GIBBS PHENOMENON FOR THE HAUSDORFF MEANS OF DOUBLE SEQUENCES [1]

FRED USTINA

(Received 31 January 1969)

Communicated by E. Strzelecki

1. Introductory Remarks

Let $g(u)$ be a regular Hausdorff weight function, and let $h_m(\psi; x_m)$ denote the m-th corresponding Hausdorff transform, evaluated at x_m, of the sequence of partial sums of the Fourier series of $\psi(x)$, where

$$(1)\qquad \begin{aligned} \psi(t) &= 0, && t = 0 \\ &= \tfrac{1}{2}(\pi - t), && 0 < t < 2\pi \\ &= \psi(t+2k\pi), && k = \pm 1, \pm 2, \pm 3, \cdots. \end{aligned}$$

In [3], Szász investigated the Gibbs phenomenon for $\psi(x)$ for these means. His main results are contained in the following two theorems:

(2) THEOREM 1. *If* $mx_m \to \tau,\ 0 < \tau \leqq \infty$, *as* $x_m \to 0$ *and* $m \to \infty$, *then*

$$h_m(\psi; x_m) \to \int_0^1 \int_0^\tau \frac{\sin su}{s}\, ds\, dg(u).$$

(3) THEOREM 2. *Taking the limit superior as* $m \to \infty$ *and* $x \to 0$,

$$\limsup h_m(\psi; x) = \max_{\tau > 0} \int_0^1 \{1 - g(u)\} \frac{\sin \tau u}{u}\, du.$$

If this maximum is attained for $\tau = \tau'$, *then*

$$\limsup h_m(\psi; x) = \lim_{mx_m \to \tau'} h_m(\psi; x_m).$$

We will extend these results to the two dimensional case. Let $g(u, v)$ be a regular Hausdorff weight function, and let $h_{m,n}(\varphi; x, y)$ denote the mn-th corresponding Hausdorff transform of the Fourier series of the function $\varphi(x, y)$, evaluated at (x, y). With $\varphi(x, y) = \psi(x)\psi(y)$, we prove

[1] The author is indebted to Professor A. E. Livingston for suggesting a thesis topic, and for supervising the preparation of a thesis, of which this paper is a part.

(4) THEOREM 3. *If $h_{m,n}(\psi; x)$ $\{h_{m,n}(\psi; y)\}$ denotes the mn-th regular Hausdorff transform of the function $\psi(x)$ $\{\psi(y)\}$, corresponding to the weight function $g(u, v)$, then*

$$h_{m,n}(\psi; x) = h_m(\psi; x)$$
$$h_{m,n}(\psi; y) = h_n(\psi; y)$$

where $h_m(\psi; x)$ $\{h_n(\psi; y)\}$ denotes the m-th $\{n\text{-th}\}$ regular one dimensional Hausdorff transform corresponding to the weight function $g(u, 1)$ $\{g(1, v)\}$.

(5) THEOREM 4. *If $mx_m \to \tau_1$, $ny_n \to \tau_2$, where $0 < \tau_i \leqq \infty$, $i = 1, 2$, and $mx_m^2 = O(1)$, $ny_n^2 = O(1)$ as $m, n \to \infty$ and $x_m, y_n \to 0$, then*

$$h_{m,n}(\varphi; x_m, y_n) \to \int_{0,0}^{1,1} \int_{0,0}^{\tau_2, \tau_1} \frac{\sin su}{s} \frac{\sin tv}{t} ds dt d^2 g(u, v).$$

(6) THEOREM 5. *Taking the limit superior as $m, n \to \infty$ and $x, y \to 0$,*

$$\limsup h_{m,n}(\varphi; x, y) = \max_{\tau_1, \tau_2 > 0} \int_{0,0}^{1,1} \frac{\sin \tau_1 u}{u} \frac{\sin \tau_2 v}{v} g(1, 1; u, v) du dv$$

where

$$g(1, 1; u, v) = g(1, 1) - g(1, v) - g(u, 1) + g(u, v).$$

If this maximum is attained for $\tau_1 = \tau_1'$, $\tau_2 = \tau_2'$, then, taking the limit as $mx_m \to \tau_1'$ and $ny_n \to \tau_2'$,

$$\limsup h_{m,n}(\varphi; x, y) = \lim h_{m,n}(\varphi; x_m, y_n).$$

The extension of Theorems 1 and 2 to the two dimensional case is, of course, trivial if $g(u, v)$ factorizes: $g(u, v) = g_1(u)g_2(v)$. We will therefore consider the case where no such factorization is possible.

Next, we quote the theorem proved in [5].

(7) THEOREM 6. *Let $f(x, y)$ be a normalized function, periodic in each variable, and of bounded variation in the sense of Hardy-Krause in the period rectangle. The Gibbs phenomenon for $f(x, y)$ at $(0, 0)$ is the same as the Gibbs phenomenon for the function*

$$\theta(f; x, y) = \frac{c}{\pi^2} \varphi(x, y) + g_1(0)\psi(x) + g_2(0)\psi(y)$$

where

$$c = f(0^+, 0^+) - f(0^+, 0^-) - f(0^-, 0^+) + f(0^-, 0^-)$$

$$g_1(y) = \frac{1}{\pi} \{f(0^+, y) - f(0^-, y)\} - \frac{c}{2\pi} \operatorname{sgn} y$$

$$g_2(x) = \frac{1}{\pi} \{f(x, 0^+) - f(x, 0^-)\} - \frac{c}{2\pi} \operatorname{sgn} x.$$

Combining this theorem with Theorems 3 and 5, and Theorem 2 due to Szász, then yields

(8) THEOREM 7. *If $f(x, y)$ is a 2π-periodic function of bounded variation in the sense of Hardy-Krause, with a discontinuity at the origin, and if $h_{m,n}(f; x, y)$ denotes the mn-*th *regular Hausdorff transform of the sequence of partial sums of the Fourier series of $f(x, y)$, relative to the Hausdorff weight function $g(u, v)$, then, taking the limit superior as $m, n \to \infty$ and $x, y \to 0$,*

$$\limsup h_{m,n}(f; x, y) = \max_{\tau_1, \tau_2} \left\{ \frac{c}{\pi^2} \int_{0,0}^{1,1} \frac{\sin \tau_1 u}{u} \frac{\sin \tau_2 v}{v} g(1, 1; u, v) du dv + g_1(0) \int_0^1 \frac{\sin \tau_1 u}{u} \{1 - g(u, 1)\} du + g_2(0) \int_0^1 \frac{\sin \tau_2 v}{v} \{1 - g(1, v)\} dv \right\},$$

where c, $g_1(y)$ and $g_2(x)$ are defined in (7).

(9) REMARKS. In general, τ_1 and τ_2 cannot be restricted to positive values in the statement of Theorem 7. Note also that the apparent restriction of the statement of Theorems 6 and 7 to the origin is readily removed by a translation of the axes.

2. Proof of Theorem 3

In the sequel, $g(u, v)$ is a regular Hausdorff weight function satisfying

$$g(u, 0) = g(u, 0^+) = g(0, v) = g(0^+, v) = 0 \tag{10}$$

so that $g(1, 1) = 1$. The restrictions on the weight function are adequate to insure that $g(u, 1)$ and $g(1, v)$ are regular Hausdorff weight functions in the one dimensional case.

If $s_k(\psi; x)$ and $s_{k,l}(\varphi; x, y)$ denote the k-th and kl-th partial sums of the Fourier series of $\psi(x)$ and $\varphi(x, y)$, then

$$\begin{aligned} s_k(\psi; x) &= \sum_{i=1}^{k} \frac{\sin ix}{i} \\ &= -\tfrac{1}{2}x + \tfrac{1}{2} \int_0^x \frac{\sin (k+\frac{1}{2}) s}{\sin \frac{1}{2} s} ds \end{aligned} \tag{11}$$

and

https://doi.org/10.1017/S1446788700006352 Published online by Cambridge University Press

$$
\begin{aligned}
s_{kl}(\varphi; x, y) &= \sum_{i=1}^{k} \frac{\sin ix}{i} \sum_{j=1}^{l} \frac{\sin jy}{j} \\
&= \left\{-\tfrac{1}{2}x+\tfrac{1}{2}\int_0^x \frac{\sin (k+\frac{1}{2})s}{\sin \frac{1}{2}s}\,ds\right\} \cdot \\
&\quad \left\{-\tfrac{1}{2}y+\tfrac{1}{2}\int_0^y \frac{\sin (l+\frac{1}{2})t}{\sin \frac{1}{2}t}\,dt\right\}
\end{aligned} \tag{12}
$$

Then by [4], theorem 5,

$$
\begin{aligned}
h_{m,n}(\psi; x) &= \sum_{k,l=0}^{m,n} \binom{n}{l}\binom{m}{k} s_k(\psi; x) \int_{0,0}^{1,1} u^k(1-u)^{m-k}v^l(1-v)^{n-l}d^2g(u, v) \\
&= \sum_{k=0}^{m} \binom{m}{k} s_k(\psi; x) \int_0^1 u^k(1-u)^{m-k}dg(u, 1) \\
&= h_m(\psi; x),
\end{aligned}
$$

where we have used the fact that $dg(u, 0) \equiv 0$. This proves half of Theorem 3. The other half is proved in a similar manner.

3. Some Preliminary Lemmas

In this section we collect a few lemmas which we will use in the proof of Theorem 4. In the case of Lemmas 1 and 3, the integral $\int_{0,0}^{y,x}\{\ \}dsdt$ is to be interpreted in the improper sense in the event that the integrand is not defined over the entire rectangle $[x, y; 0, 0]$. The improper integral clearly exists under the stated hypothesis.

Note, also, that $\psi(x)$, $\psi(y)$ and $\varphi(x, y)$ are all odd, periodic functions of period 2π. It follows that to investigate the Gibbs phenomenon for these functions, it is sufficient to investigate it in the region $0 \leqq x, y \leqq \pi$. We will assume this restriction on the variables x, y in the sequel without further explicit mention.

(13) Definition. Let

$$
\rho_1 \sin \alpha = u \sin s, \qquad \rho_1 \cos \alpha = 1-u+u \cos s
$$
$$
\rho_2 \sin \beta = v \sin t, \qquad \rho_2 \cos \beta = 1-v+v \cos t,
$$

so that $0 \leqq \rho_1, \rho_2 \leqq 1$ for $0 \leqq u, v \leqq 1$, and $0 \leqq s, t \leqq \pi$. Then also

$$
\rho_1 e^{i\alpha} = 1-u+ue^{is}
$$
$$
\rho_2 e^{i\beta} = 1-v+ve^{it}.
$$

(14) Lemma 1. *If $g(u, v)$ is a regular Hausdorff weight function and if*

$$
f(m, n; s, t, u, v) = O(s^p t^q), \qquad p, q > -1,
$$

https://doi.org/10.1017/S1446788700006352 Published online by Cambridge University Press

uniformly in m, n, u and v, and if f is continuous in u and v, then

$$\int_{0,0}^{1,1}\int_{0,0}^{y,x} f(m, n; s, t, u, v)\, ds\, dt\, d^2 g(u, v) = O(x^{p+1} y^{q+1}) = O(x^{p+1}) = O(y^{q+1}),$$

$m, m = 1, 2, 3, \cdots$.

PROOF. Lemma 1 is obvious since $0 \leqq x, y \leqq \pi$ and $g(u, v)$ is of bounded variation.

(15) LEMMA 2. *If $g(u, v)$ is a regular Hausdorff weight function, then*

$$\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m \sin m\alpha \cot \frac{s}{2}\, ds\, dt\, d^2 g(u, v) = O(y), \qquad m = 1, 2, 3, \cdots.$$

PROOF. Under the hypothesis, the integral in the lemma equals

$$y \int_0^1 \int_0^x \rho_1^m \sin m\alpha \cot \frac{s}{2}\, ds\, dg(u, 1).$$

The problem then reduces to proving that this last integral is $O(1)$, $m = 1, 2, 3, \cdots$. Observe first that

$$\begin{aligned}
&\sum_{k=0}^{m} \binom{m}{k} \sin\,(k+\tfrac{1}{2}) s\, u^k (1-u)^{m-k} \\
&\quad = \sum_{k=0}^{m} \binom{m}{k} \operatorname{Im} e^{i(k+\frac{1}{2})s} u^k (1-u)^{m-k} \\
&\quad = \operatorname{Im} \{(1-u+ue^{is})^m e^{i(s/2)}\} \qquad (16)\\
&\quad = \operatorname{Im} \rho_1^m e^{i(m\alpha+(s/2))} \\
&\quad = \rho_1^m \left\{ \sin m\alpha \cos \frac{s}{2} + \cos m\alpha \sin \frac{s}{2} \right\}.
\end{aligned}$$

Then with $s_{k,l}(\psi; x) = s_k(\psi; x)$, where $s_k(\psi; x)$ is given by (11),

$$\begin{aligned}
h_{m,n}(\psi; x) &= \sum_{k,l=0}^{m,n} \binom{n}{l} \cdot \binom{m}{k} s_{k,l}(x) \int_{0,0}^{1,1} u^k (1-u)^{m-k} v^l (1-v)^{n-l} d^2 g(u, v) \\
&= -\tfrac{1}{2}x + \tfrac{1}{2} \sum_{k=0}^{m} \binom{m}{k} \int_0^x \frac{\sin\,(k+\frac{1}{2})s}{\sin \frac{1}{2}s}\, ds \int_0^1 u^k (1-u)^{m-k} dg(u, 1) \\
&= -\tfrac{1}{2}x + \tfrac{1}{2} \int_0^1 \int_0^x \sum_{k=0}^{m} \sin\,(k+\tfrac{1}{2})s\, \frac{u^k (1-u)^{m-k}}{\sin \frac{1}{2}s}\, ds\, dg(u, 1) \\
&= -\tfrac{1}{2}x + \tfrac{1}{2} \int_0^1 \int_0^x \rho_1^m \left\{ \sin m\alpha \cot \frac{s}{2} + \cos m\alpha \right\} ds\, dg(u, 1) \\
&= \tfrac{1}{2} \int_0^1 \int_0^x \rho_1^m \sin m\alpha \cot \frac{s}{2}\, ds\, dg(u, 1) + O(1)
\end{aligned}$$

https://doi.org/10.1017/S1446788700006352 Published online by Cambridge University Press

where the last step follows the observation that $0 \leqq \rho_1 \leqq 1$. Also, the change in the order of integration and summation is clearly justified since all quantities are finite. The lemma now follows, since $h_{m,n}(\psi; x)$ is a regular transform of a bounded sequence, and hence the sequence $\{h_{m,n}(\psi; x)\}$ is bounded uniformly in m, n.

(17) Corollary 1. *If $g(u, v)$ is a regular Hausdorff weight function, then*

$$\sum_{k,l=0}^{m,n} \binom{n}{l}\binom{m}{k} \int_{0,0}^{y,x} \frac{\sin (k+\frac{1}{2})s}{\sin \frac{1}{2}s} ds\,dt \int_{0,0}^{1,1} u^k(1-u)^{m-k} v^l (1-v)^{n-l} d^2 g(u, v) = O(y),$$
$$m, n = 1, 2, 3, \cdots.$$

(18) Corollary 2. *If $g(u, v)$ is a regular Hausdorff weight function, then*

$$\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m \frac{\sin m\alpha}{s} ds\,dt\,d^2 g(u, v) = O(y), \quad m = 1, 2, 2, \cdots.$$

Proof. Corollary 1 follows immediately from the proof of Lemma 2. Corollary 2 also follows from Lemma 2 after observing that

$$\left| \cot \frac{s}{2} - \frac{2}{s} \right| \leqq \frac{2}{\pi}, \qquad 0 \leqq s \leqq \pi.$$

Then

$$\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m \sin m\alpha \cot \frac{s}{2} ds\,dt\,d^2 g(u, v)$$
$$-2 \int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m \frac{\sin m\alpha}{s} ds\,dt\,d^2 g(u, v)$$
$$= \int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m \sin m\alpha \left\{ \cot \frac{s}{2} - \frac{2}{s} \right\} ds\,dt\,d^2 g(u, v)$$
$$= O(y)$$

by Lemma 1.

(19) Remarks. Lemmas 1 and 2 are also valid under the less restricted hypothesis requiring only that $g(u, v)$ is of bounded variation (in the sense of Hardy-Krause) in the square $[1, 1; 0, 0]$, for then the Hausdorff method corresponding to $g(u,v)$ will take bounded sequences into bounded sequences. The proof of Lemma 1 is again immediate. The proof of Lemma 2 requires a minor modification. We prove the next lemma under this less restricted hypothesis.

(20) Lemma 3. If $g(u, v)$ is of bounded variation and if $\gamma(n; t, v) = O(t^p)$, $p > -1$, is continuous in v and uniformly bounded in n, v, then

$$\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m \frac{\sin m\alpha}{s} \gamma(n; t, v)\, ds\,dt\,d^2 g(u, v) = O(y^{r+1}), \quad m, n = 1, 2, 3, \cdots.$$

PROOF. As a function of bounded variation, $g(u, v)$ may be expressed as the difference of its positive and its negative variation functions, each of which is positively monotonic [2] on the square $[1, 1; 0, 0]$, and each part may then be considered separately. Hence without loss of generality, we may assume that $g(u, v)$ is already positively monotonic on the unit square. Also, since otherwise we could split $\gamma(n; t, v)$ into its positive and negative parts and treat each part separately, we may assume that $\gamma(n; t, v)$ is already non-negative. We write

$$d^2 f(n; y, u, v) = \int_0^v \gamma(n; t, v) dt \cdot d^2 g(u, v), \tag{21}$$

and since $\gamma(n; t, v) = O(t^p)$, we have

$$0 \leqq \gamma(n; t, v) \leqq M' t^p$$

for some constant M'. Then also

$$V(f_n) \leqq M y^{p+1} V(g), \quad f_n = f(n; y, u, v), \tag{22}$$

and

$$d^2 f(n; y, u, v) \leqq M y^{p+1} d^2 g(u, v), \tag{23}$$

where $(p+1)M = M'$ and $V(h)$ denotes the total variation of the function h in the unit square. Thus for each fixed y, there exists a function $f(n; y, u, v)$ for which (21) is satisfied, and the sequence of total variations of these functions, $\{V(f_n)\}$, is bounded uniformly in n for $0 \leqq y \leqq \pi$. The sequence of Hausdorff transformation methods, corresponding to the sequence of weight functions $\{f(n; y, u, v)\}$, then take bounded sequences into sequences which are bounded uniformly in n.

Let $\{s_k(x)\}$ be any non-negative, bounded sequence which is independent of y. Again, we assume the non-negative property as a matter of convenience only. Relative to the weight function $g(u, v)$, the mn-th Hausdorff transform of this sequence is given by

$$h_{m,n}(g; x) = \sum_{k,l=0}^{m,n} \binom{n}{l} \binom{m}{k} s_k(x) \int_{0,0}^{1,1} u^k (1-u)^{m-k} v^l (1-v)^{n-l} d^2 g(u, v).$$

Now relative to a function $f(i; y, u, v)$ of the sequence $\{f(n; y, u, v)\}$, it is given by

[2] A function $g(u, v)$ is said to be positively monotonic on the domain D if

$$g(a_2, b_2) - g(a_2, b_1) - g(a_1, b_2) + g(a_1, b_1) \geqq 0$$

whenever (a_1, b_1), (a_2, b_2) are in D and $a_2 \geqq a_1$ and $b_2 \geqq b_1$.

$$h_{m,n}(f_i;x) = \sum_{k,l=0}^{m,n} \binom{n}{l}\binom{m}{k} s_k(x) \int_{0,0}^{1,1} u^k(1-u)^{m-k}v^l(1-v)^{n-l}d^2f(i;y,u,v)$$
$$\leqq My^{p+1} \sum_{k,l=0}^{m,n} \binom{n}{l}\binom{m}{k} s_k(x) \int_{0,0}^{1,1} u^k(1-u)^{m-k}v^l(1-v)^{n-l}d^2g(u,v)$$
$$= My^{p+1}h_{m,n}(g;x),$$

where we have used (23) and the assumption that all the terms on the right are non-negative. Next, noting that $h_{m,n}(g;x)$ is a regular transform of a bounded sequence so that $h_{m,n}(g;x) = O(1)$, $m, n = 1, 2, 3, \cdots$, we have

$$h_{m,n}(f_i;x) = O(y^{p+1}). \tag{24}$$

To complete the proof of the lemma, for the sequence $\{s_k(x)\}$, we take the sequence $\{s_k(\psi;x)\}$, where $s_k(\psi;x)$ is given by (11). Then

$$h_{m,n}(f_n;x) = \sum_{k,l=0}^{m,n} \binom{n}{l}\binom{m}{k} \left\{-\tfrac{1}{2}x+\tfrac{1}{2}\int_0^x \frac{\sin(k+\frac{1}{2})s}{\sin\frac{1}{2}s}\,ds\right\}$$
$$\cdot \int_{0,0}^{1,1} u^k(1-u)^{m-k}v^l(1-v)^{n-l}d^2f(n;y,u,v)$$
$$= -\tfrac{1}{2}x\int_{0,0}^{1,1} d^2f(n;y,u,v)$$
$$+\tfrac{1}{2}\sum_{k=0}^{m}\binom{m}{k}\int_0^x \frac{\sin(k+\frac{1}{2})s}{\sin\frac{1}{2}s}\,ds\int_{0,0}^{1,1} u^k(1-u)^{m-k}d^2f(n;y,u,v)$$
$$= \tfrac{1}{2}\int_{0,0}^{1,1}\int_0^x \rho_1^m \sin m\alpha \cot\frac{s}{2}\,ds\,d^2f(n;y,u,v)$$
$$+\tfrac{1}{2}\int_{0,0}^{1,1}\int_0^x \rho_1^m \cos m\alpha\,ds\,d^2f(n;y,u,v)+O(y^{p+1})$$
$$= \tfrac{1}{2}\int_{0,0}^{1,1}\int_0^x \rho_1^m \sin m\alpha\left\{\cot\frac{s}{2}-\frac{2}{s}+\frac{2}{s}\right\}ds\,d^2f(n;y,u,v)+O(y^{p+1})$$
$$= \int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m \frac{\sin m\alpha}{s}\gamma(n;t,v)\,ds\,dt\,d^2g(u,v)+O(y^{p+1})$$
$$= O(y^{p+1})$$

where we have used (16), (23), (21) and (24). This completes the proof of the lemma.

(25) Corollary. *If $g(u,v)$ is of bounded variation and if $\gamma(n;t,v)$ is a function which is continuous in v and uniformly bounded in n, t and v, then*

$$\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m \frac{\sin m\alpha}{s}\gamma(n;t,v)\,ds\,dt\,d^2g(u,v) = O(y), \quad m, n = 1, 2, 3, \cdots.$$

https://doi.org/10.1017/S1446788700006352 Published online by Cambridge University Press

(26) REMARKS. In the applications which we have in mind, the functions $\gamma(n; t, v)$ are continuous in t and in v. However, for the purposes of the lemma, continuity in t is not required, and continuity in v may be relaxed to require continuity only at points of discontinuity of the function $g(u, v)$.

The next five lemmas are either due to Szász [3] or are trivial extensions of some of his results. We indicate a proof in cases where it is not immediately obvious how a result as stated here follows from his result. The number following the lemma number indicates the page of his article on which the stated result may be found.

(27) LEMMA 4. (444)

$$\rho_1^m = 1-\lambda_1 m(1-\rho_1^2) = 1+O(ms^2)$$
$$\rho_2^n = 1-\lambda_2 n(1-\rho_2^2) = 1+O(nt^2)$$

where $0 < \lambda_1, \lambda_2 < 1$.

PROOF.

$$1-\rho_1^m = (1-\rho_1)\sum_{k=1}^{m-1}\rho_1^k \leqq m(1-\rho_1) \leqq m(1-\rho_1^2).$$

Hence

$$1-\rho_1^m = \lambda_1 m(1-\rho_1^2),\ 0 < \lambda_1 < 1$$

or

$$\rho_1^m = 1-\lambda_1 m(1-\rho_1^2)$$

Now by (13),

$$1-\rho_1^2 = 4u(1-u)\sin^2\frac{s}{2} \leqq s^2,$$

so that

$$\rho_1^m = 1+O(ms^2).$$

(28) LEMMA 5. (445)

$$\sin m\alpha = \sin msu+2\cos\frac{m}{2}(\alpha+us)\sin O(mus^3)$$
$$= \sin msu+O(mus^3)$$

(29) LEMMA 6. (446)

$$\rho_1^m = e^{-(m/2)(1-\rho_1^2)}-m\beta_1(1-\rho_1^2)e^{-(m/2)(1-\rho_1^2)}$$
$$= e^{-2mu(1-u)\sin^2(s/2)}-m\beta_1(1-\rho_1^2)^2e^{-(m/2)(1-\rho_1^2)}, \quad 0 < \beta_1 < 1.$$

(30) LEMMA 7.

$$\rho_1^m = e^{-(m/2)u(1-u)s^2}+O(ms^4),$$
$$0 \leqq u \leqq 1,\ 0 \leqq s \leqq \pi \text{ and } O(ms^4) = O(1).$$

https://doi.org/10.1017/S1446788700006352 Published online by Cambridge University Press

PROOF. Observe that

$$0 \leqq \left(\frac{s}{2}\right)^2 - \sin^2\frac{s}{2} = \left(\frac{s}{2}\right)^2 - \tfrac{1}{2}(1-\cos s) = \tfrac{1}{2}\left\{\frac{s^4}{4!} - \frac{s^6}{6!} + \cdots\right\} \leqq \tfrac{1}{2}\left(\frac{s}{2}\right)^4.$$

Then if $a \geqq 0$,

$$\begin{aligned} e^{a(s/2)^2 - a\sin^2 s/2} - 1 &\leqq e^{a/2(s/2)^4} - 1 \\ &= e^{a/2(s/2)^4}\{1 - e^{-a/2(s/2)^4}\} < \frac{a}{2}\left(\frac{s}{2}\right)^4 e^{a/2(s/2)^4} \\ &= O(as^4), \quad O(as^4) = O(1). \end{aligned}$$

Then

$$e^{-a\sin^2 s/2} = e^{-a(s/2)^2} + O(as^4),$$

and with $a = 2mu(1-u)$,

(31) $$e^{-2mu(1-u)\sin^2 s/2} = e^{-(m/2)u(1-u)s^2} + O(ms^4).$$

Next,

$$\left(\frac{s}{\pi}\right)^2 \leqq \sin^2\frac{s}{2} \leqq \left(\frac{s}{2}\right)^2, \qquad 0 \leqq s \leqq \pi$$

so that

$$\left(\frac{2s}{\pi}\right)^2 u(1-u) \leqq 1-\rho_1^2 \leqq s^2u(1-u)$$

and

$$\begin{aligned} m\beta_1(1-\rho_1^2)^2 e^{-m/2(1-\rho_1^2)} & \\ &\leqq m\beta_1 s^4u^2(1-u)^2 e^{-2m(s/\pi)^2u(1-u)} \\ &= O(ms^4). \end{aligned}$$

This, together with (29) and (31), implies the lemma.

(32) LEMMA 8. (451)

$$\int_0^{x_m} e^{-(m/2)u(1-u)s^2}\frac{\sin msu}{s}\,ds \to \frac{\pi}{2},$$

uniformly in $0 < \varepsilon \leqq u \leqq 1$, *and boundedly in* $0 \leqq u \leqq 1$, *as* $m \to \infty$, $x_m \to 0$ *and* $mx_m \to \infty$.

(33) REMARKS. It is clear that the above lemmas remain valid if we replace α, m, ρ_1, s, u and x by β, n, ρ_2, t, v and y respectively, and conversely.

4. Proof of Theorem 4

In this section we prove two preliminary theorems which, taken together, will imply Theorem 4.

(34) THEOREM 4a. *If* $mx_m \to \tau_1 < \infty$, $ny_n \to \tau_2 < \infty$ *as* $m, n \to \infty$ *and* $x_m, y_n \to 0$, *then*

$$h_{m,n}(\varphi; x_m, y_n) \to \int_{0,0}^{1,1}\int_{0,0}^{\tau_2, \tau_1} \frac{\sin su}{s}\frac{\sin tv}{t}\, ds\, dt\, d^2 g(u, v).$$

PROOF. By (12) and Corollary 1 of Lemma 2,

$$h_{m,n}(\varphi; x, y) = \tfrac{1}{4}\sum_{k,l=0}^{m,n}\binom{n}{l}\binom{m}{k}\left\{xy - y\int_0^x \frac{\sin (k+\frac{1}{2})s}{\sin \frac{1}{2}s}\, ds - x\int_0^y \frac{\sin (l+\frac{1}{2})t}{\sin\frac{1}{2}t}\, dt\right.$$

$$\left. + \int_{0,0}^{y,x}\frac{\sin (k+\frac{1}{2})s}{\sin\frac{1}{2}s}\frac{\sin (l+\frac{1}{2})t}{\sin\frac{1}{2}t}\, ds\, dt\right\}\int_{0,0}^{1,1} u^k(1-u)^{m-k}v^l(1-v)^{n-l}\, d^2 g(u, v)$$

$$= \tfrac{1}{4}\int_{0,0}^{1,1}\int_{0,0}^{y,x}\sum_{k,l=0}^{m,n}\binom{n}{l}\binom{m}{k}\frac{\sin (k+\frac{1}{2})s}{\sin\frac{1}{2}s}\frac{\sin (l+\frac{1}{2})t}{\sin\frac{1}{2}t}u^k(1-u)^{m-k}v^l(1-v)^{n-l}ds\,dt\,d^2g$$

$$+O(x)+O(y)$$

$$= \tfrac{1}{4}\int_{0,0}^{1,1}\int_{0,0}^{y,x}\rho_1^m\rho_2^n\left\{\sin m\alpha\cot\frac{s}{2} + \cos m\alpha\right\}$$

$$\left\{\sin n\beta\cot\frac{t}{2} + \cos n\beta\right\} ds\,dt\,d^2g(u, v) + O(x) + O(y)$$

by (16);

$$= \tfrac{1}{4}\int_{0,0}^{1,1}\int_{0,0}^{y,x}\rho_1^m\rho_2^n\sin m\alpha\sin n\beta\cot\frac{s}{2}\cot\frac{t}{2}\, ds\,dt\,d^2g(u, v)$$

$$+ O(x) + O(y)$$

by Lemma 1 and Lemma 3, since

$$\int_{0,0}^{1,1}\int_{0,0}^{y,x}\rho_1^m\rho_2^n\sin m\alpha\cos n\beta\cot\frac{s}{2}\, ds\,dt\,d^2g(u, v)$$

$$= \int_{0,0}^{1,1}\int_{0,0}^{y,x}\rho_1^m\rho_2^n\sin m\alpha\cos n\beta\left\{\cot\frac{s}{2} - \frac{2}{s} + \frac{2}{s}\right\} ds\,dt\,d^2g(u, v)$$

$$= 2\int_{0,0}^{1,1}\int_{0,0}^{y,x}\rho_1^m\frac{\sin m\alpha}{s}\rho_2^n\cos n\beta\, ds\,dt\,d^2g(u, v)$$

$$+ \int_{0,0}^{1,1}\int_{0,0}^{y,x}\rho_1^m\rho_2^n\sin m\alpha\cos n\beta\left\{\cot\frac{s}{2} - \frac{2}{s}\right\} ds\,dt\,d^2g(u, v)$$

$$= O(y)$$

https://doi.org/10.1017/S1446788700006352 Published online by Cambridge University Press

by Lemma 3 with $p = 0$, and by Lemma 1. Then

$$
\begin{aligned}
h_{m,n}(\varphi; x, y) &= \int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m \rho_2^n \frac{\sin m\alpha}{s}\frac{\sin n\beta}{t}\, ds\, dt\, d^2 g(u, v) \\
&\quad + \tfrac{1}{2}\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m \frac{\sin m\alpha}{s}\gamma(n; t, v)\, ds\, dt\, d^2 g(u, v) \\
&\quad + \tfrac{1}{2}\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_2^n \frac{\sin n\beta}{t}\gamma(m; s, u)\, ds\, dt\, d^2 g(u, v) \qquad (35) \\
&\quad + \tfrac{1}{4}\int_{0,0}^{1,1}\int_{0,0}^{y,x} \gamma(m; s, u)\gamma(n; t, v)\, ds\, dt\, d^2 g(u, v) + O(x) + O(y) \\
&= \int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m \rho_2^n \frac{\sin m\alpha}{s}\frac{\sin n\beta}{t}\, ds\, dt\, d^2 g(u, v) + O(x) + O(y)
\end{aligned}
$$

by Lemmas 1 and 3, since

$$
\begin{aligned}
\gamma(m; s, u) &= \tfrac{1}{2}\rho_1^m \sin m\alpha \left\{\cot\frac{s}{2} - \frac{2}{s}\right\} \\
&= O(1), \qquad\qquad m = 1, 2, 3, \cdots,
\end{aligned}
$$

with $\gamma(n; t, v)$ expressed in a similar manner. Then, using Lemma 4,

$$
\begin{aligned}
h_{m,n}(\varphi; x, y) &= \int_{0,0}^{1,1}\int_{0,0}^{y,x} \{1+O(ms^2)\}\{1-O(nt^2)\}\frac{\sin m\alpha}{s}\frac{\sin n\beta}{t}\, ds\, dt\, d^2 g(u, v) \\
&\quad + O(x) + O(y) \\
&= \int_{0,0}^{1,1}\int_{0,0}^{y,x} \frac{\sin m\alpha}{s}\frac{\sin n\beta}{t}\, ds\, dt\, d^2 g(u, v) \\
&\quad + m\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_2^n \frac{\sin n\beta}{t} O(s)\, ds\, dt\, d^2 g(u, v) \\
&\quad + n\int_{0,0}^{1,1}\int_{0,0}^{y,x} \rho_1^m \frac{\sin m\alpha}{s} O(t)\, ds\, dt\, d^2 g(u, v) \\
&\quad + mn\int_{0,0}^{1,1}\int_{0,0}^{y,x} O(st)\, ds\, dt\, d^2 g(u, v) + O(x) + O(y) \\
&= \int_{0,0}^{1,1}\int_{0,0}^{y,x} \frac{\sin m\alpha}{s}\frac{\sin n\beta}{t}\, ds\, dt\, d^2 g(u, v) \\
&\quad + O(mx^2) + O(ny^2) + O(mnx^2y^2) + O(x) + O(y)
\end{aligned}
$$

by Lemmas 1 and 3 with $p = 1$. Now let $x_m, y_n \to 0$ as $m, n \to \infty$, and as $mx_m \to \tau_1 < \infty$ and $ny_n \to \tau_2 < \infty$. Then

https://doi.org/10.1017/S1446788700006352 Published online by Cambridge University Press

$$h_{m,n}(\varphi; x_m, y_n) \to \int_{0,0}^{1,1}\int_{0,0}^{y_n, y_m} \frac{\sin m\alpha}{s}\frac{\sin n\beta}{t}\, ds\, dt\, d^2 g(u, v),$$

and by Lemma 5,

$$h_{m,n}(\varphi; x_m, y_n) \to \int_{0,0}^{1,1}\int_{0,0}^{y_n, x_m} \{\sin msu + O(mus^3)\}\{\sin ntv + O(nvt^3)\} \frac{ds}{s}\frac{dt}{t}\, d^2 g(u, v)$$

$$= \int_{0,0}^{1,1}\int_{0,0}^{y_n, x_m} \frac{\sin msu}{t}\frac{\sin ntv}{s}\, ds\, dt\, d^2 g(u, v)$$

$$+ \int_{0,0}^{1,1}\int_{0,0}^{y_n, x_m} O(mus^2)\frac{\sin ntv}{t}\, ds\, dt\, d^2 g(u, v)$$

$$+ \int_{0,0}^{1,1}\int_{0,0}^{y_n, x_m} O(nvt^2)\frac{\sin msu}{s}\, ds\, dt\, d^2 g(u, v)$$

$$+ \int_{0,0}^{1,1}\int_{0,0}^{y_n, x_m} O(mnuvs^2t^2)\, ds\, dt\, d^2 g(u, v)$$

$$= \int_{0,0}^{1,1}\int_{0,0}^{ny_n, mx_m} \frac{\sin su}{s}\frac{\sin tv}{t}\, ds\, dt\, d^2 g(u, v) + O(x) + O(y)$$

$$\to \int_{0,0}^{1,1}\int_{0,0}^{\tau_2, \tau_1} \frac{\sin su}{s}\frac{\sin tv}{t}\, ds\, dt\, d^2 g(u, v)$$

since $mx_m \to \tau_1 < \infty$, $ny_n \to \tau_2 < \infty$. Here we again applied Lemmas 1 and 3, observing that under the assumed conditions, $O(mnuvs^2t^2) = O(st)$, $O(mus^2) = O(s)$ and $O(nvt^2) = O(t)$. This proves Theorem 4a.

(36) Theorem 4b. *If $mx_m \to \infty$, $ny_n \to \infty$, $mx_m^2 = O(1)$, $ny_n^2 = O(1)$ as $m, n \to \infty$ and $x_m, y_n \to 0$, then*

$$h_{m,n}(\varphi; x_m, y_n) \to \frac{\pi^2}{4}, \qquad m, n \to \infty.$$

Proof. Applying Lemma 7, we write

$$\rho_1^m \rho_2^n \frac{\sin m\alpha}{s}\frac{\sin n\beta}{t} = e^{-(m/2)u(1-u)s^2} e^{-(n/2)v(1-v)t^2} \frac{\sin m\alpha}{s}\frac{\sin n\beta}{t}$$

$$+ O(ms^3)\rho_2^n \frac{\sin n\beta}{t} + O(nt^3)\rho_1^m \frac{\sin m\alpha}{s} + O(mns^3t^3)$$

$$= \psi(m, n; s, t, u, v) + O(ms^3)\rho_2^n \frac{\sin n\beta}{t}$$

$$+ O(nt^3)\rho_1^m \frac{\sin m\alpha}{s} + O(mns^3t^3).$$

https://doi.org/10.1017/S1446788700006352 Published online by Cambridge University Press

Substituting in (35), we obtain

$$h_{m,n}(\varphi; x, y) = \int_{0,0}^{1,1}\int_{0,0}^{y,x} \psi(m, n; s, t, u, v)\, ds\, dt\, d^2g(u, v)$$
$$+\int_{0,0}^{1,1}\int_{0,0}^{y,x} O(ms^3)\rho_2^n \frac{\sin n\beta}{t}\, ds\, dt\, d^2g(u, v)$$
$$+\int_{0,0}^{1,1}\int_{0,0}^{y,x} O(nt^3)\rho_1^m \frac{\sin m\alpha}{s}\, ds\, dt\, d^2g(u, v)$$
$$+\int_{0,0}^{1,1}\int_{0,0}^{y,x} O(mns^3t^3)\, ds\, dt\, d^2g(u, v) + O(x) + O(y)$$
$$= \int_{0,0}^{1,1} \psi(m, n; s, t, u, v)\, ds\, dt\, d^2g(u, v) + O(mx^4)$$
$$+ O(ny^4) + O(mnx^4y^4) + O(x) + O(y)$$

by Lemmas 1 and 3. Now let $x_m, y_n \to 0$ as $m, n \to \infty$ in such a manner that $mx_m^2 = O(1)$, $ny_n^2 = O(1)$. Then

$$(37)\quad h_{m,n}(\varphi; x_m, y_n) \to \int_{0,0}^{1,1}\int_{0,0}^{y_m, x_n} \psi(m, n; s, t, u, v)\, ds\, dt\, d^2g(u, v), \quad m, n \to \infty.$$

We remark that this restriction still permits that $mx_m \to \infty$, $ny_n \to \infty$ as $m, n \to \infty$ and $x_m, y_n \to O$. Now by Lemma 5,

$$\sin m\alpha \sin n\beta = \sin msu \sin ntv + O(nt^3) \sin m\alpha + O(ms^3) \sin n\beta + O(mns^3t^3),$$

and so

$$\psi(m, n; s, t, u, v) = e^{-(m/2)u(1-u)s^2} e^{-(n/2)v(1-v)t^2} \frac{\sin msu}{s} \frac{\sin ntv}{t}$$
$$+ O(nt^2) e^{-(m/2)u(1-u)s^2} \frac{\sin m\alpha}{s}$$
$$+ O(ms^2) e^{-(n/2)v(1-v)t^2} \frac{\sin n\beta}{t} + O(mns^2t^2)$$
$$= \chi(m, n; s, t, u, v) + O(nt^2)\left(\rho_1^m + O(ms^4)\right) \frac{\sin m\alpha}{s}$$
$$+ O(ms^2)\left(\rho_2^n + O(nt^4)\right) \frac{\sin n\beta}{t} + O(mns^2t^2)$$
$$= \chi(m, n; s, t, u, v) + O(nt^2)\rho_1^m \frac{\sin m\alpha}{s}$$
$$+ O(ms^2)\rho_2^n \frac{\sin n\beta}{t} + O(nt^2ms^3) + (O(ms^2nt^3)$$
$$+ O(mns^2t^2).$$

https://doi.org/10.1017/S1446788700006352 Published online by Cambridge University Press

Then substituting in (37),

$$\begin{aligned} h_{m,n}(\varphi; x_m, y_n) \to & \int_{0,0}^{1,1}\int_{0,0}^{y_n, x_m} \chi(m, n; s, t, u, v)\, ds\, dt\, d^2 g(u, v) \\ & + \int_{0,0}^{1,1}\int_{0,0}^{x_n, x_m} \rho_1^m \frac{\sin m\alpha}{s} O(nt^2)\, ds\, dt\, d^2 g(u, v) \\ & + \int_{0,0}^{1,1}\int_{0,0}^{y_n, x_m} \rho_2^n \frac{\sin n\beta}{t} O(ms^2)\, ds\, dt\, d^2 g(u, v) \\ & + \int_{0,0}^{1,1}\int_{0,0}^{y_n\, x_m} O(mns^2 t^2)\, ds\, dt\, d^2 g(u, v), \qquad 0 \leqq s, t \leqq \pi, \\ = & \int_{0,0}^{1,1}\int_{0,0}^{y_n, x_m} \chi(m, n; s, t, u, v)\, ds\, dt\, d^2 g(u, v) + O(x_m) + O(y_n) \end{aligned}$$

by Lemmas 1 and 3, since by our hypothesis, $nt^2 = O(1) = ms^2$, and also $mns^2t^2 = O(1)$, $m, n \to \infty$. Then as $m, n \to \infty$ and $x_m, y_n \to 0$, taking $0 < \varepsilon_1, \varepsilon_2 < 1$,

$$\begin{aligned} h_{m,n}(\varphi; s_m, y_n) \to & \int_{0,0}^{1,1}\int_{0,0}^{y_n, x_m} \chi(m, n; s, t, u, v)\, ds\, dt\, d^2 g(u, v) \\ = & \left\{ \int_{\varepsilon_2, \varepsilon_1}^{1,1} + \int_{\varepsilon_2, 0}^{1, \varepsilon_1} + \int_{0, \varepsilon_1}^{\varepsilon_2, 1} + \int_{0,0}^{\varepsilon_2, \varepsilon_1} \right\} \\ & \int_{0,0}^{y_n, x_m} \chi(m, n; s, t, u, v)\, ds\, dt\, d^2 g(u, v) \\ = & \int_{\varepsilon_2, \varepsilon_1}^{1,1}\int_{0,0}^{y_n, x_m} \chi(m, n; s, t, u, v)\, ds\, dt\, d^2 g(u, v) \\ & + O\left(\int_{\varepsilon_2, 0}^{1, \varepsilon_1} |d^2 g(u, v)|\right) + O\left(\int_{0, \varepsilon_1}^{\varepsilon_2, 1} |d^2 g(u, v)|\right) \\ & + O\left(\int_{0,0}^{\varepsilon_2, \varepsilon_1} |d^2 g(u, v)|\right) \end{aligned}$$

since by Lemma 8,

$$\left| \int_{0,0}^{y_n, x_m} \chi(m, n; s, t, u, v)\, ds\, dt \right| < M < \infty.$$

Now let ε_1 and ε_2 tend to zero. Then, again by Lemma 8,

$$\begin{aligned} h_{m,n}(\varphi; x_m, y_n) \to & \left(\frac{\pi}{2}\right)^2 \int_{0^+, 0^+}^{1,1} d^2 g(u, v) + O\left(\int_{0^+, 0}^{1, 0^+} |d^2 g(u, v)|\right) \\ & + O\left(\int_{0, 0^+}^{0^+, 1} |d^2 g(u, v)|\right) + O\left(\int_{0,0}^{0^+, 0^+} |d^2 g(u, v)|\right) = \left(\frac{\pi}{2}\right)^2, \end{aligned}$$

https://doi.org/10.1017/S1446788700006352 Published online by Cambridge University Press

since $g(u, v)$ is a regular Hausdorff weight function, so that each of the last three terms is equal to zero. This completes the proof of Theorem 4b.

(38) REMARKS. For the case where as $m, n \to \infty$ and $x_m, y_n \to 0$, $mx_m \to \tau_1$ and $ny_n \to \tau_2$, where $\tau_1 < \infty$ and $\tau_2 < \infty$, or where $\tau_1 = \infty$ and $\tau_2 = \infty$, $mx_m^2 < \infty$, $ny_n^2 < \infty$, Theorem 4 now follows immediately from Theorems 4a and 4b. Strictly speaking, to complete the proof of Theorem 4 we should also consider the case where $\tau_1 = \infty$ and $\tau_2 < \infty$, and $\tau_1 < \infty$ and $\tau_2 = \infty$. However, it is clear that this would involve both methods of proof in a predictable manner, and it is equally obvious what the conclusion would be. We avoid the details and conclude Theorem 4.

5. Proof of Theorem 5

By Theorem 4, if $\tau_1, \tau_2 < \infty$ and $mx \to \tau_1$, $ny \to \tau_2$, as $m, n \to \infty$ and $x, y \to 0$, then

$$h_{m,n}(\varphi; x, y) \to \int_{0,0}^{1,1} \int_{0,0}^{\tau_2 v, \tau_1 u} \frac{\sin s}{s} \frac{\sin t}{t} \, ds\, dt\, d^2 g(u, v)$$
$$= \int_{0,0}^{1,1} f(u, v)\, d^2 g(u, v).$$

Integrating by parts [6, p. 38],

$$h_{m,n}(\varphi; x, y) \to fg \Big\}_{0,0}^{1,1} - \int_0^1 \left\{\frac{df}{du} f\right\}_{v=0}^1 du - \int_0^1 \left\{\frac{df}{dv} g\right\}_{u=0}^1 dv$$
$$+ \int_{0,0}^{1,1} \frac{d^2 f}{du\, dv} g\, du\, dv$$
$$= \int_{0,0}^{1,1} \frac{\sin \tau_1 u}{u} \frac{\sin \tau_2 v}{v} du\, dv - \int_{0,0}^{1,1} \frac{\sin \tau_1 u}{u} \frac{\sin \tau_2 v}{v} g(u, 1)\, du\, dv$$
$$- \int_{0,0}^{1,1} \frac{\sin \tau_1 u}{u} \frac{\sin \tau_2 v}{v} g(1, v)\, du\, dv$$
$$+ \int_{0,0}^{1,1} \frac{\sin \tau_1 u}{u} \frac{\sin \tau_2 v}{v} g(u, v)\, du\, dv$$
$$= \int_{0,0}^{1,1} \frac{\sin \tau_1 u}{u} \frac{\sin \tau_2 v}{v} g(1, 1; u, v)\, du\, dv$$

where

$$g(1, 1; u, v) = g(1, 1) - g(u, 1) - g(1, v) + g(u, v).$$

This completes the proof of Theorem 5.

References

[1] Adams, C. R., 'Hausdorff transformations for double sequences', *Bull. Amer. Math, Soc.* **39** (1933), 303—312.

[2] Hallenbach, F., *Zur Theorie der Limitierungsverfahren von Doppelfolgen.* (Inaugural-Dissertation, Rheinischen Friedrich-Wilhelms-Universität, Bonn 1933).

[3] Szász, O., 'Gibbs phenomenon for Hausdorff means', *Trans. Amer. Math. Soc.* **69** (1950), 440—456.

[4] Ustina, F., 'Hausdorff means for double sequences', *Can. Math. Bull.* **10** (1967), 347—352.

[5] Ustina, F., 'Gibbs phenomenon for functions of two variables', *Trans. Amer. Math. Soc.* **129** (1967), 124—129.

[6] Young, W. H., 'On multiple integrals', *Proc. Royal Soc., London* (A) **93** (1917), 28—41.

University of Alberta
Alberta, Canada