Theorems in the Products of Related Quantities.

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§1. Let $(x)_n$ denote the function

$$\underset{\kappa=\infty}{\mathrm{L}} \frac{(x-n+1)(x-n+2)\cdots\cdots(x-n+\kappa)}{(x+1)(x+2)\quad\cdots\cdots\quad(x+\kappa)}\cdot\kappa^n$$

then

$$\frac{(x+r)_n}{\underline{|r}\,\underline{|0}} - \frac{(x+r-1)_n}{\underline{|r-1}\,\underline{|1}} + \frac{(x+r-2)_n}{\underline{|r-2}\,\underline{|2}} - \cdots\cdots\cdots$$

$$\cdots\cdots + (-1)^r \frac{(x)_n}{\underline{|0}\,\underline{|r}} = \frac{n\cdot n-1\cdot n-2\cdots n-r+1}{\underline{|r}}(x)_{n-r} \quad \ldots (1)$$

In Gamma Functions the above may be written.

$$\frac{\Gamma(x)}{\Gamma(x-n)} - {}_rC_1\frac{\Gamma(x-1)}{\Gamma(x-n-1)} + {}_rC_2\frac{\Gamma(x-2)}{\Gamma(x-n-2)} - \cdots\cdots\cdots$$

$$\cdots\cdots + (-1)^r\frac{\Gamma(x-r)}{\Gamma(x-n-r)} = (n)_r\frac{\Gamma(x-r)}{\Gamma(x-n)} \quad \ldots (2)$$

By using the theorem (1) I shall obtain a purely algebraical proof of the well-known theorem

$$F_1(\alpha, \beta, \gamma) = \frac{\Pi(\gamma-1)\,.\,\Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1)\Pi(\gamma-\beta-1)}$$

where Π denotes Gauss's Π Function and $F_1(\alpha, \beta, \gamma)$ denotes the Hypergeometric Series in which the element $x_z = 1$.

It can be deduced from (1) that

$$1 - {}_rC_1\frac{(x-n)_1}{(x)_1} + {}_rC_2\frac{(x-n)_2}{(x)_2} - \cdots\cdots = \frac{(n)_r}{(x)_r} \quad \text{a pretty analogy with} \ldots (3)$$

$$1 - {}_rC_1\frac{(x-n)^1}{x^1} + {}_rC_2\frac{(x-n)^2}{x^2} - \cdots\cdots = \frac{(n)^r}{(x)^r}$$

r is not necessarily an integer in (3) and (1).*

* See §7.

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§ 2. **A fundamental property of the function $(x)_n$ is**

$$(x)_n \times (x-n)_m = (x)_{n+m}$$

whence we get $(x)_{n-r} \times (x-n+r)_s \times (x+r-s)_{r-s} = (x+r-s)_n$

Now the $(s+1)^{th}$ term on the left side of (1) $= (-1)^r \dfrac{(x+r-s)_n}{\underline{|r-s}\,\underline{|s}}$

which may be written

$$(-1)^s \frac{(x)_{n-r}(x-n+r)_s \;\; (x+r-s)_{r-s}}{\underline{|r-s}\;\;\underline{|s}}$$

Since $(x+r-s)_{r-s}$ —when r and s are both integers—may be written in the form

$$(x+1)(x+2)(x+3)\cdots(x+r-s) = (-1)^{r-s}(-x-1)_{r-s}$$

$\therefore$ the $(s+1)^{th}$ term $= (-1)^r \dfrac{(x)_{n-r}(x-n+r) \;\; (-x-1)_{r-s}}{\underline{|r-s}\;\;\underline{|s}}$

The expression on the left side of (1) may be written

$$(-1)^r \frac{(x)_{n-r}}{\underline{|r}} \Bigg\{ (-x-1)_r + \frac{\underline{|r}}{\underline{|r-1}\,\underline{|1}}(-x-1)_{r-1}(x-n+r)_1$$
$$+ \frac{\underline{|r}}{\underline{|r-2}\,\underline{|2}}(-x-1)_{r-2}(x-n+r)_2 + \cdots + (-1)^r(x-n+r)_r \Bigg\} \quad (4)$$

By Vandermonde's theorem* the expression with the large bracket

$$= (-x-1+x-n+r)_r = (r-n-1)_r$$

Expression (4) becomes

$$(-1)^r \frac{(x)_{n-r}}{\underline{|r}}(-n-1)_r = \frac{n\cdot n-1\cdot n-2\cdots n-r+1}{\underline{|r}}(x)_{n-r}$$

which proves theorem (1).

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* See § 7.

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§3. Now $$(x)_n = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}$$

$$\because\ \Gamma(x+1) = \underset{\kappa=\infty}{\mathrm{L}} \frac{1\cdot 2\cdot 3\cdot 4\cdots\kappa}{(x+1)(x+2)\cdots(x+\kappa)}\kappa^x$$

Replacing $(\ )_n$ by Gamma Functions, the theorem (1), after multiplication throughout by $\underline{|r}$, becomes

$$\frac{\Gamma(x+r+1)}{\Gamma(x+r-n+1)} - {}_rC_1\frac{\Gamma(x+r)}{\Gamma(x+r-n)} + {}_rC_2\frac{\Gamma(x+r-1)}{\Gamma(x+r-n-1)} - \cdots\cdots$$
$$\cdots\cdots + (-1)^r\frac{\Gamma(x+1)}{\Gamma(x-n+1)} = (n)_r\frac{\Gamma(x+1)}{\Gamma(x-n+r+1)} \qquad (5)$$

substitute y for $x+r+1$, then (5) becomes

$$\frac{\Gamma(y)}{\Gamma(y-n)} - {}_rC_1\frac{\Gamma(y-1)}{\Gamma(y-n-1)} + {}_rC_2\frac{\Gamma(y-2)}{\Gamma(y-n-2)} - \cdots\cdots$$
$$\cdots\cdots + (-1)^r\frac{\Gamma(y-r)}{\Gamma(y-n-r)} = (n)_r\frac{\Gamma(y-r)}{\Gamma(y-n)}$$

Remembering that $\Gamma(y) = (y-1)\Gamma(y-1)$ on division throughout by $\dfrac{\Gamma(y)}{\Gamma(y-n)}$ we have

$$1 - {}_rC_1\frac{(y-n-1)_1}{(y-1)_1} + {}_rC_2\frac{(y-n-1)_2}{(y-1)_2} - {}_rC_3\frac{(y-n-1)_3}{(y-1)_3} + \cdots = (n)_r\frac{1}{(y-1)_r}$$

this may be written

$$1 - r\cdot\frac{(x-n)_1}{(x)_1} + \frac{r\cdot r-1}{2\,!}\frac{(x-n)_2}{(x)_2} - \frac{r\cdot r-1\cdot r-2}{3\,!}\frac{(x-n)_3}{(x)_3} + \cdots = \frac{(n)_r}{(x)_r} \qquad (6)$$

analogous to the Binomial Expansion

$$1 - r\cdot\frac{(x-n)}{(x)} + \frac{r\cdot r-1}{2\,!}\frac{(x-n)^2}{(x)^2} - \cdots = \frac{(n)^r}{(x)^r}$$

The Expansion (5) has been obtained on the supposition that r is a positive integer; but it will be shown later to hold for negative and fractional values of r.

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§ 4. To consider the expansion in general of $f(x+y)$ in the form $\quad P_0+P_1(x)_1+P_2(x)_2+\cdots+P_r(x)_r+\cdots$ where $P_0 \cdot P_1 \cdot P_2 \cdots$ are functions of y only or constants. Assume that $f(x+y)$ is capable of being expanded in a convergent series of the above form then

$$f(x+y)=P_0+P_1(x)_1+P_2(x)_2+\cdots+P_r(x)_r+\cdots$$

By giving x the values $0 \cdot 1 \cdot 2 \cdot 3 \cdots$ in succession we obtain the following equations to determine $P_0 \cdot P_1 \cdot P_2 \cdots$

$$f(y) \quad = P_0$$

$$f(y+1)=P_0+P_1$$

$$f(y+2)=P_0+2P_1+2 \cdot 1 \ P_2$$

....

$$f(y+r)=P_0+r \cdot P_1+r \cdot r-1 \ P_2+\ldots \underline{|r} \cdot P$$

....

From which we obtain

$$P_0=\frac{f(y)}{\underline{|0}\ \underline{|0}}$$

$$P_1=\frac{f(y+1)}{\underline{|1}\ \underline{|0}}-\frac{f(y)}{\underline{|0}\ \underline{|1}}$$

....

$$P_r=\frac{f(y+r)}{\underline{|r}\ \underline{|0}}-\frac{f(y+r-1)}{\underline{|r-1}\ \underline{|1}}+\frac{f(y+r-2)}{\underline{|r-2}\ \underline{|2}}-\cdots+(-1)^r\frac{f(y)}{\underline{|0}\ \underline{|r}}$$

....

which is that

$$f(x+y)=\sum_{r=0}^{r=\infty}\sum_{s=0}^{s=r}(-1)^s\frac{f(y+r-s)}{\underline{|r-s}\ \underline{|s}}(x)_r \qquad \ldots \qquad \ldots \qquad (7)$$

subject to the convergence of the series.

§5. The expansion of $(x+y)_n$, n being unrestricted.

The coefficient of x_r will be

$$P_r \equiv \frac{(y+r)_n}{\lfloor n \lfloor 0} - \frac{(y+r-1)_n}{\lfloor n-1 \lfloor 1} + \frac{(y+r-2)_n}{\lfloor n-2 \lfloor 2} - \cdots\cdots + (-1)^r \frac{(y)_n}{\lfloor 0 \lfloor n} \equiv \frac{n \cdot n-1 \cdots n-r+1}{\lfloor r} y_{n-r}$$

(by Theorem (1)).

$$\therefore \quad (x+y)_n = y_n + n y_{n-1} x_1 + \frac{n \cdot n-1}{\lfloor 2} y_{n-2} x_2 + \cdots\cdots + \frac{n \cdot n-1 \cdots x-r+1}{\lfloor r} y_{n-r} x_r + \cdots\cdots \quad (8)$$

This is the generalised form of Vandermonde's Theorem; the proof depends, as will be seen on reference to §2, No. 4, on Vandermonde's Theorem for positive integral values of the suffix.

To expand a^x in a series of form (7)

we have $$P_r = \frac{a^r}{\lfloor r \lfloor 0} - \frac{a^{r-1}}{\lfloor r-1 \lfloor 1} + \cdots + (-1)^r \frac{a^0}{\lfloor 0 \lfloor r} \equiv \frac{(a-1)^r}{\lfloor r}$$

$$\therefore \quad a^x = 1 + (a-1)(x)_1 + \frac{(a-1)^2}{\lfloor 2}(x)_2 + \cdots + \frac{(a-1)^r}{\lfloor r}(x)_r + \cdots\cdots$$

this is a well known particular case of the Binomial Expansion.

To expand $\dfrac{1}{x+a}$

we have $$P_r = \frac{1}{\lfloor r}\left\{\frac{1}{a+r} - {}_rC_1 \frac{1}{a+r-1} + {}_rC_2 \frac{1}{a+r-2} - \cdots + (-1)^r \frac{1}{a}\right\}$$

$$= \frac{1}{\lfloor r} \cdot \frac{\lfloor r}{(a+r)(a+r-1)\cdots(a+1)a \cdot}$$

$$\therefore \quad \frac{1}{x+a} = \frac{1}{a} + \frac{(x)_1}{a \cdot a+1} + \frac{(x)_2}{a \cdot a+1 \cdot a+2} + \cdots + \frac{(x)_r}{a \cdot a+1 \cdots a+r} + \cdots$$

This is a special case of Vandermonde's Theorem for negative integral values of the suffix. The functions which can be expanded in series of form (7) seem very restricted in number.

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§ 6. Writing

$$(x+y)_n = (y)_n + n\cdot (y)_{n-1}(x)_1 + \frac{n\cdot n-1}{\underline{|2}\ \underline{|r}}(y)_{n-2}(x)_2 + \cdots$$

$$\cdots\cdots + \frac{n\cdot n-1\cdots n-r+1}{\underline{|r}}(y)_{n-r}(x)_r + \cdots\cdots$$

divide both sides by $(y)_n$.

Then $\dfrac{(x+y)_n}{(y)_n} = 1 + n\cdot\dfrac{(y)_{n-1}(x)_1}{(y)_n} + \dfrac{n\cdot n-1}{\underline{|2}\ \underline{|r}}\ \dfrac{(y)_{n-2}(x)_2}{(y)_n} + \cdots$

Now it is easily seen that $\dfrac{(y)_{n-1}}{(y)_n} = \dfrac{1}{y-n+1}$

$$\cdots\cdots\cdots\cdots$$

$$\frac{(y)_{n-r}}{(y)_n} = \frac{1}{(y-n+1)_r}$$

and $\dfrac{(x+y)_n}{(y)_n} = \dfrac{\Pi(x+y)}{\Pi(x+y-n)}\cdot\dfrac{\Pi(y-n)}{\Pi(x)}$ where Π denotes Gauss's Π Function. Therefore

$$\frac{\Pi(x+y)\cdot\Pi(y-n)}{\Pi(x+y-n)\ \Pi(x)} = 1 + n\cdot\frac{(x)_1}{(y-n+1)_1}$$

$$+\frac{n\cdot n-1}{2\,!}\ \frac{(x)_2}{(y-n+1)(y-n+2)} + \cdots + \frac{n\cdots n-r+1}{r\,!}\ \frac{(x)_r}{(y-n+1)_r} + \cdots$$

Replacing n by $-\alpha$, x by $-\beta$, and $y-n+1$ by γ we have

$$\frac{\Pi(\gamma-\alpha-\beta-1)\cdot\Pi(\gamma-1)}{\Pi(\gamma-\beta-1)\Pi(\gamma-\alpha-1)} = 1 + \frac{\alpha\cdot\beta}{1\cdot\gamma} + \frac{\alpha\cdot\alpha+1\cdot\beta\cdot\beta+1}{1\cdot 2\cdot\gamma\cdot\gamma+1} + \cdots \qquad (9)$$

$$= F_1(\alpha,\ \beta,\ \gamma)$$

§ 7. If in § 2, result (4), we had assumed the truth of Vandermonde's Theorem for unrestricted values of the suffix, Theorems (1), (2), and (3) would have been proved for all values of r. Since we have proved Vandermonde's Theorem for unrestricted values of the suffix, the proofs of §§ 2 and 3 may be repeated with r unrestricted. The use of $(-1)^r$ in § 2 can easily be avoided. When r is unrestricted, $(r)_r$ must be used instead of $\underline{|r}$.