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DEFICIENCIES OF CERTAIN REAL UNIFORM ALGEBRAS

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Introduction. Let U be a complex uniform algebra, Z and ∂Z its maximal ideal space and its Šilov boundary, respectively. The Dirichlet (respectively Arens–Singer) deficiency of U is the codimension in $C_R(\partial Z)$ of the closure of Re U (respectively of the real linear span of $\log|U^{-1}|$). Algebras with finite Dirichlet deficiency have many interesting properties, especially when the Arens–Singer deficiency is zero. (See, e.g. [**5**].) By a real uniform algebra we mean a real commutative Banach algebra A with identity 1, and norm $\|\ \|$ such that $\|f^2\| = \|f\|^2$ for each f in A.

By considering the complexification B of A, we show in §1 that the Šilov boundary of A exists, thus giving a valid proof of a result claimed by Alling in [**1**, Theorems 3.13 and 3.16]. This enables us to define the Dirichlet and the Arens–Singer deficiencies of A. Next, we introduce the concepts of imaginary Dirichlet deficiency and inverse Arens–Singer deficiency of A. It turns out easily that the Dirichlet (Arens–Singer) deficiency of B is the sum of the Dirichlet (Arens–Singer) and the imaginary Dirichlet (inverse Arens–Singer) deficiencies of A (Proposition 1.3). As an example, we consider the standard algebras on compact bordered non-orientable Klein surfaces, and compute their Dirichlet and imaginary Dirichlet deficiencies in terms of the first Betti numbers of the surfaces (Example 1.4).

In § 2, we study the following real subalgebras of a complex uniform algebra U. Let $\{z_1, \ldots, z_q\}$ be a finite subset of Z and D_k a continuous (possibly trivial) point derivation of U at z_k, for each k. Let

$$A_q = \{f \text{ in } U : f(z_k) \text{ and } D_k(f) \text{ real for } 1 \leqq k \leqq q\}.$$

In §3, we calculate the Dirichlet, the Arens–Singer, the imaginary Dirichlet and the inverse Arens–Singer deficiencies of A_q in terms of the deficiencies of U, the number of Gleason parts in Z to which the z_k's belong, the number of points z_k which belong to ∂Z, and the number of nontrivial point derivations D_k (Theorems 3.3 and 3.4). This tells us about the possibilities of approximating continuous real-valued functions on ∂Z by various real-valued functions associated with A_q like Re A_q, Im A_q, etc.

1. Šilov boundary and deficiencies of A.

Let A be a commutative real Banach algebra with identity $1 \neq 0$, and Y its maximal ideal space. Each f

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in A defines two real-valued functions, $\operatorname{Re} f$ and $|f|$, on Y. (See [**1**, §3].) Let Y be given the weakest topology making all $|f|$, f in A, continuous. Note that for f in A, $g = \exp f$ belongs to A^{-1}, and $\operatorname{Re} f = \log |g|$. Since $|g|$ is always positive, $\operatorname{Re} f$ is also continuous in this topology on Y. A general reference for real Banach algebras is [**6**].

Let us consider now the complexification

$$B \equiv \{1 \otimes f + i \otimes g : f, g \text{ in } A\}$$

of A, together with a norm which makes the natural **R**-injection of A into B an isometry. Then B is a commutative complex Banach algebra with identity $1 \otimes 1$. Let f in A be identified with $1 \otimes f$ in B. Let X be the maximal ideal space of B and cx^* the map from X to Y which restricts a maximal ideal of B to that of A. Define the involution σ of B by $\sigma\,(1 \otimes f + i \otimes g) = 1 \otimes f - i \otimes g$. Now, σ induces an endomorphism τ of X, where

$$\tau(x) = \{1 \otimes f + i \otimes g : 1 \otimes f - i \otimes g \text{ in } x\},$$

for each x in X. Clearly $cx^* \circ \tau = cx^*$, and if ∂X is the Šilov boundary of B, $\tau(\partial X) = \partial X$.

Now, for each f in A and M in X, $|(1 \otimes f)(M)| = |f|(cx^*(M))$. Since cx^* maps X onto Y, we have for each f in A,

$$||1 \otimes f\,||_\infty = \sup_{N \text{ in } Y} |\,f\,|\;(N),$$

where $||\ \ ||_\infty$ is the spectral norm for B.

For f in A, let $||f||_\infty = ||1 \otimes f||_\infty$. We shall henceforth work with this norm on A. It is, therefore, necessary to know when the original norm $||\ \ ||$ on A coincides with this norm. By the spectral radius formula for B,

$$\begin{aligned} ||1 \otimes f\,||_\infty &= \lim_{n\to\infty} ||(1 \otimes f\,)^n||^{1/n} \\ &= \lim_{n\to\infty} ||1 \otimes f^{\,n}||^{1/n} \\ &= \lim_{n\to\infty} ||f^{\,n}||^{1/n}. \end{aligned}$$

Thus, the two norms $||\ \ ||$ and $||\ \ ||_\infty$ for A coincide if and only if $||f^2|| = ||f||^2$ for every f in A. We shall assume this property from now on and not distinguish between the two norms for A.

A *boundary* Y_0 of A is a subset of Y such that for each f in A,

$$||f\,|| = \sup_{N \text{ in } Y_0} |\,f\,|\;(N).$$

PROPOSITION 1.0. *Let ∂X be the Šilov boundary of B. Then $cx^*(\partial X)$ is the smallest closed boundary of A.*

Proof. First, since ∂X is compact, cx^* is continuous and Y is Hausdorff,

$cx^*(\partial X)$ is closed. Also, for f in A,

$$\|f\| = \|1 \otimes f\|_\infty = \sup_{M \text{ in } \partial X} |(1 \otimes f)(M)| = \sup_{N \text{ in } cx^*(\partial X)} |f|(N).$$

Thus, $cx^*(\partial X)$ is a closed boundary of A. To prove it is the smallest such boundary, it is enough to show that if Y_0 is a closed boundary of A, then $X_0 \equiv (cx^*)^{-1}(Y_0)$ is a boundary for B. Assume for a moment that X_0 is not a boundary for B. Then there exists a b in B such that for some x_0 in X, $b(x_0) = 1$, but $|b| \leqq \epsilon < 1$ on X_0. Hence $|b + \sigma(b)| \leqq 2\epsilon$ and $|b\sigma(b)| \leqq \epsilon^2$ on X_0. Since $b + \sigma(b)$ and $b\sigma(b)$ belong to A, and since Y_0 is a boundary of A, these inequalities are valid on X, in particular at x_0. Thus,

$$|\sigma(b)(x_0)| = |b(x_0)\sigma(b)(x_0)| \leqq \epsilon^2, \text{ and } 1 - \epsilon^2 \leqq |b(x_0)| - |\sigma(b)(x_0)| \leqq |(b + \sigma(b))(x_0)| \leqq 2\epsilon.$$

By considering a high enough power of b, we can make ϵ arbitrarily small. This contradicts $1 - \epsilon^2 \leqq 2\epsilon$.

Let $\partial Y \equiv cx^*(\partial X)$. Then ∂Y is called the *Šilov boundary* of A. We would like to remark here that the existence of the Šilov boundary for real commutative Banach algebras was claimed by Alling in [**1**, Theorem 3.13], but it seems that the proof he indicated there as well as the proof of Theorem 3.16 of [**1**] cannot be justified.

Let $C_R(\partial Y)$ denote the space of all real-valued continuous functions on ∂Y. The codimension of the uniform closure in $C_R(\partial Y)$ of

$$(\operatorname{Re} A)\ (\partial Y) \equiv \{\operatorname{Re} f \text{ restricted to } \partial Y : f \text{ in } A\}$$

is called the *Dirichlet deficiency* of A; and the codimension of the uniform closure of the real linear span of

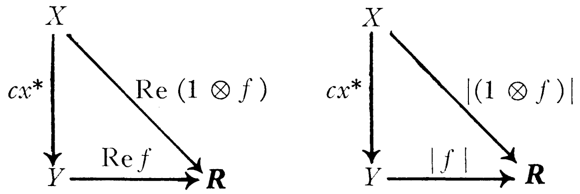
$$(\log |A^{-1}|)(\partial Y) \equiv \{\log |f| \text{ restricted to } \partial Y : f \text{ in } A^{-1}\}$$

in $C_R(\partial Y)$ is called the *Arens–Singer deficiency of A*.

Let $C(\partial X)$ denote the space of all complex-valued continuous functions on ∂X. For h in $C(\partial X)$, let $\sigma(h)(x) = \bar{h}(\tau(x))$, for each x in ∂X. Then, $\sigma^2 =$ identity, $\sigma(c_1h_1 + c_2h_2) = \bar{c}_1\sigma(h_1) + \bar{c}_2\sigma(h_2)$, and $\sigma(h_1h_2) = \sigma(h_1)\sigma(h_2)$, for c_1 and c_2 complex numbers and h_1 and h_2 in $C(\partial X)$. Let $B(\partial X)$ and $A(\partial X)$ be the sets of restrictions of elements in B and A, respectively, to ∂X. Then σ maps $B(\partial X)$ into itself, and $A(\partial X) = \{h \text{ in } B(\partial X) : \sigma(h) = h\}$. Correspondingly, let $C_R(\partial X)^s \equiv \{u \text{ in } C_R(\partial X) : u \circ \tau = u\}$ be the set of all *symmetric* (w.r.t. τ) *elements* of $C_R(\partial X)$.

PROPOSITION 1.1. *The Dirichlet (respectively Arens–Singer) deficiency of A equals the codimension of the closure of* Re $(A\ (\partial X))$ *(respectively of the real linear span of* $\log |A(X)^{-1}|$) *in* $C_R(\partial X)^s$.

Proof. Notice that for each f in A, the following two diagrams commute:



Since the map cx^* is both continuous and open [**1**, Corollary 3.3 and Lemma 3.9] the proof of the proposition follows easily.

The above proposition lets us introduce the concepts of imaginary Dirichlet and inverse Arens–Singer deficiencies of A. Let $C_R(\partial X)^a$ denote the set of all *antisymmetric* (w.r.t. τ) *elements of* $C_R(\partial X)$: i.e., $\{u \text{ in } C_R(\partial X) : u \circ \tau = -u\}$.

Definition 1.2. The codimension of the uniform closure of Im $(A(\partial X))$ in $C_R(\partial X)^a$ will be called *the imaginary Dirichlet deficiency of* A. The codimension in $C_R(\partial X)^a$ of the uniform closure of the real linear span of log $|A_{-1}(\partial X)|$, where $A_{-1}(\partial X) = \{h \text{ in } B(\partial X)^{-1} : \sigma(h) = h^{-1}\}$, will be called *the inverse Arens–Singer deficiency* of A.

Note that Im $(A(\partial X))$ is contained in $\log|A_{-1}(\partial X)|$, since f in $A(\partial X)$ and $g \equiv \exp(-if)$ give Im $f = \log|g|$. Hence the inverse Arens–Singer deficiency of A is less than or equal to the imaginary Dirichlet deficiency of A.

PROPOSITION 1.3. *The Dirichlet* (*respectively Arens–Singer*) *deficiency of* B *is equal to the sum of the Dirichlet* (*respectively Arens–Singer*) *and the imaginary Dirichelt* (*respectively inverse Arens–Singer*) *deficiencies of* A.

Proof. Since $C_R(\partial X) = C_R(\partial X)^s \oplus C_R(\partial X)^a$, it is enough to show that

$$\text{cl. Re } B(\partial X) = \text{cl. Re } A(\partial X) \oplus \text{cl. Im } A(\partial X)$$

and that

$$\text{cl. } \langle \log|B(\partial X)^{-1}| \rangle = \text{cl. } \langle \log|A(\partial X)^{-1}| \rangle \oplus \text{cl. } \langle \log|A_{-1}(\partial X)| \rangle$$

where cl. denotes the uniform closure and $\langle\ \rangle$ denotes the real linear span. For f and g in A, let $b = 1 \otimes f + i \otimes g$. Then Re b = Re f − Im g, and $\log|b| = \frac{1}{2}\log|b\sigma(b)| + \frac{1}{2}\log|b\sigma(b)^{-1}|$. The result follows by taking the real linear span and the uniform closure.

Example 1.4. Let Y be a compact non-orientable Klein surface with a non-empty boundary ∂Y, and let A be the standard algebra associated with Y. (See [**1**, § 2].) Let (X, p, τ) be the orienting double of Y, where X is a compact Riemann surface with boundary ∂X, p is a covering morphism such that $p^{-1}(\partial Y) = \partial X$, and τ is an antianalytic involution of X which commutes with p. If c is the first Betti number of Y, then the first Betti number of X is $2c - 1$.

If B is the standard algebra associated with X, then B is the complexification of A. (See [**1**, §4].) It is well-known [**9**, Lemma 1] that the Dirichlet deficiency of B is $2c - 1$, while its Arens–Singer deficiency is 0. It was proved recently in [**3**, Theorem 4.2] that there exists a basis $\{Z_1, \ldots, Z_{2c-1}\}$ of B^{-1} modulo exp B such that $\sigma(Z_j) = Z_j$ for $1 \leqq j \leqq c - 1$, $\sigma(Z_j) = Z_{j-(c-1)} Z_j^{-1}$ for $c \leqq j \leqq 2c - 2$, and $\sigma(Z_{2c-1}) = -Z_{2c-1}^{-1}$, where $\sigma(f) = \bar{f} \circ \tau$. Let us now define

$$u_j = \begin{cases} \log|Z_j|, \text{ if } 1 \leqq j \leqq c - 1, \text{ or } j = 2c - 1 \\ \log|Z_j| - \tfrac{1}{2}\log|Z_{j-(c-1)}|, \text{ if } c \leqq j \leqq 2c - 2. \end{cases}$$

Then cl. $\langle \text{Re } B, u_1, \ldots, u_{2c-1} \rangle = C_R(\partial X)$, and $u_j = u_j \circ \tau$ for $1 \leqq j \leqq c - 1$, whereas $u_j = -u_j \circ \tau$ for $c \leqq j \leqq 2c - 1$. It follows that the Dirichlet deficiency of A is $c - 1$ (cf. [**2**, Theorem 5.7]), and that the imaginary Dirichlet deficiency of A is c (cf. [**3**, Theorem 3.6]).

2. Some real subalgebras of a complex algebra. Let U be a complex uniform algebra with norm $\| \ \|$. Let Z and ∂Z be its maximal ideal space and its Šilov boundary respectively. In this section we consider the following real subalgebras of U. Let $\{z_1, \ldots, z_q\}$ be a specified finite subset of q points in Z, and let D_k be a continuous (possibly trivial) point derivation of U at z_k for each k. Define

$$A_q = \{f \text{ in } U : f(z_k) \text{ and } D_k(f) \text{ real for } 1 \leqq k \leqq q\}.$$

Then A_q is a real uniform algebra. Let Y and ∂Y be its maximal ideal space and its Šilov boundary respectively. We shall assume throughout that the Dirichlet deficiency of U is finite and would like to compute the Dirichlet, the imaginary Dirichlet, the Arens–Singer and the inverse Arens–Singer deficiencies of A_q. Consider now the restriction map j from Z to Y. We shall show that we can identify Y and ∂Y with Z and ∂Z respectively, by means of this map. For this purpose, we need the following crucial lemma.

LEMMA 2.1. (i) *There exist* $f_1^*, \ldots, f_q^*$ *in* U *such that* $(f_m^*)(z_k) = \delta_{m,k}$, *and* $D_k(f_m^*) = 0$, *for* $1 \leqq m, k \leqq q$.

(ii) *Of the given* q *continuous point derivations* $D_1, \ldots, D_q$, *let* $D_{k_1}, \ldots, D_{k_p}$ *be the only non-trivial ones. Then there exist* $g_1^*, \ldots, g_q^*$ *in* U *such that* $(g_m^*)(z_k) = 0$, *for* $1 \leqq m, k \leqq q$, *and* $D_{k_i}(g_m^*) = \delta_{m,k_i}$, *for* $1 \leqq m \leqq q$, $1 \leqq i \leqq p$.

Proof. (i) First we show that given two points a and b in Z, and two derivations D_a and D_b at a and b respectively, there exists f in U such that $f(a) = 1$, $f(b) = D_a(f) = D_b(f) = 0$. Surely there exists h in U such that $h(a) = 1$ and $h(b) = 0$. Then it suffices to take $f \equiv 2h^2 - h^4$. Denote this function as $f_{a,b}$. Fix now m, $1 \leqq m \leqq q$, and let

$$f_m^* \equiv \prod_{j=1,\, j\neq m}^{q} f_{z_m, z_j}.$$

(ii) Again, fix m, $1 \leqq m \leqq q$. For $j \neq m$, $1 \leqq j \leqq q$, there exists f_j in U

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such that $f_j(z_j) = 0$ and $f_j(z_m) = 1$. Also, let f_m be in U such that $f_m(z_m) = 0$, and $D_m(f_m) = 1$, if D_m is non-trivial.

Now let

$$g_m^* \equiv f_m \cdot \prod_{j=1,\, j\geqq m}^{q} f_j^2 .$$

It is easy to verify that these f_m^* and g_m^* are as required.

We shall fix these functions f_m^* and g_m^*, $1 \leqq m \leqq q$, as obtained in the above lemma, once and for all.

PROPOSITION 2.2. *The restriction map j from Z to Y is one-to-one and onto. Moreover, $\partial Y = j(\partial Z)$.*

Proof. Let $z \neq z'$ be in Z. As in (i) of Lemma 2.1 find f in U such that $f(z) = 1$, $f(z') = 0$, $f(z_k) = 0$, for $1 \leqq k \leqq q$ and $z \neq z_k$, and $D_k(f) = 0$ for $1 \leqq k \leqq q$. Clearly, this f is in A_q, it belongs to $j(z')$ but not to $j(z)$. Thus j is one-to-one. In order to show that j is onto, it is enough to prove that if N is a maximal ideal of A_q, then the ideal generated by N in U is proper. For this purpose we quote an algebraic result. Let R be a commutative ring with $1 \neq 0$, and S a subring of R containing 1. As an S-module, let R be finitely generated. If I is a proper ideal of $\boldsymbol{S}$, then the ideal generated by I in R is also proper. We now show that U is finitely generated as an A_q-module. If f is a function in U, $f(z_k) = c_k$ and $D_k(f) = d_k$, then

$$f = h + \sum_{k=1}^{q} (\operatorname{Im} c_k)\, i f_k^* + \sum_{k=1}^{q} (\operatorname{Im} d_k)\, i g_k^*,$$

where $h = f - i \sum_{k=1}^{q} (\operatorname{Im} c_k) f_k^* - i\sum_{k=1}^{q} (\operatorname{Im} d_k) g_k^*$. Since $h(z_k) = \operatorname{Re} c_k$, and $D_k(h) = \operatorname{Re} d_k$, h belongs to A_q, and it follows that the functions 1, $i f_m^*$ and $i g_m^*$ generate U as an A_q-module.

Finally, we show that $j(\partial Z) = \partial Y$. Now, $j(\partial Z)$ is a closed subset of Y, and it is a boundary for A_q:

$$\sup_{y \text{ in } Y} |f|(y) \leqq ||f|| = \sup_{z \text{ in } \partial Z} |f(z)| = \sup_{z \text{ in } \partial Z} |f|(j(z)).$$

Since ∂Y is the smallest closed boundary for A_q, it is contained in $j(\partial Z)$. Conversely, to show that ∂Y contains $j(\partial Z)$, note that a point z_0 in Z belongs to ∂Z if and only if for every neighbourhood V of z_0, there is f in U such that the set on which f attains its maximum modulus is contained in V. Let now z_0 be in ∂Z and V a neighbourhood of z_0. We prove that there exists g in A_q which satisfies the above condition. Since j is continuous, it will then follow that $j(z_0)$ belongs to ∂Y. First, assume that $z_0 \neq z_k$, for $1 \leqq k \leqq q$. Since Z is Hausdorff, we can assume without loss of generality that no z_k belongs to V. Let f be in U such that $\max_{z \text{ in } V} |f(z)| = 1$, but $|f(z)| < 1$ for z outside V. Let $f(z_k) = c_k$, $1 \leqq k \leqq q$. Then $|c_k| < 1$ for each k, and hence $(1 - \bar{c}_k f)$ is invertible in U. Define

$$f_k \equiv (f - c_k)/(1 - \bar{c}_k f).$$

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Then f_k is in U, $|f_k(z)| = 1$ if $|f(z)| = 1$, and $|f_k(z)| < 1$ if $|f(z)| < 1$. If we let

$$g \equiv \prod_{k=1}^{q} f_k^{\,2},$$

then $g(z_k) = D_k(g) = 0$ for each k, and we are done. Now let $z_0 = z_1$ say. In this case we can assume that V does not contain any z_k, for $2 \leqq k \leqq q$. First, let $D_1(f) \neq 0$. Then since D_1 is nontrivial and since the Dirichlet deficiency of U is finite, it follows from [**4**, Théorème 2] that the Gleason part P of z_1 is nontrivial. Now, if $|f(z_1)| = 1$, then f is constant on P, and hence V contains P. Since P is nontrivial, there exists a neighbourhood V_1 of z_1 not containing P. Then the function corresponding to $V \cap V_1$ has absolute value less than 1 at z_1. Hence we can assume that $|f(z_1)| = |c_1| < 1$. Thus again the function g constructed above works. Next, let $D_1(f) = 0$. Here, let $g \equiv \exp(-is)g'$, where $g' = f$ if $q = 1$, and $g' = \prod_{k=2}^{q} f_k^{\,2}$, if $q \geqq 2$, and $g'(z_1) = r\exp(is)$.

We thus see that the restriction map j is a homeomorphism of Z onto Y and it maps ∂Z onto ∂Y. Hence we can and shall identify Y and ∂Y with Z and ∂Z respectively. Let B be the complexification of A_q, and X and ∂X its maximal ideal space and its Šilov boundary respectively.

PROPOSITION 2.3. *X is homeomorphic to two copies of Z, pasted together at the real locus of Z* (considered as the maximal ideal space of A_q), *viz.*, $\{z_1, \ldots, z_q\}$.

Proof. For z in Z, let $s(z)$ be the complex homomorphism of B such that $(1 \otimes f)(s(z)) = f(z)$ for each f in A_q. Then s is a continuous section of cx^* over Z. For z in Z, $(cx^*)^{-1}(z) = \{x_0, x_1\}$, where $\tau(x_0) = x_1$. Hence X is the union of $s(Z)$ and $\tau(s(Z))$; and z belongs to the real locus of Z if and only if the inverse image of z consists of a single point of X. It is clear that s is one-to-one, and hence a homeomorphism into X. The result now follows.

Since X is homeomorphic to two copies of Z glued together at certain points and since the values of functions in B on one copy determine their values on the other copy, we can make the following identifications which will be useful in computing the various deficiencies of A_q in the next section. First, $C_R(\partial X)^s$ can be identified with $C_R(\partial Z)$, and $C_R(\partial X)^a$ with

$$C_R^{\,0}(\partial Z) \equiv \{u \text{ in } C_R(\partial Z) : u = 0 \text{ at each } z_k \text{ in } \partial Z,\ 1 \leqq k \leqq q\}.$$

Also, $A_q(\partial X)$ can be identified with $A_q(\partial Z) \equiv \{f \text{ restricted to } \partial Z : f \text{ in } A_q\}$. The following simple result allows us to identify $(A_q)_{-1}(\partial X)$ with $(A_q)_{-1}(\partial Z) \equiv \{f + ig \text{ restricted to } \partial Z : f \text{ and } g \text{ in } A_q,\ f^2 + g^2 = 1\}$: Let A be a real commutative algebra with $1 \neq 0$, and let B be its complexification. Then $b = 1 \otimes f + i \otimes g$ is invertible in B if and only if $f^2 + g^2$ is invertible in A; and if b is in B^{-1}, then $\sigma(b) = b^{-1}$ if and only if $f^2 + g^2 = 1$.

3. Deficiencies of A_q and the Gleason parts. Let the Dirichlet deficiency of U be d and the Arens–Singer deficiency a. We can assume without loss of generality that the first s of the q points $z_1, \ldots, z_q$ belong to ∂Z and the last $q - s$ do not, for some s, $0 \leqq s \leqq q$. Let $D_{k_1}, \ldots, D_{k_p}$ be the only nontrivial ones among the q point derivations $D_1, \ldots, D_q$, for some p, $0 \leqq p \leqq q$. Finally, let the points $z_1, \ldots, z_q$ belong to r different Gleason parts in Z. We shall determine the various deficiencies of A_q in terms of d, a, q, s, p and r. The functions f_k^* and g_k^*, $1 \leqq k \leqq q$ constructed in Lemma 2.1 will turn out to be very useful, as is seen from the following proposition.

PROPOSITION 3.1. (i) $\langle \operatorname{Re} A_q, \operatorname{Im} f_1^*, \ldots, \operatorname{Im} f_q^*, \operatorname{Im} g_{k_1}^*, \ldots, \operatorname{Im} g_{k_p}^* \rangle = \operatorname{Re} U$.
(ii) $\langle \log |A_q^{-1}|, \operatorname{Im} f_1^*, \ldots, \operatorname{Im} f_q^*, \operatorname{Im} g_{k_1}^*, \ldots, \operatorname{Im} g_{k_p}^* \rangle = \langle \log |U^{-1}| \rangle$.
(iii) $\langle \operatorname{Im} A_q, \operatorname{Re} f_{s+1}^*, \ldots, \operatorname{Re} f_q^*, \operatorname{Re} g_{k_1}^*, \ldots, \operatorname{Re} g_{k_p}^* \rangle = \operatorname{Re} U \cap C_R^0(\partial Z)$.
(iv) $\langle \log |(A_q)_{-1}|, \operatorname{Re} f_{s+1}^*, \ldots, \operatorname{Re} f_q^*, \operatorname{Re} g_{k_1}^*, \ldots, \operatorname{Re} g_{k_p}^* \rangle = \langle \log |U^{-1}| \rangle \cap C_R^0(\partial Z)$.

Proof. Let f be in U, $f(z_m) = c_m$ and $D_m(f) = d_m$, $1 \leqq m \leqq q$. Then

$$\left[f - i \sum_{m=1}^{q} (\operatorname{Im} c_m) f_m^* - i \sum_{m=1}^{q} (\operatorname{Im} d_{k_m}) g_{k_m}^* \right]$$

and

$$\left[if - i \sum_{m=1}^{q} (\operatorname{Re} c_m) f_m^* - i \sum_{m=1}^{q} (\operatorname{Re} d_{k_m}) g_{k_m}^* \right]$$

both belong to A_q. From this (i) and (iii) follow by considering the real and imaginary parts.

Now, let f be in U^{-1}, $f(z_m) = r_m \exp(is_m)$ and $D_m(f) = r_m' \exp(is_m')$. Then $f \cdot \exp(-i \sum_{m=1}^{q} s_m f_m^* - i \sum_{m=1}^{p} t_{k_m} g_{k_m}^*)$ where $t_{k_m} = r_{k_m}' \sin(s_{k_m}' - s_{k_m})/r_{k_m}$, belongs to A_q^{-1}. This gives (ii).

As for (iv), let v belong to $\langle \log |U^{-1}| \rangle \cap C_R^0(\partial Z)$. Then, by (ii), there exists f in A_q^{-1} and real numbers $a, a_1, \ldots, a_q, b_{k_1}, \ldots, b_{k_p}$ such that

$$v = a \log |f| + \sum_{m=1}^{q} a_m \operatorname{Im} f_m^* + \sum_{m=1}^{p} b_{k_m} \operatorname{Im} g_{k_m}^*,$$

and $v = 0$ at $z_1, \ldots, z_s$. Moreover, since $\operatorname{Im} f_m^* = \operatorname{Im} g_{k_m}^* = 0$ at $z_1, \ldots, z_s$ for each m, $|f| = 1$ at $z_1, \ldots, z_s$. Let $f(z_m) = r_m \exp(is_m)$ and $D_m(f) = r_m' \exp(is_m')$. Consider now

$$g = f \cdot \exp\left(- \sum_{m=s+1}^{q} (\log r_m) f_m^* - \sum_{m=1}^{p} t_{k_m} g_{k_m}^* \right),$$

where $t_{k_m} = r_{k_m}'/r_{k_m}$. Then g belongs to A_q^{-1}, $g = \pm 1$ at $z_1, \ldots, z_q$, and $D_m(g) = 0$ for each m. Now,

$$ig = [i/2(g - g^{-1})] + i[1/2(g + g^{-1})],$$

where $(i/2)(g - g^{-1})$ and $(1/2)(g + g^{-1})$ both belong to A_q and the sum of their squares is 1. Hence ig belongs to $(A_q)_{-1}$. The rest follows immediately since $\mathrm{Im}\, f_m{}^*$ and $\mathrm{Im}\, g_{k_m}{}^*$ also belong to $\log |(A_q)_{-1}|$, for each m.

We conclude from (i) and (ii) of Proposition 3.1 that the Dirichlet (respectively Arens–Singer) deficiency of A_q is at most $d + q + p$ (respectively $a + q + p$). In order to determine them actually, we need the following lemma.

LEMMA 3.2. *Let P be a Gleason part in Z, z a point in P, and z' a point outside P. Then, for every positive ϵ there exists g in U such that $g(z) = 1$, $g(z') = 0$, $|\mathrm{Re}\, g| \leqq 2$, and $|\mathrm{Im}\, g| < \epsilon$.*

Proof. We know that

$$\sup \{|f(z)| : f \text{ in } U, ||f|| \leqq 1, f(z') = 0\} = 1.$$

By the Riemann mapping theorem, there exists a one-to-one complex analytic function φ on the closed unit disk such that $\varphi(0) = 0$, φ maps $[-1, 1]$ to the reals, $\varphi(1) = 2$, $|\mathrm{Re}\, \varphi| \leqq 2$ and $|\mathrm{Im}\, \varphi| < \epsilon$. Let $\varphi(s) = 1$. Now there exists f in U such that $||f|| \leqq 1$, $f(z') = 0$, and $f(z) = s_1$, for some s_1, $s < s_1 \leqq 1$. Then $1 < \varphi(s_1) = k$, say. Since φ is analytic, it can be approximated by polynomials, and hence $\varphi \circ f$ belongs to U. Then $g = (1/k)\ \varphi \circ f$ has the required properties.

THEOREM 3.3. *The Dirichlet (respectively Arens–Singer) deficiency of A_q is $d + q - r + p$ (respectively $a + p$), where d (respectively a) is the Dirichlet (respectively Arens–Singer) deficiency of U, r is the number of distinct Gleason parts to which the q points $z_1, \ldots, z_q$ belong, and p is the number of non-trivial point derivations among $D_1, \ldots, D_q$.*

Proof. We use the identifications introduced at the end of § 2, and take our starting point as (i) and (ii) of Proposition 3.1. Let $z_1, \ldots, z_t$ belong to a Gleason part P, and $z_{t+1}, \ldots, z_q$ be outside P. We shall show that

(i) $\mathrm{Im}\, f_t{}^*$ belongs to the closure of $\mathrm{Re}\, A_q$, $\mathrm{Im}\, f_m{}^*$, $1 \leqq m \leqq t - 1$, and $\mathrm{Im}\, g_{k_m}{}^*$, $1 \leqq m \leqq p$;

(ii) for any k, $1 \leqq k \leqq t - 1$, $\mathrm{Im}\, f_k{}^*$ does not belong to the closure of $\mathrm{Re}\, A_q$, $\mathrm{Im}\, f_m{}^*$, $1 \leqq m \leqq q$, $m \neq t, k$, and $\mathrm{Im}\, g_{k_m}{}^*$, $1 \leqq m \leqq p$; whereas it does belong to $\langle \log |A_q{}^{-1}| \rangle$; and that

(iii) $\mathrm{Im}\, g_{k_j}{}^*$ does not belong to the closure of $\langle \log |A_q{}^{-1}| \rangle$, and $\mathrm{Im}\, g_{k_m}{}^*$, $m \neq j$, $1 \leqq m \leqq p$.

Since $\mathrm{Re}\, A_q$ is contained in $\log |A_q{}^{-1}|$, (i) shows that corresponding to each of the r Gleason parts to which the q points $z_1, \ldots, z_q$ belong, we can eliminate one of the functions $\mathrm{Im}\, f_1{}^*, \ldots, \mathrm{Im}\, f_q{}^*$; (ii) shows that we can eliminate exactly one such function for the Dirichlet deficiency, whereas we can eliminate all these for the Arens–Singer deficiency; and (iii) shows that we cannot eliminate any of $\mathrm{Im}\, g_{k_1}{}^*, \ldots, \mathrm{Im}\, g_{k_p}{}^*$.

In order to prove (i), we construct a sequence $(f_n)_n$ in U such that $(\mathrm{Im}\, f_n)(z_t)$

$= 1$, $(\operatorname{Im} f_n)(z_m) = 0$ for $t + 1 \leqq m \leqq q$, and $|\operatorname{Re} f_n| \leqq 1/n$ for each n. First, note that the map which sends $(w_{t+1}, \ldots, w_q)$ to $\prod_{m=t+1}^{q} w_m$, where $w_m = x_m + iy_m$ is a complex number, is continuous. Hence, given a positive integer n there exists a positive number ϵ such that if $|x_m| \leqq 2$ and $|y_m| < \epsilon$ for each m, then $|\operatorname{Im}(\prod_{m=t+1}^{q} w_m)| \leqq 1/n$. Now, let $z = z_t$, and $z' = z_m$, $t + 1 \leqq m \leqq q$, in Lemma 3.2, and get functions g_m in U such that $g_m(z_t) = 1$, $g_m(z_m) = 0$, $|\operatorname{Re} g_m| \leqq 2$, and $|\operatorname{Im} g_m| < \epsilon$. Take then $f_n = i \prod_{m=t+1}^{q} g_m$. If we define

$$f_n' = f_n - i\left[f_t^* + \sum_{m=1}^{t-1} (\operatorname{Im} f_n(z_m))f_m^* + \sum_{m=1}^{p} (\operatorname{Im} D_{k_m}(f_n))g_{k_m}^*\right],$$

then f_n' belongs to A_q, and since $(\operatorname{Re} f_n)_n$ tends to zero,

$$\left(\operatorname{Re} f_n' - \sum_{m=1}^{t-1} (\operatorname{Im} f_n(z_m)) \operatorname{Im} f_m^* - \sum_{m=1}^{p} (\operatorname{Im} D_{k_m}(f_n)) \operatorname{Im} g_{k_m}^*\right)_n$$

tends to $\operatorname{Im} f_t^*$. This proves (i).

As for (ii), let $1 \leqq k \leqq t - 1$, and assume for a moment that

$$\operatorname{Im} f_k^* = \lim_n \operatorname{Re} f_n + \sum_{m \neq t,k} t_m \operatorname{Im} f_m^* + \sum_{m=1}^{p} s_m \operatorname{Im} g_{k_m}^*,$$

where f_n is in A_q, and t_m and s_m are real numbers. If we let

$$g \equiv i\left(-f_k^* + \sum_{m \neq t,k} t_m f_m^* + \sum_{m=1}^{p} s_m g_{k_m}^*\right),$$

then $(\operatorname{Re}(f_n - g))_n$ tends to zero. Since $((f_n - g)(z_t))_n$ also tends to zero, and z_k belongs to the same Gleason part as z_t, $((f_n - g)(z_k))_n$ must also tend to zero. But $((f_n - g)(z_k))_n$ tends to i, which is a contradiction. Finally, $\operatorname{Im} f_k^* = (1/2\pi) \log|\exp(-2\pi i f_k^*)|$, which is in $\langle \log |A_q^{-1}|\rangle$. This proves (ii).

As for (iii), let, if possible,

$$\operatorname{Im} g_{k_j}^* = \lim_n u_n + \sum_{m \neq j} s_m \operatorname{Im} g_{k_m}^*,$$

where

$$u_n = \sum_{m=1}^{j_n} t_{n,m} \log |f_{n,m}|$$

for some $f_{n,m}$ in A_q^{-1}, and $t_{n,m}$ and s_m real numbers. If we let

$$g \equiv i\left(-g_{k_j}^* + \sum_{m \neq j} s_m g_{k_m}^*\right),$$

then $(u_n)_n$ tends to $\operatorname{Re} g$.

If D is a continuous point derivation of any uniform algebra U at z, then we show that the map T from $\langle \log |U^{-1}|\rangle$ to $\mathbf{C}$ given by

$$T\left(\sum_{j=1}^{n} a_j \log |f_j|\right) = \sum_{j=1}^{n} a_j \frac{D(f_j)}{f_j(z)}$$

is well-defined. If D is trivial, then so is the map T. If D is nontrivial, then by [**4**, Théorème 1], there exists a representing measure m for z and a function F in $H^\infty(m)$ with $\int F\,dm = 0$ such that $D(f) = \int f\bar{F}\,dm$ for every f in U. From this it follows immediately that $D(f) = 2\int \operatorname{Re} f\,\bar{F}\,dm$ for every f in U. Thus, if $(f_n)_n$ is a sequence in U such that $(\operatorname{Re} f_n)_n$ tends to zero, then so does $(D(f_n))_n$. If $(f_n)_n$ is a sequence in U^{-1} such that $(\log|f_n|)_n$ tends to zero, then by considering $\operatorname{Re}(f_n - f_n^{-1})$ it again follows that $(D(f_n)/f_n(z))_n$ tends to zero. Now let $f_1, \ldots, f_n$ be in U^{-1}, and $a_1, \ldots, a_n$ real numbers such that

$$\sum_{j=1}^{n} a_j \log|f_n| = 0.$$

Then, by Dirichlet's theorem on Diophantine approximation, given a positive integer k, there exists a positive integer q_k such that $q_k a_j$ differs from an integer, say $p_{j,k}$, by less than $1/k$, $j = 1, \ldots, n$. Let $g_k \equiv \prod_{j=1}^{n} f_j^{p_{k,n}}$. Then $(\log|g_k|)_k$ tends to zero, and hence so does $(D(g_k)/g_k(z))_k$. But

$$\left|\sum_{j=1}^{n} a_j \frac{D(f_j)}{f_j(z)}\right| \leqq \frac{1}{k}\sum_{j=1}^{n}\left|\frac{D(f_j)}{f_j(z)}\right| + \left|\frac{D(g_k)}{g_k(z)}\right|.$$

Thus $\sum_{j=1}^{n} a_j D(f_j)/f_j(z) = 0$, and the map T is well defined.

Of course, T, restricted to $\operatorname{Re} U$, is continuous. Now, in the case at hand, the Dirichlet deficiency of U is finite, hence cl. $\operatorname{Re} U$ has finite codimension in cl. $\langle \log|U^{-1}|\rangle$. Hence the map T is actually continuous. Taking $D = D_{kj}$ and $z = z_{kj}$, it now follows that $(\sum_{m=1}^{j_n} t_{n,m} D_{kj}(f_{n,m})/f_{n,m}(z_{kj}))_n$ tends to $D_{kj}(g)$. But each term of this sequence is real since each $f_{n,m}$ is in A_q^{-1}, while $D_{kj}(g) = -i$. This gives the required contradiction.

THEOREM 3.4. *The imaginary Dirichlet (respectively inverse Arens–Singer) deficiency of A_q is $d + q - s + p$ (respectively $a + q - s + p$), where d, a and p are as in Theorem 3.3, and s is the number of points among $z_1, \ldots, z_q$ which belong to the Šilov boundary of U.*

Proof. Recall that we have identified $C_R(\partial X)^a$ with $C_R{}^0(\partial Z)$. Then, (iii) and (iv) of Proposition 3.1 show that the imaginary Dirichlet (respectively inverse Arens–Singer) deficiency of A_q is at most $d + q - s + p$ (respectively $a + q - s + p$). Since $\operatorname{Im} A_q$ is contained in $\log|(A_q)_{-1}|$, we need only show that $\operatorname{Re} f_{s+1}^*, \ldots, \operatorname{Re} f_q^*, \operatorname{Re} g_{k_1}^*, \ldots, \operatorname{Re} g_{k_p}^*$ are linearly independent over cl. $\langle \log|(A_q)_{-1}|\rangle$. Let

$$u = \sum_{m=s+1}^{q} t_m \operatorname{Re} f_m^* + \sum_{m=1}^{p} s_m \operatorname{Re} g_{k_m}^*$$

where $u = \lim_n u_n$, with $u_n = \sum_{m=1}^{j_n} t_{n,m} \log|f_{n,m}|$, and $f_{n,m}$ in $(A_q)_{-1}$.

First, $|f_{n,m}(z_k)| = 1$ for $1 \leqq k \leqq q$. This gives $0 = u(z_k) = t_k$ for $s + 1 \leqq k \leqq q$. Secondly, if we let $g = \sum_{m=1}^{p} s_m g_{k_m}^*$, $(u_n)_n$ tends to $\operatorname{Re} g$, and as in the proof of Theorem 3.4,

$$\left(\sum_{m=1}^{j_n} t_{n,m} \frac{D_{kj}(f_{n,m})}{f_{n,m}(z_{kj})}\right)_n$$

tends to $D_{k_j}(g) = s_j$, for $1 \leqq j \leqq p$. Now we show that if f is in $(A_q)_{-1}$, then the real part of $D_k(f)/f(z_k)$ is zero for each k. Let $f = g + ih$, where g and h are in A_q and $g^2 + h^2 = 1$. Thus,

$$D_k(f)/f(z_k) = [D_k(g) + i\,D_k(h)]/[g(z_k) + ih(z_k)].$$

Since $D_k(g), D_k(h), g(z_k)$ and $h(z_k)$ are all real, Re $[D_k(f)/f(z_k)] = g(z_k)D_k(g) + h(z_k)D_k(h) = \frac{1}{2}D_k(g^2 + h^2) = 0$. This shows that $s_j = 0$ for $1 \leqq j \leqq p$, and we are done.

COROLLARY 3.5. *If k is the Dirichlet (respectively Arens–Singer) deficiency of A_q, then the Dirichlet (respectively Arens–Singer) deficiency of its complexification B is $2k + r - s$ (respectively $2k + q - s$).*

Proof. The result follows from Proposition 1.3 and Theorems 3.3 and 3.4.

Example 3.6. In the case of a standard algebra on a compact bordered nonorientable Klein surface, the Dirichlet deficiency is less than or equal to half the Dirichlet deficiency of its complexification. (See Example 1.4.) In view of Corollary 3.5, we can construct an algebra A_q for which the Dirichlet deficiency is strictly greater than half the Dirichlet deficiency of its complexification. We only have to find a complex uniform algebra with finite Dirichlet deficiency such that one of the Gleason parts in its maximal ideal space contains at least two points of its Šilov boundary. An example of such an algebra is given by the subalgebra of the standard algebra on the unit disk consisting of functions which satisfy $f(1) = f(\frac{1}{2})$ and $f(-1) = f(-\frac{1}{2})$.

Added in proof. The referee has kindly pointed out that Lemma 3.2 and much of the proof of Theorem 3.3 can be essentially found in *Peak points for hypo-Dirichlet algebras*, Proc. Amer. Math. Soc. *26* (1970), 431–436, by S. J. Sidney. (See Lemma 3 and Remark 8.)

REFERENCES

1. N. L. Alling, *Real Banach algebras and nonorientable Klein surfaces. I*, J. Reine Angew. Math. *241* (1970), 200–208.
2. ——— *Analytic and harmonic obstruction on non-orientable Klein surfaces*, Pacific J. Math. *37* (1971), 1–19.
3. N. L. Alling and B. V. Limaye, *Ideal theory on non-orientable Klein surfaces*, Ark. Mat. *10* (1972), 277–292.
4. J. Chaumat, *Dérivations ponctuelles continues dans les algèbres de fonctions*, C. R. Acad. Sc. Paris Ser. A-B *269* (1969), 347–350.
5. T. W. Gamelin, *Uniform algebras* (Prentice-Hall, Englewood Cliffs, N.J. 1969).
6. L. Ingelstam, *Real Banach algebras*, Ark. Mat. *5* (1965), 239–270.
7. L. H. Loomis, *An introduction to abstract harmonic analysis* (Van Nostrand, Princeton, N.J., 1953).
8. C. E. Rickart, *General theory of Banach algebras* (Van Nostrand, Princeton, N.J., 1960).
9. J. Wermer, *Analytic disks in maximal ideal spaces*, Amer. J. Math. *86* (1964), 161–170.

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