ON INTEGRAL OPERATORS ASSOCIATED WITH POISSON TRANSFORMS AND THE OPERATOR H_α

G. O. OKIKIOLU

(Received 23 July 1965)

1. Introduction

In this paper we study certain operators allied to the Poisson operator and the transforms $H_\alpha(f)$ considered by the author in [2]. We define the integrals $\psi_\alpha^{(a)}(f)$ and $\theta_\alpha^{(a)}(f)$ as follows:

$$\psi_\alpha^{(a)}(f)(x) = \frac{1}{\varphi(1-\alpha)}\int_{-\infty}^{\infty} f(t+x)\,\frac{\sin(\alpha\tan^{-1} t/a)}{(a^2+t^2)^{\frac{1}{2}\alpha}}\,dt,$$

$$\theta_\alpha^{(a)}(f)(x) = \frac{1}{\varphi(1-\alpha)}\int_{-\infty}^{\infty} f(t+x)\,\frac{\cos(\alpha\tan^{-1} t/a)}{(a^2+t^2)^{\frac{1}{2}\alpha}}\,dt,$$

where $\varphi(\alpha) = 2\Gamma(\alpha)\sin\frac{1}{2}\pi\alpha$, and the principal value of $\tan^{-1} x$ (lying between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$) is taken throughout. Further, we define the integrals $\Psi_\alpha^{(a)}(f)$ and $\Theta_\alpha^{(a)}(f)$ which will be employed later in obtaining inversion processes for $\psi_\alpha^{(a)}$ and $\theta_\alpha^{(a)}$. We have

$$\Psi_\alpha^{(a)}(f)(x) = \frac{1}{\varphi(1-\alpha)}\int_{-\infty}^{\infty} f(t)\left\{\frac{\sin(\alpha\tan^{-1}(t-x)/a)}{(a^2+(t-x)^2)^{\frac{1}{2}\alpha}} - \frac{\sin(\alpha\tan^{-1} t/a)}{(a^2+t^2)^{\frac{1}{2}\alpha}}\right\}dt,$$

$$\Theta_\alpha^{(a)}(f)(x) = \frac{1}{\varphi(1-\alpha)}\int_{-\infty}^{\infty} f(t)\left\{\frac{\cos(\alpha\tan^{-1}(t-x)/a)}{(a^2+(t-x)^2)^{\frac{1}{2}\alpha}} - \frac{\cos(\alpha\tan^{-1} t/a)}{(a^2+t^2)^{\frac{1}{2}\alpha}}\right\}dt.$$

The Poisson operator and its "conjugate" are given respectively by

$$P_a(f)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{a}{a^2+(t-x)^2}\,f(t)\,dt$$

and

$$Q_a(f)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t-x}{a^2+(t-x)^2}\,f(t)\,dt.$$

Next we define the transform $H_\alpha(f)$ and operators related to it. These are given by

$$H_\alpha(f)(x) = \frac{1}{\varphi(\alpha)}\int_{-\infty}^{\infty} \frac{|t-x|^\alpha}{t-x} f(t)\,dt,$$

$$K_\alpha(f)(x) = \frac{1}{\varphi(\alpha)}\int_{-\infty}^{\infty} |t-x|^{\alpha-1} f(t)\,dt,$$

$$L_\alpha(f)(x) = \frac{1}{\varphi(\alpha)}\int_{-\infty}^{\infty} \left\{\frac{|t-x|^\alpha}{t-x} - \frac{|t|^\alpha}{t}\right\} f(t)\,dt$$

and

$$M_\alpha(f)(x) = \frac{1}{\varphi(\alpha)}\int_{-\infty}^{\infty} \{|t-x|^{\alpha-1} - |t|^{\alpha-1}\} f(t)\,dt.$$

Finally we define the Hilbert transform by

$$H(f)(x) = \frac{1}{\pi}\,(\text{P.V.})\int_{-\infty}^{\infty} \frac{f(t)}{t-x}\,dt.$$

In what follows we shall use the known properties of the transforms $H_\alpha(f)$ and $K_\alpha(f)$ in obtaining identities involving the operators $\psi_\alpha^{(a)}(f)$ and $\theta_\alpha^{(a)}(f)$. These identities are then applied in deriving inversion processes for the new operators. In the formulae given here, the integrals $P_a(f)$ and $Q_a(f)$ are expressed in terms of $\psi_\alpha^{(a)}(f)$ and $\theta_\alpha^{(a)}(f)$. The function f may then be given in terms of the new operators by applying processes similar to those given in [4]. It will be observed that the relationship between the pair $(\psi_\alpha^{(a)}, \theta_\alpha^{(a)})$ and the pair (P_a, Q_a) is analogous to that between (H_α, K_α) and (H, I), where I denotes the identity operator.

The space $L^p(-\infty, \infty)$ will be denoted by L^p, the pair of numbers p and p' will be connected by the equation $1/p+1/p' = 1$, and the norm

$$\left(\int_{-\infty}^{\infty} |f(t)|^p\,dt\right)^{1/p}$$

will be denoted by $||f||_p$.

2. Preliminary results

In this section we obtain certain properties of the operators P_a, Q_a, H_α and K_α which will be applied later.

THEOREM 1. *Let $f \in L^p$, $p > 1$. Then $P_a(f) \in L^q$ and $Q_a(f) \in L^q$ for $q \geqq p$. Further, for $g \in L^{q'}$, we have*

$$\int_{-\infty}^{\infty} g(t) P_a(f)(t)dt = \int_{-\infty}^{\infty} f(t) P_a(g)(t)dt$$

and

$$\int_{-\infty}^{\infty} g(t) Q_a(f)(t)dt = -\int_{-\infty}^{\infty} f(t) Q_a(g)(t)dt.$$

Proof. It is obvious that the function $h_1(t) = a/(a^2+t^2)$ belongs to L^q for $q \geqq 1$. Hence by applying a well-known inequality (see Lemma β page 97 of [6]), we clearly have $P_a(f) \in L^q$ for $q \geqq p$. Now the function $h_2(t) = t/(a^2+t^2)$ belongs to L^q for $q > 1$. Hence by applying the above inequality again, it follows that $Q_a(f) \in L^q$ for $q > p$. In Theorem 2 (below), we prove that $Q_a(f) = P_a\{H(f)\}$. This result together with Theorem 101 of [6] and the fact that $P_a(f) \in L^p$ shows that $Q_a(f) \in L^p$ also.

By the absolute convergence of the integrals involved, we clearly have

$$\int_{-\infty}^{\infty} g(t)dt \int_{-\infty}^{\infty} f(x)h_1(x-t)dx = \int_{-\infty}^{\infty} f(x)dx \int_{-\infty}^{\infty} g(t)h_1(t-x)dt.$$

This proves the first product formula. The second result can be obtained similarly by considering $h_2(t)$.

Theorem 2. *Let $f \in L^p$, $p > 1$, and a and b be positive numbers. Then we have*

(i) $P_a\{P_b(f)\} = -Q_a\{Q_b(f)\} = P_{a+b}(f)$;

(ii) $P_a\{Q_b(f)\} = Q_a\{P_b(f)\} = Q_{a+b}(f)$;

(iii) $P_a\{H(f)\} = Q_a(f)$ and (iv) $Q_a\{H(f)\} = -P_a(f)$.

Proof. The result $P_a\{P_b(f)\} = P_{a+b}(f)$ is a well-known property of Poisson transforms (see [5]). Since the other results of (i) and (ii) can be obtained by applying (iii) and (iv), we shall only prove the latter. It is known that if $a > 0$ and

$$h_1(t) = a/(a^2+t^2), \quad h_2(t) = t/(a^2+t^2),$$

then

$$H(h_1)(x) = -h_2(x) \quad \text{and} \quad H(h_2)(x) = h_1(x)$$

(see [6] page 121). Hence by applying the product formula for Hilbert transforms (Theorem 102 of [6]), we clearly have

$$\int_{-\infty}^{\infty} h_1(t-x)f(t)dt = -\int_{-\infty}^{\infty} h_2(t-x)H(f)(t)dt,$$

and

$$\int_{-\infty}^{\infty} h_2(t-x)f(t)dt = \int_{-\infty}^{\infty} h_1(t-x)H(f)(t)dt.$$

This proves (iii) and (iv).

Note: By making suitable interchanges in the order of integration, it is not difficult to show that

$$P_a\{H(f)\} = H\{P_a(f)\} \quad \text{and} \quad Q_a\{H(f)\} = H\{Q_a(f)\}.$$

We note here the following convergence property of the operators P_a and Q_a.

(1) $$\lim_{a\to 0+} \|f-P_a(f)\|_p = 0,$$

(2) $$\lim_{a\to 0+} \|H(f)-Q_a(f)\|_p = 0.$$

In the next two theorems we state the results involving $H_\alpha(f)$ and $K_\alpha(f)$ to be applied in this paper.

THEOREM 3. *Let $f \in L^p$, $p > 1$, let $0 < \alpha < 1/p$ and let $1/r = 1/p-\alpha$. Then $H_\alpha(f) \in L^r$, $K_\alpha(f) \in L^r$, and for $g \in L^{r'}$, we have*

$$\int_{-\infty}^{\infty} g(t)H_\alpha(f)(t)dt = -\int_{-\infty}^{\infty} f(t)H_\alpha(g)(t)dt$$

and

$$\int_{-\infty}^{\infty} g(t)K_\alpha(f)(t)dt = \int_{-\infty}^{\infty} f(t)K_\alpha(g)(t)dt.$$

The resuls $H_\alpha(f) \in L^r$ and $K_\alpha(f) \in L^r$ follow as in the proofs of similar results for the Riemann-Liouville and Weyl fractional integrals (see Theorem 383 of [1]). The product formulae follow from the absolute convergence of the ingerals involved.

The inversion processes for $\psi_\alpha^{(a)}$ and $\theta_\alpha^{(a)}$ will be derived from corresponding results for H_α and K_α. The latter results have been established by the author (Theorem 6 of [3]), and we state them in the next theorem.

THEOREM 4. *Let $f \in L^p, p > 1$, and let $0 < \alpha < 1/p$. Then*

(i) $$L_{1-\alpha}\{K_\alpha(f)\}(x) = -\cot \tfrac{1}{2}\pi\alpha \int_0^x f(t)dt,$$

(ii) $$M_{1-\alpha}\{H_\alpha(f)\}(x) = -\tan \tfrac{1}{2}\pi\alpha \int_0^x f(t)dt.$$

Note: The results of Theorem 4 are alternative forms of the inversion process given in [2].

The following lemma will be applied later.

LEMMA 1. *Let y be a fixed number and let*

$$k_1(x) = |x-y|^{\alpha-1}-|x|^{\alpha-1}, \quad k_2(x) = \frac{|x-y|^\alpha}{x-y} - \frac{|x|^\alpha}{x}.$$

Then for $1-1/q < \alpha < 2-1/q$, we have $k_1 \in L^q$ and $k_2 \in L^q$.

PROOF. It is clearly sufficient to prove the result for either k_1 or k_2. We consider k_1 and assume without loss of generality that $y = 1$. The integral

$$\int_0^1 |t-1|^{\alpha-1}-|t|^{\alpha-1}|^q dt$$

clearly converges if $(\alpha-1)q+1 > 0$ (i.e. if $\alpha > 1-1/q$).

https://doi.org/10.1017/S1446788700004262 Published online by Cambridge University Press

Now let

$$I = \int_1^\infty |(t-1)^{\alpha-1}-t^{\alpha-1}|^q\,dt.$$

On making obvious changes of variables in the integral, we have

$$I = \int_0^1 \left|\frac{(1-t)^{\alpha-1}-1}{t^{\alpha+2/q-1}}\right|^q dt.$$

Hence the integral I converges if its integrand is $O(t^\gamma)$ as $t\to 0$, where $\gamma+1>0$. Now

$$\lim_{t\to 0}\frac{(1-t)^{\alpha-1}-1}{t^{\alpha-1+2/q}} = \lim_{t\to 0}\frac{(1-\alpha)(1-t)^{\alpha-2}}{(\alpha-1+2/q)t^{\alpha-2+2/q}}.$$

Hence $\gamma = -(\alpha-2+2/q)q$, and this implies that $2-1/q>\alpha$.

3. Representation theorems for $\psi_\alpha^{(a)}$ and $\theta_\alpha^{(a)}$

We shall now express $\psi_\alpha^{(a)}$, $\theta_\alpha^{(a)}$, $\Psi_\alpha^{(a)}$ and $\Theta_\alpha^{(a)}$ in terms of the Poisson operators and the H_α-transforms. The following lemmas will be employed in the proofs of the theorems.

LEMMA 2. *Let $h_1(t)=a/(a^2+t^2)$, $h_2(t)=t/(a^2+t^2)$ and let $a>0$. Then we have*

(i) $$H_{1-\alpha}(h_1)(x) = -\cot\tfrac{1}{2}\pi\alpha\, K_{1-\alpha}(h_2)(x) = -\Gamma(\alpha)\frac{\sin(\alpha\tan^{-1}x/a)}{(a^2+x^2)^{\frac{1}{2}\alpha}},$$

(ii) $$H_{1-\alpha}(h_2)(x) = \cot\tfrac{1}{2}\pi\alpha\, K_{1-\alpha}(h_1)(x) = \Gamma(\alpha)\frac{\cos(\alpha\tan^{-1}x/a)}{(a^2+x^2)^{\frac{1}{2}\alpha}}.$$

PROOF. Let $\hat{f}$ denote the Fourier transform of a function f. Then by proceeding as in the proof of the identity (5) of [2], it follows that if $f\in L^2$ and if $\hat{f}\in L$, then

$$H_\alpha(f)(x) = -\frac{i}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{|t|^{1-\alpha}}{t}\hat{f}(t)e^{-ixt}\,dt.$$

The expression given here incorporates a correction in the sign of (5) of [2], and the redundant condition $f\in L$ is excluded. Also, by considering $f(x+t)+f(x-t)$ and proceeding similarly, we have

$$K_\alpha(f)(x) = \frac{\cot\frac{1}{2}\pi\alpha}{\sqrt{2\pi}}\int_{-\infty}^\infty |t|^{-\alpha}\hat{f}(t)e^{-ixt}\,dt.$$

Now it is well-known that

$$\hat{h}_1(x) = \sqrt{\frac{\pi}{2}}\,e^{-a|x|} \quad\text{and}\quad \hat{h}_2(x) = i\sqrt{\frac{\pi}{2}}\,e^{-a|x|}|x|/x.$$

Hence by using the identities given above, we have

$$H_{1-\alpha}(h_1)(x) = -(\cot \tfrac{1}{2}\pi\alpha)K_{1-\alpha}(h_2)(x) = -\tfrac{1}{2}i\int_{-\infty}^{\infty} \frac{|t|^\alpha}{t}\, e^{-a|t|}\, e^{-ixt}\, dt$$

$$= -\int_0^\infty t^{\alpha-1}\, e^{-at} \sin xt\, dt,$$

$$H_{1-\alpha}(h_2)(x) = (\cot \tfrac{1}{2}\pi\alpha)K_{1-\alpha}(h_1)(x) = \tfrac{1}{2}\int_{-\infty}^{\infty} |t|^{\alpha-1}\, e^{-a|t|}\, e^{-ixt}\, dt$$

$$= \int_0^\infty t^{\alpha-1}\, e^{-at} \cos xt\, dt.$$

The integrals can now be evaluated by known methods (e.q. contour integration) to give the required result.

Lemma 3. *Let y be a fixed real number, let $a > 0$ and let*

$$k(x) = \frac{|x-y|^{1-\alpha}}{x-y} - \frac{|x|^{1-\alpha}}{x}, \qquad k_0(x) = |x-y|^{-\alpha} - |x|^{-\alpha}.$$

Then

$$P_a(k)(x) = -(\cot \tfrac{1}{2}\pi\alpha)Q_a(k_0)(x)$$

$$= \frac{\Gamma(\alpha)\varphi(1-\alpha)}{\pi}\left\{\frac{\sin(\alpha \tan^{-1}(x-y)/a)}{(a^2+(x-y)^2)^{\frac{1}{2}\alpha}} - \frac{\sin(\alpha\tan^{-1}x/a)}{(a^2+x^2)^{\frac{1}{2}\alpha}}\right\},$$

$$Q_a(k)(x) = (\cot \tfrac{1}{2}\pi\alpha)P_a(k_0)(x)$$

$$= \frac{\Gamma(\alpha)\varphi(1-\alpha)}{\pi}\left\{\frac{\cos(\alpha \tan^{-1}(x-y)/a)}{(a^2+(x-y)^2)^{\frac{1}{2}\alpha}} - \frac{\cos(\alpha\tan^{-1}x/a)}{(a^2+x^2)^{\frac{1}{2}\alpha}}\right\}.$$

Proof. The results of Lemma 2 can be written in the form

$$\frac{1}{\varphi(1-\alpha)}\int_{-\infty}^{\infty} \frac{|t-x|^{1-\alpha}}{t-x}\,\frac{a}{a^2+t^2}\,dt = -\frac{\cot\frac{1}{2}\pi\alpha}{\varphi(1-\alpha)}\int_{-\infty}^{\infty}|t-x|^{\alpha-1}\,\frac{t}{a^2+t^2}\,dt$$

$$= -\Gamma(\alpha)\,\frac{\sin(\alpha\tan^{-1}x/a)}{(a^2+x^2)^{\frac{1}{2}\alpha}},$$

$$\frac{1}{\varphi(1-\alpha)}\int_{-\infty}^{\infty} \frac{|t-x|^{1-\alpha}}{t-x}\,\frac{t}{a^2+t^2}\,dt = \frac{\cot\frac{1}{2}\pi\alpha}{\varphi(1-\alpha)}\int_{-\infty}^{\infty}|t-x|^{\alpha-1}\,\frac{a}{a^2+t^2}\,dt$$

$$= \Gamma(\alpha)\,\frac{\cos(\alpha\tan^{-1}x/a)}{(a^2+x^2)^{\frac{1}{2}\alpha}}.$$

The results of Lemma 3 are easily deduced from these identities.

https://doi.org/10.1017/S1446788700004262 Published online by Cambridge University Press

THEOREM 5. *Let* $f \in L^p$, $p > 1$, *let* $1-1/p < \alpha < 1$ *and let* $a > 0$. *Then*

(i) $$\psi_\alpha^{(a)}(f) = (\sin \tfrac{1}{2}\pi\alpha)\, P_a\{H_{1-\alpha}(f)\} = (\cos \tfrac{1}{2}\pi\alpha) Q_a\{K_{1-\alpha}(f)\},$$

(ii) $$\theta_\alpha^{(a)}(f) = -(\sin \tfrac{1}{2}\pi\alpha) Q_a\{H_{1-\alpha}(f)\} = (\cos \tfrac{1}{2}\pi\alpha) P_a\{K_{1-\alpha}(f)\}.$$

PROOF. Let $h_1(x)$ and $h_2(x)$ be defined as in Lemma 2. Since $0 < 1-\alpha < 1/p$, by applying the product formulae of Theorem 3, we have

$$\int_{-\infty}^{\infty} h_1(t-x) H_{1-\alpha}(f)(t)dt = -\int_{-\infty}^{\infty} f(t) H_{1-\alpha}(h_1)(t-x)dt,$$
$$\int_{-\infty}^{\infty} h_1(t-x) K_{1-\alpha}(f)(t)dt = \int_{-\infty}^{\infty} f(t) K_{1-\alpha}(h_1)(t-x)dt.$$

The results

$$\psi_\alpha^{(a)}(f) = (\sin \tfrac{1}{2}\pi\alpha) P_a\{H_{1-\alpha}(f)\} \quad \text{and} \quad \theta_\alpha^{(a)}(f) = (\cos \tfrac{1}{2}\pi\alpha) P_a\{K_{1-\alpha}(f)\}$$

are obtained from these integrals by applying Lemma 2.

The other results of the theorem are obtained by applying the product formula with $h_2(x)$ in place of $h_1(x)$.

REMARK 1. From the results of Theorems 2 and 3, it follows by applying the identities of Theorem 5, that if the conditions of the latter theorem are satisfied, then

$$\psi_\alpha^{(a)}(f) \in L^s \quad \text{and} \quad \theta_\alpha^{(a)}(f) \in L^s,$$

where $s \geqq q$ and $1/q = 1/p+\alpha-1$. Also, by the absolute convergence of the integrals involved, or by applying Theorems 1 and 3, we have

$$\int_{-\infty}^{\infty} g(t) \psi_\alpha^{(a)}(f)(t)dt = -\int_{-\infty}^{\infty} f(t) \psi_\alpha^{(a)}(g)(t)dt$$

and

$$\int_{-\infty}^{\infty} g(t) \theta_\alpha^{(a)}(f)(t)dt = \int_{-\infty}^{\infty} f(t) \theta_\alpha^{(a)}(g)(t)dt.$$

THEOREM 6. *Let* $f \in L^p$, $p > 1$, *let* $0 < \alpha < 1-1/p$ *and let* $a > 0$. *Then*

(i) $$\Psi_\alpha^{(a)}(f) = (\sin \tfrac{1}{2}\pi\alpha) L_{1-\alpha}\{P_a(f)\} = (\cos \tfrac{1}{2}\pi\alpha) M_{1-\alpha}\{Q_a(f)\},$$

(ii) $$\Theta_\alpha^{(a)}(f) = -(\sin \tfrac{1}{2}\pi\alpha) L_{1-\alpha}\{Q_a(f)\} = (\cos \tfrac{1}{2}\pi\alpha) M_{1-\alpha}\{P_a(f)\}.$$

PROOF. Let $k(x)$ and $k_0(x)$ be defined as in Lemma 3. From Lemma 1, it follows that $k \in L^{p'}$ and $k_0 \in L^{p'}$ if we have

$$1-1/p' < 1-\alpha < 2-1/p' \quad (\text{i.e. } -1/p < \alpha < 1-1/p).$$

If we also have $\alpha > 0$, then the functions $P_a(k)$, $P_a(k_0)$, $Q_a(k)$ and $Q_a(k_0)$ are given by Lemma 3. Hence by applying the product formulae of Theorem 1, we have

https://doi.org/10.1017/S1446788700004262 Published online by Cambridge University Press

$$\int_{-\infty}^{\infty} k(t) P_a(f)(t)dt = \int_{-\infty}^{\infty} f(t) P_a(k)(t)dt,$$
$$\int_{-\infty}^{\infty} k(t) Q_a(f)(t)dt = -\int_{-\infty}^{\infty} f(t) Q_a(k)(t)dt.$$

The results

$$\Psi_\alpha^{(a)}(f) = (\sin \tfrac{1}{2}\pi\alpha) L_{1-\alpha}\{P_a(f)\} \quad \text{and} \quad \Theta_\alpha^{(a)}(f) = -(\sin \tfrac{1}{2}\pi\alpha) L_{1-\alpha}\{Q_a(f)\}$$

now follow by applying Lemma 3. The other results of the theorem are obtained by considering $k_0(x)$ in place of $k(x)$.

In the next section we shall require the result that the operators P_a and Q_a commute with $H_{1-\alpha}$ and $K_{1-\alpha}$. Hence we prove

THEOREM 7. *Let $f \in L^p$, $p > 1$ and let $0 < \gamma < 1/p$. Then*

(i) $H_\gamma\{P_a(f)\} = P_a\{H_\gamma(f)\}$, (ii) $H_\gamma\{Q_a(f)\} = Q_a\{H_\gamma(f)\}$,

(iii) $K_\gamma\{P_a(f)\} = P_a\{K_\gamma(f)\}$, (iv) $K_\gamma\{Q_a(f)\} = Q_a\{K_\gamma(f)\}$.

PROOF. It is clearly sufficient to consider any one of the four identities. When one of them has been established, the proofs of the others follow similarly. Consider (iii), let c and c_1 be numbers such that

$$-\infty < -c_1 < c < \infty,$$

and let $h_1(x) = a/(a^2+x^2)$. Then it is clear that

$$\int_{-c_1}^{c} |t-x|^{\gamma-1} dt \int_{-\infty}^{\infty} f(y+t) h_1(y) dy = \int_{-\infty}^{\infty} h_1(y) dy \int_{-c_1}^{c} f(y+t)|t-x|^{\gamma-1} dt$$
$$= \int_{-\infty}^{\infty} h_1(y-x) dy \int_{-c_1-x}^{c-x} f(y+t)|t|^{\gamma-1} dt.$$

Also, it follows from Theorem 3 that

$$\int_{-c_1-x}^{c-x} f(t+y)|t|^{\gamma-1} dt \quad \text{and} \quad \int_{-\infty}^{\infty} f(t+y)|t|^{\gamma-1} dt$$

are both members of $L^r(1/r = 1/p-\alpha)$, so that

$$\lim_{c_1, c\to\infty} \left\| \int_{-c_1-x}^{c-x} f(t+y)|t|^{\gamma-1} dt - \int_{-\infty}^{\infty} f(t+y)|t|^{\gamma-1} dt \right\|_r = 0.$$

Hence since $h_1 \in L^{r'}$, the required result follows by letting c_1 and c tend to ∞ in the above equation.

4. The inversion process

We shall now obtain results expressing the P and Q operators in terms of the $\psi_\alpha^{(a)}$ and $\theta_\alpha^{(a)}$ operators. As indicated in the introduction, f may then be obtained by processes like those given in [4] or by the limiting processes given below.

THEOREM 8. *Let $f \in L^p$, $p > 1$, let $1-1/p < \alpha < 1$ and let a and b be positive numbers. Then we have*

(i) $$\Psi^{(a)}_{1-\alpha}\{\psi^{(b)}_\alpha(f)\} = -\Theta^{(a)}_{1-\alpha}\{\theta^{(b)}_\alpha(f)\} = -\tfrac{1}{2}(\sin \pi\alpha)\int_0^x Q_{a+b}(f)(t)dt,$$

(ii) $$\Psi^{(a)}_{1-\alpha}\{\theta^{(b)}_\alpha(f)\} = \Theta^{(a)}_{1-\alpha}\{\psi^{(b)}_\alpha(f)\} = -\tfrac{1}{2}(\sin \pi\alpha)\int_0^x P_{a+b}(f)(t)dt.$$

PROOF. Let β satisfy $1-1/p < \beta < 1$. Then it follows from Remark 1 that $\psi^{(b)}_\beta(f) \in L^s$ and $\theta^{(b)}_\beta(f) \in L^s$, where $s \geqq q$ and $1/q = 1/p+\beta-1$. Now if $0 < \gamma < 1-(1/p+\beta-1) = 2-\beta-1/p$, then

$$\Psi^{(a)}_\gamma\{\psi^{(b)}_\beta(f)\},\ \Psi^{(a)}_\gamma\{\theta^{(b)}_\beta(f)\},\ \Theta^{(a)}_\gamma\{\theta^{(b)}_\beta(f)\}\ \text{and}\ \Theta^{(a)}_\gamma(\{\psi^{(b)}_\beta(f)\}$$

can be obtained from Theorem 6. Hence by substituting for $\psi^{(b)}_\beta(f)$ and $\theta^{(b)}_\beta(f)$ from expressions similar to those given in Theorem 5 and using Theorems 2, 6 and 7, we have the following:

$$\begin{aligned}\Psi^{(a)}_\gamma\{\psi^{(b)}_\beta(f)\} &= -\Theta^{(a)}_\gamma\{\theta^{(b)}_\beta(f)\}\\ &= (\sin \tfrac{1}{2}\pi\beta)(\cos \tfrac{1}{2}\pi\gamma)M_{1-\gamma}\{H_{1-\beta}(Q_{a+b}(f))\}\\ &= (\sin \tfrac{1}{2}\pi\gamma)(\cos \tfrac{1}{2}\pi\beta)L_{1-\gamma}\{K_{1-\beta}(Q_{a+b}(f))\}\end{aligned} \tag{3}$$

$$\begin{aligned}\Psi^{(a)}_\gamma\{\theta^{(b)}_\beta(f)\} &= \Theta^{(a)}_\gamma\{\psi^{(b)}_\beta(f)\}\\ &= (\sin \tfrac{1}{2}\pi\gamma)(\cos \tfrac{1}{2}\pi\beta)L_{1-\gamma}\{K_{1-\beta}(P_{a+b}(f))\}\\ &= (\sin \tfrac{1}{2}\pi\beta)(\cos \tfrac{1}{2}\pi\gamma)M_{1-\gamma}\{H_{1-\beta}(P_{a+b}(f))\}.\end{aligned} \tag{4}$$

The inversion formulae of Theorem 8 follow from Theorem 4 by taking

$$1-\gamma = \beta = \alpha.$$

REMARK 2. By letting a and b tend to 0 in Theorem 8 and applying the results (1) and (2) of section 1, we have

(i) $$\begin{aligned}\lim_{a,b\to 0+} \Psi^{(a)}_{1-\alpha}\{\psi^{(b)}_\alpha(f)\} &= -\lim_{a,b\to 0+} \Theta^{(a)}_{1-\alpha}\{\theta^{(b)}_\alpha(f)\}\\ &= -\tfrac{1}{2}(\sin \pi\alpha)\int_0^x H(f)(t)dt,\end{aligned}$$

(ii) $$\begin{aligned}\lim_{a,b\to 0+} \Psi^{(a)}_{1-\alpha}\{\theta^{(b)}_\alpha(f)\} &= \lim_{a,b\to 0+} \Theta^{(a)}_{1-\alpha}\{\psi^{(b)}_\alpha(f)\}\\ &= -\tfrac{1}{2}(\sin \pi\alpha)\int_0^x f(t)dt.\end{aligned}$$

References

[1] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities* (Cambridge University Press, 1934).

[2] G. O. Okikiolu, A Generalisation of the Hilbert transform, *Journal London Math. Soc.* 40 (1965), 27–30.

[3] G. O. Okikiolu, Fourier transforms and the operator H'_α, *Proc. Cambridge Phil. Soc.* 62 (1966) 73—78.
[4] H. Pollard, The Poisson transformation, *Trans. American Math. Soc.* 78 (1955), 541—550.
[5] H. Pollard, The semi-group property of Poisson transformation and Snow's inversion formula, *Proc. American Math. Soc.* 14 (1963), 285—290.
[6] E. C. Titchmarsh, *The theory of Fourier integrals* (Oxford Clarendon Press, 1948).

Mathematics Division
University of Sussex, Brighton,
England
and
University of East Anglia
Norwich