussions.

Definition 1. Let X and Y be random variables with distribution functions $F(x)$ and $G(x)$ and survival functions $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$, respectively.

1. X is said to be smaller than Y in the usual stochastic order (denoted by $X \le_{\text{st}} Y$) if $\bar{F}(t) \le \bar{G}(t)$ for all t.

2. X is said to be smaller than Y in the hazard rate order (denoted by $X \le_{\text{hr}} Y$) if $\bar{G}(t)/\bar{F}(t)$ is increasing in t.

3. X is said to be smaller than Y in the reversed hazard rate order (denoted by $X \le_{\text{rh}} Y$) if $G(t)/F(t)$ is increasing in t.

4. X is said to be smaller than Y in the likelihood ratio order (denoted by $X \le_{\text{lr}} Y$) if $g(x)/f(x)$ is increasing in the union of their supports, where $f(x)$ and $g(x)$ are the densities of $F(x)$ and $G(x)$, respectively.

2. Residual lifetime of the system

Let T be the lifetime of a reliability system of order n and $X_1, \ldots, X_n$ denote the lifetimes of its components. We assume that $X_1, \ldots, X_n$ are i.i.d. according to a common underlying continuous distribution F. We assume that the signature of the system has the form (i), that is,

$$s = (0, \ldots, 0, s_i, s_{i+1}, \ldots, s_n) \quad \text{for } i = 2, \ldots, n.$$

In this section we study the reliability and stochastic properties of the residual lifetime of the system, that is, the variable

$$(T - t \mid X_{j:n} = t) \quad \text{for } j = 1, \ldots, i-1.$$

We denote

$$A_i = \{\pi \in \mathcal{P}_n : T = X_{i:n}$$

when $X_{\pi 1} < \cdots < X_{\pi n}\}$, where $\mathcal{P}_n$ is the set of permutations of the set $\{1, \ldots, n\}$ and $\pi = (\pi 1, \ldots, \pi n)$ is a permutation in $\mathcal{P}_n$. Then,

$$\begin{aligned}
&\mathbb{P}\{X_{k:n} \le x,\ X_{j:n} \le t,\ T = X_{k:n}\} \\
&\quad = \sum_{\pi} \mathbb{P}\{X_{k:n} \le x,\ X_{j:n} \le t,\ T = X_{k:n}, X_{\pi 1} < \cdots < X_{\pi n}\} \\
&\quad = \sum_{\pi \in A_k} \mathbb{P}\{X_{\pi k} \le x,\ X_{\pi j} \le t,\ X_{\pi 1} < \cdots < X_{\pi n}\} \\
&\quad = |A_k| \mathbb{P}\{X_k \le x,\ X_j \le t,\ X_1 < \cdots < X_n\} \\
&\quad = \frac{|A_k|}{n!} \mathbb{P}\{X_{k:n} \le x,\ X_{j:n} \le t\} \\
&\quad = \mathbb{P}\{T = X_{k:n}\} \mathbb{P}\{X_{k:n} \le x,\ X_{j:n} \le t\}.
\end{aligned}$$

Therefore, $(X_{k:n}, X_{j:n})$ is independent of the event $(T = X_{k:n})$. Hence, for all $x \ge 0$ and $t > 0$, we have (see also Navarro *et al.* (2013))

$$\begin{aligned}
\mathbb{P}\{T - t > x \mid X_{j:n} = t\} &= \sum_{k=i}^{n} \mathbb{P}\{T - t > x,\ T = X_{k:n} \mid X_{j:n} = t\} \\
&= \sum_{k=i}^{n} \mathbb{P}\{X_{k:n} - t > x,\ T = X_{k:n} \mid X_{j:n} = t\} \\
&= \sum_{k=i}^{n} \mathbb{P}\{T = X_{k:n}\} \mathbb{P}\{X_{k:n} - t > x \mid X_{j:n} = t\} \\
&= \sum_{k=i}^{n} s_k \mathbb{P}\{X_{k:n} - t > x \mid X_{j:n} = t\}.
\end{aligned}$$

Remark 1. The random variable $(X_{k:n} - t \mid X_{j:n} = t)$ has the following properties.

1. For $j < k$, we have (Navarro *et al.* (2013) and Nama and Asadi (2013))

$$(X_{k:n} - t \mid X_{j:n} = t) \stackrel{\mathrm{D}}{=} X^t_{k-j:n-j},$$

where $X^t_{k-j:n-j}$ denotes the $(k-j)$th order statistic in a random sample of size $(n-j)$ from the left truncated distribution with survival function $\bar{F}_t(x) = \bar{F}(x+t)/\bar{F}(t)$.

2. For $j_1 \le j_2 < k$, we have (Nama and Asadi (2013))

$$X^t_{k-j_2\,:\,n-j_2} \le_{\text{lr}} X^t_{k-j_1\,:\,n-j_1},$$
$$(X_{k\,:\,n} - t \mid X_{j_2\,:\,n} = t) \le_{\text{lr}} (X_{k\,:\,n} - t \mid X_{j_1\,:\,n} = t);$$

   thus, $\mathbb{P}\{X_{k\,:\,n} - t > x \mid X_{j\,:\,n} = t\}$ is a decreasing function of j for all $t > 0$ and $x > 0$.

3. If the Xs are IFR (increasing failure rate) then, for $j < k$ and all $x > 0$, $\mathbb{P}\{X_{k\,:\,n} - t > x \mid X_{j\,:\,n} = t\}$ is a decreasing function of $t > 0$ (Nama and Asadi (2013)).

Remark 2. It is well known that for $k_1 \le k_2$, we have $X_{k_1\,:\,n} \le_{\text{lr}} X_{k_2\,:\,n}$, see Shaked and Shanthikumar (2007, Theorem 1.C.37). Therefore, for $k_1 \le k_2$, we have $(X_{k_1\,:\,n} - t \mid X_{j\,:\,n} = t) \le_{\text{lr}} (X_{k_2\,:\,n} - t \mid X_{j\,:\,n} = t)$. Moreover, $\mathbb{P}\{X_{k\,:\,n} - t > x \mid X_{j\,:\,n} = t\}$ is an increasing function of k for all $t > 0$ and $x > 0$.

Theorem 1. *Let*

$$\boldsymbol{s}_1 = (0, \dots, 0, s_{1,i}, s_{1,i+1}, \dots, s_{1,n}) \quad \textit{and} \quad \boldsymbol{s}_2 = (0, \dots, 0, s_{2,i}, s_{2,i+1}, \dots, s_{2,n})$$

be the signatures of two coherent systems $T_1 = \phi_1(X_1, \dots, X_n)$ and $T_2 = \phi_2(X_1, \dots, X_n)$ whose lifetimes $X_1, \dots, X_n$ are i.i.d. with a common continuous distribution function F.

1. *If $\boldsymbol{s}_1 \le_{st} \boldsymbol{s}_2$ then $(T_1 - t \mid X_{j\,:\,n} = t) \le_{st} (T_2 - t \mid X_{j\,:\,n} = t)$.*
2. *If $\boldsymbol{s}_1 \le_{rh} \boldsymbol{s}_2$ then $(T_1 - t \mid X_{j\,:\,n} = t) \le_{rh} (T_2 - t \mid X_{j\,:\,n} = t)$.*
3. *If $\boldsymbol{s}_1 \le_{lr} \boldsymbol{s}_2$ then $(T_1 - t \mid X_{j\,:\,n} = t) \le_{lr} (T_2 - t \mid X_{j\,:\,n} = t)$.*

Proof. (a) Define the function $h_{j,t,x}(k) = \mathbb{P}\{X_{k\,:\,n} - t > x \mid X_{j\,:\,n} = t\}$. Then, from Remark 2, we have $h_{j,t,x}(k) \le h_{j,t,x}(k+1)$, that is, $h_{j,t,x}$ is an increasing function of k. This implies that

$$\begin{aligned}
\mathbb{P}\{T_1 - t > x \mid X_{j\,:\,n} = t\} &= \sum_{k=i}^{n} s_{1,k}\mathbb{P}\{X_{k\,:\,n} - t > x \mid X_{j\,:\,n} = t\} \\
&\le \sum_{k=i}^{n} s_{2,k}\mathbb{P}\{X_{k\,:\,n} - t > x \mid X_{j\,:\,n} = t\} \quad (\text{since } \boldsymbol{s}_1 \le_{\text{st}} \boldsymbol{s}_2) \\
&= \mathbb{P}\{T_2 - t > x \mid X_{j\,:\,n} = t\},
\end{aligned}$$

where the last inequality follows from the fact that $h_{j,t,x}$ is an increasing function of k, and from Relation (1.A.7) of Shaked and Shanthikumar (2007). Note that in the above equations, $s_{1,k}$ and $s_{2,k}$ are precisely the probabilities that the kth ordered failure will cause the failure of the systems T_1 and T_2, respectively, for $k = i, i+1, \dots, n$.

(b) Now, assume that

$$H_1(x) = \mathbb{P}\{T_1 - t \le x \mid X_{j\,:\,n} = t\} \quad \text{and} \quad H_2(x) = \mathbb{P}\{T_2 - t \le x \mid X_{j\,:\,n} = t\}.$$

We need to prove that $H_1(x)/H_2(x)$ is a decreasing function of x, that is, if $x_1 < x_2$,

$$\frac{H_1(x_1)}{H_2(x_1)} \ge \frac{H_1(x_2)}{H_2(x_2)},$$

equivalently,

$$\frac{\sum_{k=i}^{n} s_{1,k}\mathbb{P}\{X_{k:n}-t\le x_1 \mid X_{j:n}=t\}}{\sum_{k=i}^{n} s_{2,k}\mathbb{P}\{X_{k:n}-t\le x_1 \mid X_{j:n}=t\}} \ge \frac{\sum_{k=i}^{n} s_{1,k}\mathbb{P}\{X_{k:n}-t\le x_2 \mid X_{j:n}=t\}}{\sum_{k=i}^{n} s_{2,k}\mathbb{P}\{X_{k:n}-t\le x_2 \mid X_{j:n}=t\}}$$

or

$$\frac{\sum_{k=i}^{n} s_{1,k}\mathbb{P}\{X_{k:n}-t\le x_1 \mid X_{j:n}=t\}}{\sum_{k=i}^{n} s_{1,k}\mathbb{P}\{X_{k:n}-t\le x_2 \mid X_{j:n}=t\}} \ge \frac{\sum_{k=i}^{n} s_{2,k}\mathbb{P}\{X_{k:n}-t\le x_1 \mid X_{j:n}=t\}}{\sum_{k=i}^{n} s_{2,k}\mathbb{P}\{X_{k:n}-t\le x_2 \mid X_{j:n}=t\}}. \tag{2}$$

Take $\alpha(k)=\mathbb{P}\{X_{k:n}-t\le x_1 \mid X_{j:n}=t\}$ and $\beta(k)=\mathbb{P}\{X_{k:n}-t\le x_2 \mid X_{j:n}=t\}$. Furthermore, let F_j be a discrete distribution with probability mass function $\boldsymbol{s}_j$ for $j=1,2$. Under the assumptions of the theorem it is easy to show that

$$\frac{\alpha(k)}{\beta(k)}=\frac{\mathbb{P}\{X_{k:n}-t\le x_1 \mid X_{j:n}=t\}}{\mathbb{P}\{X_{k:n}-t\le x_2 \mid X_{j:n}=t\}}$$

is a decreasing function of k, that is,

$$\frac{\alpha(k)}{\beta(k)}\ge\frac{\alpha(k+1)}{\beta(k+1)},$$

which is equivalent to

$$\frac{\mathbb{P}\{X_{k:n}-t\le x_1 \mid X_{j:n}=t\}}{\mathbb{P}\{X_{k:n}-t\le x_2 \mid X_{j:n}=t\}} \ge \frac{\mathbb{P}\{X_{k+1:n}-t\le x_1 \mid X_{j:n}=t\}}{\mathbb{P}\{X_{k+1:n}-t\le x_2 \mid X_{j:n}=t\}}$$

or

$$\frac{\mathbb{P}\{X_{k:n}-t\le x_1 \mid X_{j:n}=t\}}{\mathbb{P}\{X_{k+1:n}-t\le x_1 \mid X_{j:n}=t\}} \ge \frac{\mathbb{P}\{X_{k:n}-t\le x_2 \mid X_{j:n}=t\}}{\mathbb{P}\{X_{k+1:n}-t\le x_2 \mid X_{j:n}=t\}},$$

where the last inequality holds since

$$(X_{k:n}-t \mid X_{j:n}=t)\le_{\mathrm{rh}}(X_{k+1:n}-t \mid X_{j:n}=t). \tag{3}$$

It is clear that $\beta(k)=\mathbb{P}\{X_{k:n}-t\le x_2 \mid X_{j:n}=t\}$ is decreasing in k since (3) holds. The required result in (2) then follows from Theorem 1.B.50 of Shaked and Shanthikumar (2007) under the assumption that $\boldsymbol{s}_1\le_{\mathrm{rh}}\boldsymbol{s}_2$.

(c) For $l=1,2$, assume that

$$\mathbb{P}\{T_l-t>x \mid X_{j:n}=t\}=\sum_{k=i}^{n}s_{l,k}\mathbb{P}\{X_{k:n}-t>x \mid X_{j:n}=t\}, \qquad l=1,2,$$

and let $h_{l,j}(x)$ and $f_{j,k}(x)$ denote, respectively, the conditional densities of T_l and $X_{k:n}$, conditioned on $X_{j:n}=t$. Then, clearly,

$$h_{l,j}(x)=\sum_{k=i}^{n}s_{l,k}(t)f_{j,k}(x), \qquad l=1,2.$$

Now, we need to prove that $h_{1,j}(x)/h_{2,j}(x)$ is a decreasing function of x, i.e.

$$x_1<x_2\to\frac{h_{1,j}(x_1)}{h_{2,j}(x_1)}\ge\frac{h_{1,j}(x_2)}{h_{2,j}(x_2)},$$

which is equivalent to showing that, for $x_1<x_2$,

$$\sum_{k=i}^{n}\sum_{m=i}^{n}s_{1,k}(t)s_{2,m}(t)f_{j,k}(x_1)f_{j,m}(x_2)-\sum_{k=i}^{n}\sum_{j=i}^{n}s_{1,k}(t)s_{2,m}(t)f_{j,k}(x_2)f_{j,m}(x_1)\ge 0.$$

The left-hand side of the above inequality is

$$\begin{aligned}
&\sum_{k=i}^{n}\sum_{m=i}^{n} s_{1,k}s_{2,m}\{f_{j,k}(x_1)f_{j,m}(x_2) - f_{j,k}(x_2)f_{j,m}(x_1)\} \\
&\quad = \sum_{k=i}^{n}\sum_{m=i}^{k} s_{1,k}s_{2,m}\{f_{j,k}(x_1)f_{j,m}(x_2) - f_{j,k}(x_2)f_{j,m}(x_1)\} \\
&\qquad + \sum_{k=i}^{n}\sum_{m=k}^{n} s_{1,k}s_{2,m}\{f_{j,k}(x_1)f_{j,m}(x_2) - f_{j,k}(x_2)f_{j,m}(x_1)\} \\
&\quad = \sum_{m=i}^{n}\sum_{k=m}^{n} s_{1,k}s_{2,m}\{f_{j,k}(x_1)f_{j,m}(x_2) - f_{j,k}(x_2)f_{j,m}(x_1)\} \\
&\qquad + \sum_{k=i}^{n}\sum_{m=k}^{n} s_{1,k}s_{2,m}\{f_{j,k}(x_1)f_{j,m}(x_2) - f_{j,k}(x_2)f_{j,m}(x_1)\} \\
&\quad = \sum_{k=i}^{n}\sum_{m=k}^{n} s_{1,m}s_{2,k}\{f_{j,m}(x_1)f_{j,k}(x_2) - f_{j,m}(x_2)f_{j,k}(x_1)\} \\
&\qquad + \sum_{k=i}^{n}\sum_{m=k}^{n} s_{1,k}s_{2,m}\{f_{j,k}(x_1)f_{j,m}(x_2) - f_{j,k}(x_2)f_{j,m}(x_1)\} \\
&\quad = \sum_{k=i}^{n}\sum_{m=k}^{n} \{s_{1,k}s_{2,m} - s_{1,m}s_{2,k}\}\{f_{j,k}(x_1)f_{j,m}(x_2) - f_{j,k}(x_2)f_{j,m}(x_1)\} \\
&\quad \geq 0.
\end{aligned}$$

The last inequality follows because, for $k \leq m$, $(X_{k\,:\,n}-t \mid X_{j\,:\,n} = t) \leq_{\mathrm{lr}} (X_{m\,:\,n}-t \mid X_{j\,:\,n} = t)$ and $\boldsymbol{s}_1 \leq_{\mathrm{lr}} \boldsymbol{s}_2$.

Next, we prove that when the lifetimes of components of the system are IFR, then $\mathbb{P}\{T - t > x \mid X_{j\,:\,n} = t\}$ is a decreasing function of time t.

Theorem 2. *If the Xs are IFR then, for all $x > 0$, $\mathbb{P}\{T - t > x \mid X_{j\,:\,n} = t\}$ is a decreasing function of $t > 0$.*

Proof. From Part 3 of Remark 1, it is known that for all $x > 0$, $\mathbb{P}\{X_{k\,:\,n} - t > x \mid X_{j\,:\,n} = t\}$ is a decreasing function of $t > 0$, and so

$$\mathbb{P}\{T - t > x \mid X_{j\,:\,n} = t\} = \sum_{k=i}^{n} s_k \mathbb{P}\{X_{k\,:\,n} - t > x \mid X_{j\,:\,n} = t\}$$

is a decreasing function of $t > 0$.

Now, we shall consider two coherent systems with n components each having their lifetimes to be i.i.d. continuous random variables with distribution functions F and G, respectively.

Theorem 3. *Let T_X (T_Y) be the lifetime of a coherent system with n components whose lifetimes X_i (Y_i), $i = 1, \ldots, n$, are i.i.d. random variables with an absolutely continuous distribution function F (G). If $X_1 \leq_{hr} Y_1$ then $(T_X - t \mid X_{j\,:\,n} = t) \leq_{st} (T_Y - t \mid Y_{j\,:\,n} = t)$.*

Proof. If $X_1 \le_{\mathrm{hr}} Y_1$ then, for all $x, t \ge 0$, we have

$$\frac{\bar{G}(t)}{\bar{F}(t)} \le \frac{\bar{G}(t+x)}{\bar{F}(t+x)},$$

which is equivalent to $\bar{F}_t(x) \le \bar{G}_t(x)$. Then, Theorem 1.A.23 of Shaked and Shanthikumar (2007) implies that

$$(X_{k:n} - t \mid X_{j:n} = t) \overset{\mathrm{D}}{=} X^t_{k-j:n-j} \le_{\mathrm{st}} Y^t_{k-j:n-j} \overset{\mathrm{D}}{=} (Y_{k:n} - t \mid Y_{j:n} = t).$$

Moreover, the signature vector is distribution-free which means $\boldsymbol{s}^F = \boldsymbol{s}^G$, where $\boldsymbol{s}^F = (0, \ldots, s^F_i, \ldots, s^F_n)$ ($\boldsymbol{s}^G = (0, \ldots, s^G_i, \ldots, s^G_n)$) is the signature vector of T_X (T_Y). Therefore,

$$\begin{aligned}
\mathbb{P}\{T_X - t > x \mid X_{j:n} = t\} &= \sum_{k=i}^{n} s^F_k \mathbb{P}\{X_{k:n} - t > x \mid X_{j:n} = t\} \\
&\le \sum_{k=i}^{n} s^F_k \mathbb{P}\{Y_{k:n} - t > x \mid Y_{j:n} = t\} \\
&= \sum_{k=i}^{n} s^G_k \mathbb{P}\{Y_{k:n} - t > x \mid Y_{j:n} = t\} \\
&= \mathbb{P}\{T_Y - t > x \mid Y_{j:n} = t\}.
\end{aligned}$$

Remark 3. If we consider the mean residual life $M_{j,T}(t) = \mathbb{E}(T - t \mid X_{j:n} = t)$ then

$$M_{j,T}(t) = \mathbb{E}(T - t \mid X_{j:n} = t) = \sum_{k=i}^{n} s_k \mathbb{E}(X_{k:n} - t \mid X_{j:n} = t).$$

It is clear from Remark 1 that, for fixed n and k and for all $t > 0$, $M_{j,T}(t)$ is a decreasing function in j. Also, from Theorem 2, it is clear that if the lifetimes of components of the system are IFR then $M_{j,T}(t)$ is a decreasing function of time (in t). Furthermore, under the assumption of Theorem 3, we have $M^F_{j,T}(t) \le M^G_{j,T}(t)$.

3. Inactivity time of the system

Let T be the lifetime of a reliability system of order n and $X_1, \ldots, X_n$ be the lifetimes of its components. Suppose $X_1, \ldots, X_n$ are i.i.d. variables having a common underlying continuous distribution F. Now, we assume that the signature of the system has the form (ii), that is,

$$\boldsymbol{s} = (s_1, \ldots, s_i, 0, \ldots, 0) \quad \text{for } i = 1, \ldots, n-1.$$

In this section we study the reliability and stochastic properties of the inactivity time of the system, that is, the variable

$$(t - T \mid X_{j:n} = t) \quad \text{for } j = i, \ldots, n.$$

As in the case of residual lifetime, for $t > 0$ and $0 \le x \le t$, we have

$$\begin{aligned}
\mathbb{P}\{t - T > x \mid X_{j:n} = t\} &= \sum_{k=i}^{n} \mathbb{P}\{t - T > x, T = X_{k:n} \mid X_{j:n} = t\} \\
&= \sum_{k=i}^{n} \mathbb{P}\{t - X_{k:n} > x, T = X_{k:n} \mid X_{j:n} = t\}
\end{aligned}$$

$$= \sum_{k=i}^{n} \mathbb{P}\{T = X_{k\,:\,n}\}\mathbb{P}\{t - X_{k\,:\,n} > x \mid X_{j\,:\,n} = t\}$$
$$= \sum_{k=i}^{n} s_k \mathbb{P}\{t - X_{k\,:\,n} > x \mid X_{j\,:\,n} = t\}.$$

Remark 4. (1) As in Remark 1, for $j > k$, we can easily show that (Nama and Asadi (2013))

$$(t - X_{k\,:\,n} \mid X_{j\,:\,n} = t) \stackrel{\mathrm{D}}{=} X^t_{j-k\,:\,j-1},$$

where $X^t_{j-k\,:\,j-1}$ denotes the $(j-k)$th order statistic in a random sample of size $(j-1)$ from the right truncated distribution with the survival function $\bar{F}_t(x) = F(x-t)/F(t)$, $0 < x < t$. Also, for $k < j_1 \le j_2$, we have (Nama and Asadi (2013))

$$X^t_{j_1-k\,:\,j_1-1} \le_{\mathrm{lr}} X^t_{j_2-k\,:\,j_2-1}, \qquad (t - X_{k\,:\,n} \mid X_{j_1\,:\,n} = t) \le_{\mathrm{lr}} (t - X_{k\,:\,n} \mid X_{j_2\,:\,n} = t),$$

and, thus, $\mathbb{P}\{t - X_{k\,:\,n} > x \mid X_{j\,:\,n} = t\}$ is an increasing function of j for all $t > 0$ and $x > 0$.

(2) Also, as in Remark 2, we have $(t - X_{k_1\,:\,n} \mid X_{j\,:\,n} = t) \ge_{\mathrm{lr}} (t - X_{k_2\,:\,n} \mid X_{j\,:\,n} = t)$ and so $\mathbb{P}\{t - X_{k\,:\,n} > x \mid X_{j\,:\,n} = t\}$ is a decreasing function of k for all $t > 0$ and $x > 0$.

From Remark 4, Theorem 4 follows readily and its proof is similar to that of Theorem 1.

Theorem 4. *Let*

$$\boldsymbol{s}_1 = (0, \dots, 0, s_{1,i}, s_{1,i+1}, \dots, s_{1,n}) \quad \textit{and} \quad \boldsymbol{s}_2 = (0, \dots, 0, s_{2,i}, s_{2,i+1}, \dots, s_{2,n})$$

be the signatures of two coherent systems $T_1 = \phi_1(X_1, \dots, X_n)$ and $T_2 = \phi_2(X_1, \dots, X_n)$, with component lifetimes $X_1, \dots, X_n$ being i.i.d. with common continuous distribution function F.

1. *If $\boldsymbol{s}_1 \le_{st} \boldsymbol{s}_2$ then $(T_1 - t \mid X_{j\,:\,n} = t) \ge_{st} (T_2 - t \mid X_{j\,:\,n} = t)$.*
2. *If $\boldsymbol{s}_1 \le_{hr} \boldsymbol{s}_2$ then $(T_1 - t \mid X_{j\,:\,n} = t) \ge_{hr} (T_2 - t \mid X_{j\,:\,n} = t)$.*
3. *If $\boldsymbol{s}_1 \le_{lr} \boldsymbol{s}_2$ then $(T_1 - t \mid X_{j\,:\,n} = t) \ge_{lr} (T_2 - t \mid X_{j\,:\,n} = t)$.*

If the Xs are DRHR (decreasing reversed hazard rate) then, for $j > k$ and all $x > 0$, $\mathbb{P}\{t - X_{k\,:\,n} > x \mid X_{j\,:\,n} = t\}$ is an increasing function of $t > 0$ (Nama and Asadi (2013)). The following theorem and its proof are similar to that of Theorem 2.

Theorem 5. *If the Xs are DRHR then, for all $x > 0$, $\mathbb{P}\{t - T > x \mid X_{j\,:\,n} = t\}$ is an increasing function of $t > 0$.*

Finally, we have the following theorem in the spirit of Theorem 3.

Theorem 6. *Let T_X (T_Y) be the lifetime of a coherent system with n components whose lifetimes X_i (Y_i), $i = 1, \dots, n$, are i.i.d. random variables with an absolutely continuous distribution function F (G). If $X_1 \le_{rh} Y_1$ then $(T_X - t \mid X_{j\,:\,n} = t) \ge_{st} (T_Y - t \mid Y_{j\,:\,n} = t)$.*

Remark 5. Consider the mean inactivity time function $K_{j,T}(t) = \mathbb{E}(t - T \mid X_{j\,:\,n} = t)$, then

$$K_{j,T}(t) = \mathbb{E}(t - T \mid X_{j\,:\,n} = t) = \sum_{k=i}^{n} s_k \mathbb{E}(t - X_{k\,:\,n} \mid X_{j\,:\,n} = t).$$

It is clear that for fixed n and k and for all $t > 0$, $M_{j,T}(t)$ is an increasing function in j. Also, from Theorem 5, it is clear that if the components of the system are DRHR then $M_{j,T}(t)$ is an increasing function of time (in t). Furthermore, under the assumption of Theorem 6, $M^F_{j,T}(t) \geq M^G_{j,T}(t)$.

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