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Holomorphic Frames for Weakly Converging Holomorphic Vector Bundles

Georgios D. Daskalopoulos and Richard A. Wentworth

Abstract. Using a modification of Webster's proof of the Newlander–Nirenberg theorem, it is shown that, for a weakly convergent sequence of integrable unitary connections on a complex vector bundle over a complex manifold, there is a subsequence of local holomorphic frames that converges strongly in an appropriate Holder class.

Perhaps the most useful analytic tool in gauge theory is the Uhlenbeck compactness theorem for sequences of unitary connections on hermitian vector bundles [U1]. Given connections $\{D_j\}$ on a bundle E over a compact manifold M, the result returns a subsequence converging weakly in L^p_1, up to unitary gauge transformations, provided the original sequence has uniform L^p bounds on curvature, where $2p > \dim M$. With further assumptions, *e.g.,* if the connections are solutions to the Yang–Mills equations, convergence away from some singular set can also be obtained in the critical case $2p = \dim M$.

When M is a complex manifold and the connections satisfy an integrability condition, the theorem implies weak L^p_1 convergence of the induced holomorphic structures D''_j on E. If M is assumed to be Kähler, weak L^p_1 convergence away from a singular set can in fact be obtained from the more natural assumptions of L^2 bounded curvature and bounded Hermitian–Einstein tensor (Uhlenbeck, personal communication). In any case, on complex manifolds it is useful for applications to have control on local holomorphic frames, since then one may use techniques from several complex variables. The purpose of this note is to show that under the circumstances described above one may find local holomorphic trivializations of E which also converge with the optimal regularity, provided $p > \dim M$.

The argument we give is based largely on Webster's proof of the Newlander–Nirenberg theorem [W]. A notable difference is the somewhat more linear character of the problem for vector bundles. For this reason, the proof in [W] may be adapted to the weak L^p_1 convergence that is natural to Uhlenbeck compactness, whereas stronger control of derivatives is generally required for holomorphic structures on manifolds.

For background on connections on hermitian vector bundles we refer the reader to [K].

Theorem 1 *Let $\{D_j\}$ be a sequence of integrable unitary connections on a complex vector bundle E over a complex manifold M of complex dimension n. Assume that $D_j \to$*

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D_∞ weakly in $L^p_{1,\mathrm{loc}}(M)$ for some integrable connection D_∞ and some $p > 2n$. Then for each $x \in M$ there is:

(1) *a coordinate neighborhood Ω of x,*
(2) *a sequence $\{\mathbf{s}_j\}$ of D''_j-holomorphic frames on Ω,*
(3) *a D''_∞-holomorphic frame $\mathbf{s}_\infty$ on Ω,*
(4) *and a subsequence $\{j_k\} \subset \{j\}$,*

such that $\mathbf{s}_{j_k} \to \mathbf{s}_\infty$ weakly in $L^p_2(\Omega)$ and strongly in $C^{1,\alpha}(\Omega)$ for $0 < \alpha < 1 - 2n/p$.

In the following, B_r will denote the open ball of radius r about the origin in $\mathbb{C}^n$. For k a non-negative integer, and α a real number $0 < \alpha < 1$, $\|\cdot\|_{k+\alpha;r}$ will denote the $C^{k,\alpha}$ norm on B_r. Theorem 1 is a consequence of the following:

Proposition 2 *Fix a positive integer R and a real number α, $0 < \alpha < 1$. Given $r > 0$ there are constants $\theta > 0$, $B > 0$, and $r' > 0$, $0 < r' < r$, such that the following holds: if a is any $R \times R$-matrix valued $(0,1)$-form on B_r satisfying:*

- $\bar\partial a + a \wedge a = 0$,
- $\|a\|_{\alpha;r} \le \theta$,

then there exists an $R \times R$-matrix valued function G on $B_{r'}$ satisfying:

- $\bar\partial G + aG = 0$,
- $\|G\|_{1+\alpha;r'} \le B$,
- $\inf_{B_{r'}} |\det G| \ge B^{-1}$.

Assuming the result above, let us give the:

Proof of Theorem 1 Choose a coordinate neighborhood centered at x, identified with B_r for some $r > 0$, and over which there exists a D''_∞-holomorphic trivialization of E. With respect to this trivialization we may regard $a_j = D''_j - D''_\infty$ as matrix valued $(0,1)$-forms satisfying $\bar\partial a_j + a_j \wedge a_j = 0$. By the compactness of the embedding $L^p_1 \hookrightarrow C^\alpha$ for $0 < \alpha < 1 - 2n/p$ and the weak convergence $a_j \to 0$ in $L^p_{1,\mathrm{loc}}$, we may assume $\|a_j\|_{\alpha;r} \to 0$. Hence, by Proposition 2, for each sufficiently large j we may find matrix-valued functions G_j satisfying:

$$\bar\partial G_j + a_j G_j = 0, \tag{1}$$

$$\|G_j\|_{1+\alpha;r'} \le B, \tag{2}$$

$$\inf_{B_{r'}} |\det G_j| \ge B^{-1}, \tag{3}$$

for some B and $r' > 0$ independent of j. In particular, the column vectors of G_j are linearly independent and define D''_j-holomorphic frames on $B_{r'}$. By (2) and the elliptic estimate for $\bar\partial$ applied to (1), it follows that the G_j are bounded in $L^p_{2,\mathrm{loc}}(B_{r'})$ uniformly in j. After passing to a subsequence, we may assume that there is some G such that $G_j \to G$ weakly in $L^p_{2,\mathrm{loc}}(B_{r'})$ and strongly in $C^{1,\alpha}_{\mathrm{loc}}(B_{r'})$. In particular, again using (1), $\bar\partial G = 0$. Finally, by the uniform bound (3), G is invertible on $B_{r'}$ and so its column vectors define a D''_∞-holomorphic frame. This completes the proof. ■

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It remains to prove Proposition 2. We will need the following:

Lemma 3 *Suppose T_j is a sequence of $R \times R$ complex matrices with $|T_j| \leq 1/2$ and $\sum_{j=1}^{\infty} |T_j| = C < \infty$. Set $S_k = (\mathbf{I}+T_1)(\mathbf{I}+T_2)\cdots(\mathbf{I}+T_k)$ where $\mathbf{I}$ is the $R \times R$ identity matrix. Then $|\det S_k| \geq e^{-2RC}$ for all k.*

Proof For each T_j we have:

$$|\det(\mathbf{I} + T_j)| \geq (1 - |T_j|)^R,$$
$$\log|\det(\mathbf{I} + T_j)| \geq R\log(1 - |T_j|),$$

Since $\log(1 - x) \geq -2x$ for $0 \leq x \leq 1/2$, $\log|\det(\mathbf{I} + T_j)| \geq -2R|T_j|$. Hence:

$$\log|\det S_k| = \sum_{j=1}^{k} \log|\det(\mathbf{I} + T_j)| \geq -2R\sum_{j=1}^{k} |T_j| \geq -2RC. \qquad \blacksquare$$

Recall the Leray–Koppelman operators P and Q for matrix valued $(0,1)$- and $(0,2)$-forms on B_r, respectively. Given σ, $0 < \sigma < 1$, have the following important properties [W, eq. (1.7) and Lemma 2.2]:

$$\varphi = \overline{\partial}P(\varphi) + Q(\overline{\partial}\varphi) \tag{4}$$
$$\|P(\varphi)\|_{1+\alpha;r(1-\sigma)} \leq rK\|\varphi\|_{\alpha;r} \tag{5}$$
$$\|Q(\psi)\|_{1+\alpha;r(1-\sigma)} \leq rK\|\psi\|_{\alpha;r} \tag{6}$$

where $K = c_\alpha\sigma^{-s}$ for c_α constant and $s > 0$ an integer.

With these preliminaries we now give:

Proof of Proposition 2 Set $a_0 = a$, $h_0 = 0$. Define sequences a_j, h_j recursively, where h_j are $R \times R$ matrix-valued functions defined on

$$h_{j+1} = -P(a_j), \tag{7}$$
$$g_{j+1} = \mathbf{I} + h_{j+1}, \tag{8}$$
$$a_{j+1} = (g_{j+1})^{-1}(\overline{\partial}g_{j+1} + a_j g_{j+1}). \tag{9}$$

The initial bound θ on a will be chosen presently so that $\sup_{B_{r'}} |h_j| \leq 1/4$. Hence, $g_j = \mathbf{I} + h_j$ will be uniformly invertible. Also, notice that with this definition the integrability condition $\overline{\partial}a_j + a_j \wedge a_j = 0$ is satisfied for all j. Following [W], set $\sigma_j = 4^{-j-1}$ and $r_{j+1} = r_j(1 - \sigma_j)$ with $r_0 = r$. It follows that the r_j are decreasing and that $r' = \lim_{j\to\infty} r_j > 0$. Recalling the constants $K_j = c_\alpha\sigma_j^{-s}$ in (5) and (6), and using (7), we have:

$$\|h_{j+1}\|_{1+\alpha;r_{j+1}} \leq rK_j\|a_j\|_{\alpha;r_j}\ . \tag{10}$$

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From (8) and (9) we have:

$$a_{j+1} = (g_{j+1})^{-1}(\overline{\partial} h_{j+1} + a_j + a_j h_{j+1})$$

and by (4) and (7):

$$\overline{\partial} h_{j+1} + a_j = Q(\overline{\partial} a_j) = -Q(a_j \wedge a_j).$$

Assuming the uniform invertibility of g_{j+1} mentioned above, it follows from (6) and (10) that there is a constant C independent of j such that:

$$\|a_{j+1}\|_{\alpha;r_{j+1}} \leq CK_j \|a_j\|^2_{\alpha;r_j}. \tag{11}$$

After absorbing constants into the definition of K_j, (10) and (11) may be written:

$$\|h_{j+1}\|_{1+\alpha;r_{j+1}} \leq K_j \|a_j\|_{\alpha;r_j} \tag{12}$$

$$\|a_{j+1}\|_{\alpha;r_{j+1}} \leq K_j \|a_j\|^2_{\alpha;r_j}. \tag{13}$$

Moreover, there is a constant b (*e.g.*, $b = 4^s$) such that $K_{j+1} \leq bK_j$ for all j. Next, we define: $\theta_j = K_j \|a_j\|_{\alpha;r_j}$. Then by assumption: $\theta_0 = K_0 \|a\|_{\alpha;r} \leq K_0 \theta$. We assume that θ has been chosen so small that $bK_0\theta \leq 1/4$, say. We then deduce inductively, using (13), that:

$$\theta_{j+1} \leq b\theta_j^2 \leq \theta_j/4. \tag{14}$$

It follows that $\theta_j \to 0$. Furthermore, we can rewrite (12) and (13) as:

$$\|h_{j+1}\|_{1+\alpha;r_{j+1}} \leq \theta_j \tag{15}$$

$$\|a_{j+1}\|_{\alpha;r_{j+1}} \leq \theta_j \|a_j\|_{\alpha;r_j}. \tag{16}$$

It follows from (16) that $\|a_j\|_{\alpha;r'} \to 0$. Notice also that $\|h_j\|_{1+\alpha;r_j} \leq 1/4$ for all j. Hence, the g_j are uniformly invertible, as desired. We now define gauge transformations:

$$G_k = g_1 g_2 \cdots g_k. \tag{17}$$

First, note that $|G_k|$ is uniformly bounded. Indeed, $|G_k| \leq \prod_{j=1}^k |g_j| \leq \prod_{j=1}^k (1+\theta_j)$, by (8) and (15), and the right-hand side converges as $k \to \infty$ by (14). The derivatives $|\nabla G_k|$ are similarly bounded:

$$|\nabla G_k| = \Big|\sum_{j=1}^k g_1 \cdots g_{j-1} \nabla g_j g_{j+1} \cdots g_k\Big| \leq \sum_{j=1}^k |g_1| \cdots |g_{j-1}| |\nabla g_j| |g_{j+1}| \cdots |g_k|$$

$$\leq \Big(\sum_{j=1}^k \theta_j\Big) \prod_{j=1}^k (1+\theta_j),$$

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which also converges as $k \to \infty$. In particular, we have a bound on $\|G_k\|_{\alpha;r'}$ that is uniform in k. Next, from (17) we have: $G_{k+1} = G_k g_{k+1} = G_k + G_k h_{k+1}$, so by (15),

$$\|G_{k+1} - G_k\|_{\alpha;r'} \leq c\|G_k\|_{\alpha;r'}\|h_{k+1}\|_{\alpha;r'} \leq C\theta_k,$$

for a constant C independent of k. It follows again by (14) that G_k converges in $C^\alpha(B_{r'})$ to some G. To improve the convergence, use the definition (9) to write:

$$\overline{\partial} G_k + aG_k - G_k a_k = 0, \tag{18}$$

for all k. Hence,

$$\|\overline{\partial} G_j - \overline{\partial} G_k\|_{\alpha;r'} \leq C\big(\|G_j - G_k\|_{\alpha;r'} + \|a_j - a_k\|_{\alpha;r'}\big),$$

and since $\|a_k\|_{\alpha;r'} \to 0$ and $\|G_j - G_k\|_{\alpha;r'} \to 0$ it follows that $\overline{\partial} G_k$ converges in $C^\alpha(B_{r'})$. By the elliptic estimate for $\overline{\partial}$, $G_k \to G$ in $C^{1,\alpha}(B_{r'})$, and moreover $\overline{\partial} G + aG = 0$ (*cf.* (18)). Finally, we claim that G is nonsingular. Indeed, it follows from the convergence of G_k that $\det G_k \to \det G$. By definitions (8) and (17), $G_k = \prod_{j=1}^{k}(1 + h_j)$, where according to the estimates (14) and (15), $|h_i| \leq 1/2$ and $\sum_{j=1}^{\infty} |h_j| < \infty$. The claim now follows immediately from Lemma 3. Since r', the $C^{1,\alpha}$ bound on G, and the bound on the determinant all stem from the initial choice of θ, which in turn depends only on r, the proof of the Proposition is complete. ■

References

[K] S. Kobayashi, *Differential Geometry of Complex Vector Bundles.* Publications of the Mathematical Society of Japan 15, Princeton University Press, Princeton, NJ, 1987.

[U1] K. Uhlenbeck, *Connections with L^p bounds on curvature.* Comm. Math. Phys. **83**(1982), 31–42.

[W] S. Webster, *A new proof of the Newlander-Nirenberg theorem.* Math. Z. **201**(1989), 303–316.

Department of Mathematics
Brown University
Providence, RI 02912
U.S.A.
e-mail: daskal@math.brown.edu

Department of Mathematics
Johns Hopkins University
Baltimore, MD 21218
U.S.A.
e-mail: wentworth@jhu.edu

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