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VALUATION RINGS AND INTEGRAL CLOSURE

BY

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ABSTRACT. A famous theorem of Krull's is that the integral closure of an integral domain D is the intersection of the valuation domains that contain D. An example is given to show that the same result need not hold for the integral closure of a ring with zero divisors.

In what follows, all rings are assumed to be commutative with multiplicative identity $1 \neq 0$. For ring R, the *total quotient ring* of R is the ring $T(R)$ consisting of all fractions of the form r/s where $r,\ s \in R$ and s is *regular*; i.e. $as = 0$ implies $a = 0$. As in forming the quotient field of an integral domain, we identify r with $r/1$ and set r/s equal to u/v whenever $ru = sv$.

For a pair of rings $R \subset T$, an element $t \in T$ is said to be *integral* over R if there is a monic polynomial f with coefficients in R such that $f(t) = 0$. The set of all such elements of T forms a ring R' which is referred to as the *integral closure* of R in T.

1. **Valuations**. In the classical setting a valuation domain is a ring $V = \{t \in K \mid v(t) \geqq 0\}$ where v is a map from a field K to a totally ordered abelian group G (together with symbol ∞) such that for all $x,\ y \in K$

$$v(xy) = v(x) + v(y), \tag{1}$$

$$v(x + y) \geqq \min\{v(x),\ v(y)\}, \tag{2}$$

$$v(1) = 0,\ v(0) = \infty. \tag{3}$$

It is a straightforward exercise to show $\mathrm{Im}(v)/\{\infty\}$ is a group so we may assume v is surjective.

The definition above can be extended to arbitrary rings by setting $V = \{t \in T \mid v(t) \geqq 0\}$ where v is a map from a ring T to $G \cup \{\infty\}$ such that v satisfies properties (1) - (3) above. Unlike the case for fields, $\mathrm{Im}(v)/\{\infty\}$ need not form a group. In the terminology of [5] we say that in general v is a *paravaluation* on T and V a *paravaluation ring* of T. In the event v is surjective (or more precisely, $\mathrm{Im}(v)/\{\infty\}$ is a group), v is said to be a *valuation* on T and V a *valuation ring* of T. When $T = T(V)$, we drop the reference to T. With these definitions, if we take G to be the

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trivial group $G = \{0\}$, a trivial valuation can be put on any ring R. Specifically, let P be a (fixed) prime ideal of R and set $v(x) = \infty$ if $x \in P$ and $v(x) = 0$ if not. With this valuation, R is a valuation ring of itself. In particular any total quotient ring is a valuation ring.

Using properties (1) and (2), it is straightforward to show that if V is a paravaluation ring of T, then V is integrally closed in T. Hence an intersection of such rings is also integrally closed.

In 1932, W. Krull [6] proved that for an integral domain R with quotient field $K = qf(R)$, the integral closure of R is the intersection of the valuation domains (of K) which contain R. In [8, Théoreme 8], P. Samuel showed that for a pair of rings $R \subset T$, the integral closure of R in T is the intersection of (what he called) the dominated polynomial rings of T which contain R. Later, M. Griffin proved that Samuel's dominated polynomial rings are what we call paravaluation rings [2, Proposition 2]. Hence, Samuel's result can be restated in terms of paravaluation rings of T. J. Huckaba [H1] gave a different proof of this result in the case $T = T(R)$ and more recently J. Gräter [1] has given a new and shorter proof for arbitrary T.

Also in [8], Samuel provided the following example. Let $R = K$ be a field and $T = K[X]$ be the polynomial ring in one indeterminate over K. Then certainly R is integrally closed in T and by setting $v(f) = -\deg f$ for each nonzero polynomial f we have a paravaluation on T with corresponding paravaluation ring R. However, as there are no integrally closed rings strictly contained between R and T, R is not an intersection of valuation rings of T. Hence Krull's result does not hold for arbitrary pairs of rings $R \subset T$. An open question has been whether Krull's result might hold under the assumption that $T = T(R)$ is the total quotient ring of R (see [5, p. 82]). In Example 3 we shall show that even in this setting, Samuel's result is still the best possible without further restrictions on R. In so doing we reinforce a statement made by Griffin [2, p. 34] that "... little is gained in terms of good behavior by restricting the study of valuations to valuations of total quotient rings."

2. **The Example**. To construct our example we make use of a generalization of the $A + B$ construction as found in [2, §8] and [5, §§26, 27]. Before presenting the example we describe the construction and provide two general results.

Let D be a domain and $\mathcal{P}$ be a set of prime ideals of D such that $\cap_{P_\alpha \in \mathcal{P}} P_\alpha = (0)$. Index $\mathcal{P}$ by $\mathcal{A}$ and let $I = \mathcal{A} \times N$ where N is the set of natural numbers. Let B be the D-module $B = \Sigma K_i$ where for each $i = (\alpha, n)$ $K_i = qf(D/P_\alpha)$. Finally, let R be the direct sum $R = D \oplus B$ with multiplication defined by $(r, b,)(s, c) = (rs, rc + sb + bc)$. Here we are viewing B both as a D-module and as a ring without unit. In this way it is routine to verify that $T(R) = D_S \oplus B$ where $S + D/\cup P_\alpha$.

Our first result describes the valuation rings of $T(R)$ which contain R.

Proposition 1. *(cf. [5, Theorem 26.5]). With $\mathcal{R}$ and $\mathcal{D}$ as above, $\mathcal{V}$ is a valuation ring containing $\mathcal{R}$ if and only if $\mathcal{V} = \mathcal{W} \oplus \mathcal{B}$ where $\mathcal{W}$ is a valuation ring of $\mathcal{D}_S$ containing $\mathcal{D}$.*

PROOF. Let W be a valuation ring of D_S containing D corresponding to a valuation $w : D_S \longrightarrow G \cup \{\infty\}$. Define $v : T(R) \longrightarrow G \cup \{\infty\}$ by setting $v((r, b,)) = w(r)$. It is routine to verify that v is a valuation on $T(R)$ and that $V = W \oplus B$ contains R.

Let V be a valuation ring of $T(R)$ containing R and let $v : T(R) \longrightarrow G \cup \{\infty\}$ be the corresponding valuation. If $G = \{0\}$, then $V = T(R)$ so we set $W = D_S$. Hence we assume that $G \neq \{0\}$. By arguing as in [5, Theorem 26.5] we have that $v((0, b)) = \infty$ for all $b \in B$ since $B \subset R \subset V$. Hence by setting $w(r) = v((r, 0))$ we obtain a valuation on D_S such that $V = W \oplus B$.

With this we have the following useful corollary.

COROLLARY 2. *With $\mathcal{R}$ and $\mathcal{D}$ as above, the integral closure of $\mathcal{R}$ in $\mathcal{T}(\mathcal{R})$ is an intersection of valuation rings (of $\mathcal{T}(\mathcal{R})$) if and only if the integral closure of $\mathcal{D}$ in $\mathcal{D}_S$ is an intersection of valuation rings of $\mathcal{D}_S$.*

With the corollary we are ready to construct our example of a (reduced) integrally closed ring R which is not the intersection of the valuation rings which contain it.

EXAMPLE 3. (cf. [3, Lemma 8]). Let $D = K[X, XY, XY^2, \ldots]$ and let $\mathcal{P}$ be the set of primes which do not contain X. Form the ring $R = D \oplus B$.

The proof that R is integrally closed and not the intersection of the valuation rings which contain it will be established in a series of claims.

CLAIM 1. $S = K \cup \{X, X^2, X^3, \ldots\}$ so that $D_S = K[X, 1/X, Y]$.

If $f(X, Y)$ is a polynomial with constant term nonzero, then obviously f is in a prime ideal Q of $K[X, Y]$ which does not contain X. Then $Q \cap D$ is a prime of D missing X. Now if $f(X, Y)$ is not a power of X, it has a highest common power of X. Factor out this power to get $f(X, Y) = X^n g(X, Y)$. As before, take a prime Q of $K[X, Y]$ containing g but not X. Then $Q \cap D$ is a prime of D missing X but containing f. Hence $S = K \cup \{X, X^2, X^3, \ldots\}$ and $D_S = K[X, 1/X, Y]$.

CLAIM 2. R is paravaluation ring of $T(R)$.

Let $G = Z \times Z$ be the cartesian product of the integers with themselves in the lexicographic order. Define $w(X) = (1, 0)$, $w(Y) = (0, -1)$, $w(k) = 0$ for $k \in K/\{0\}$ and extend to D_S using properties (1) and (2) of a paravaluation. Then it is easy to see that w is a paravaluation on D_S and not a valuation. Moreover, D is the corresponding paravaluation ring. As in Proposition 1, we extend w to $T(R)$ by defining $v((r, b)) = w(r)$. In so doing R becomes a paravaluation ring of $T(R)$.

CLAIM 3. Every valuation ring of $T(R)$ containing R contains $(Y, 0)$.

Noting Corollary 2, it suffices to show that every valuation ring of D_S which contains D contains $K[X, Y]$. Let W be a paravaluation ring of D_S such that $Y \notin W$. We will show that W is not a valuation ring.

Let w be the valuation corresponding to W. Since $K \subset R$, $w(a) = 0$ for all $a \in K/\{0\}$. Hence, using the properties of a paravaluation, it is a straightforward

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exercise to show that w is completely determined by the values of $w(X)$ and $w(Y)$. As $Y \notin W$, $w(Y) < 0$. However, since $XY^n \in D$ for all n, $w(XY^n) > 0$. In other words, $0 < -nw(Y) < w(X)$ for all $n > 0$. Since $1/Y \notin D_S$, there can be no $f \in W$ such that $w(f) = -w(Y)$ and so W is a paravaluation ring which is not a valuation. In fact, by mapping $w(Y)$ to $(0, -1)$ and $w(X)$ to $(1, 0)$ it is not hard to show that the value group of w is isomorphic to $Z \times Z$.

REMARK. Krull's result is known to hold for some types of rings. For example, if R is noetherian, then Krull's theorem holds for R and every ring S between R and $T(R)$. A proof of this is available through a more general result of J. Marot. In [7], Marot introduced the concept of what is now known as a Marot ring [5, p. 31]. A ring R is a Marot ring if every regular ideal of R is generated by its set of regular elements. A regular ideal is simply one which contains a regular element. Marot proved that in the event R is a Marot ring, then the integral closure of R is the intersection of the valuation rings that contain R. This result is a consequence of the facts that any overring of a Marot ring is also Marot and a paravaluation ring which is also a Marot ring is a valuation ring. Proofs of these results can be found in [5, §7].

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