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On Lie algebra obstructions

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A basic problem in the theory of Lie algebra extensions concerns a given homomorphism χ of a Lie algebra L into the Lie algebra of outer derivations of a Lie algebra B. In analogy with the theory of group extensions, Mori and Hochschild developed the concept of an obstruction to χ being the homomorphism defined by some Lie algebra extension of B by L. This note considers an alternative approach to this theory, which is particularly simple when applied to the problem of realizing arbitrary three-cohomology classes of L as obstructions. The approach is analogous to one for groups, which was given recently by Gruenberg.

The theory of extensions of linear algebras (see for example [4] - [7]) contains a number of basic theorems all of which are analogous to results in the theory of group extensions (see for example [2]). In particular, one of the basic problems in the case of Lie algebras concerns a given homomorphism $\chi : L \to \mathrm{Der}B/\mu B$ of a Lie algebra L into the Lie algebra of outer derivations of a Lie algebra B. In analogy with the theory of groups, Mori [10] and Hochschild [6] developed a theory of *obstructions* to χ being the homomorphisms arising from some Lie algebra extension $0 \to B \to E \to L \to 0$. This note considers an alternative approach to this theory for Lie algebras, which is particularly simple when applied to the problem of realizing 3-cohomology classes as obstructions.

This approach is the direct analogue of one given recently by Gruenberg [3] in the case of groups. Gruenberg's method is based on his "resolution by relations" [2] for the cohomology of groups. The present note

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uses a similar resolution for Lie algebras, which has been considered in a different connection by this author in [8].

1. The resolution for Lie algebras

Consider Lie algebras over a given commutative ground ring Λ ; let $U(L)$ denote the universal enveloping algebra of a Lie algebra L . We consider throughout a Lie algebra L which is free as a Λ-module, and which has a presentation

$$0 \to R \to F \overset{\pi}{\to} L \to 0$$

in which *both* F and R are free Lie algebras. (If Λ is a field, the Širšov-Witt theorem implies that every Lie algebra over Λ has such properties for *all* free presentations.)

The following facts appear as special cases in [8]: Let $\pi_* : U(F) \to U(L)$ be the induced epimorphism of enveloping algebras, and let $\underline{\underline{r}} = \text{Ker } \pi_*$. If $\underline{\underline{f}}$ denotes the augmentation ideal of $U(F)$, let

$$X_{2n} = \underline{\underline{r}}^n/\underline{\underline{r}}^{n+1} , \quad X_{2n-1} = \underline{\underline{r}}^{n-1}\underline{\underline{f}}/\underline{\underline{r}}^n\underline{\underline{f}} , \quad n > 0 , \quad \text{where} \quad \underline{\underline{r}}^0 = U(F) .$$

Consider the sequence

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to U(L) \to \Lambda \to 0$$

obtained by letting $U(L) \to \Lambda$ be the augmentation mapping, $X_1 \to U(L)$ be the map induced by π_* , and $X_n \to X_{n-1}$ $(n > 1)$ be induced by inclusion. Then this sequence is a *free resolution* of Λ over $U(L)$. Further, if X and Y are sets of free Lie algebra generators for F and R respectively, then Y is a free basis for $\underline{\underline{r}}$ as left $U(F)$-module, and a free $U(L)$-basis Y_n for X_n is obtained by letting

$$Y_{2n} = \left\{y_1 \cdots y_n + \underline{\underline{r}}^{n+1} : y_i \in Y\right\} ,$$

and

$$Y_{2n-1} = \left\{y_1 \cdots y_{n-1}\, x + \underline{\underline{r}}^n \underline{\underline{f}} : y_i \in Y , \quad x \in X\right\} .$$

Now let A be any L-module. Then, for the purpose of the obstruction theory, we note in particular that $H^3(L;A) \cong \text{Ker } i_3^*/\text{Im } i_2^*$ where i_2^* , i_3^* are the following maps induced by inclusion:

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$$\mathrm{Hom}_{U(L)}(\underline{\underline{r}}/\underline{\underline{r}}^2, A) \xrightarrow{i_2^*} \mathrm{Hom}_{U(L)}(\underline{\underline{rf}}/\underline{\underline{r}}^2\underline{\underline{f}}, A) \xrightarrow{i_3^*} \mathrm{Hom}_{U(L)}(\underline{\underline{r}}^2/\underline{\underline{r}}^3, A) \ .$$

The above notation will be referred to later.

2. Definition of obstruction

Let $\chi : L \to \mathrm{Der}\, B/\mu B$ be a homomorphism of L into the Lie algebra $\mathrm{Der}B$ of derivations of a Lie algebra B, modulo the ideal μB of inner derivations of B, and let the *centre* K_B of B be given the L-module structure induced by χ. Since both F and R are free Lie algebras, there exists a commutative diagram of algebra homomorphisms:

$$\begin{array}{ccccccccc}
K_B & \longrightarrow & B & \xrightarrow{\mu} & \mathrm{Der}B & \longrightarrow & \mathrm{Der}B/\mu B & \longrightarrow & 0 \\
 & & \eta\uparrow & & \xi\uparrow & & \chi\uparrow & & \\
0 & \longrightarrow & R & \longrightarrow & F & \xrightarrow{\pi} & L & \longrightarrow & 0 \ ,
\end{array}$$

in which $\mu_b = -\,\mathrm{ad}\, b \quad (b \in B)$.

If $r \in R$, $w \in F$, one checks that the element $r \circ w = \eta_{[rw]} + \xi_w(\eta_r)$ lies in K_B . Hence one obtains a $U(L)$-linear map

$$\phi : \underline{\underline{rf}}/\underline{\underline{r}}^2\underline{\underline{f}} \to K_B$$

by letting $\phi(yx + \underline{\underline{r}}^2\underline{\underline{f}}) = y \circ x$ for $y \in Y$, $x \in X$.

THEOREM 2.1. *The mapping* ϕ *is an element of* $\mathrm{Ker}\ i_3^*$ *, whose cohomology class* $\Phi = \Phi(\chi)$ *in* $H^3(L;K_B)$ *is independent of the choice of homomorphisms* ξ *,* η *. (*ϕ *will be called the obstruction determined by* ξ *,* η *, and* Φ *the obstruction class of* χ *.)*

Proof. Firstly it is easy to check two formulae:

(1) $\qquad [r_1 r_2] \circ w = 0 \qquad (r_1, r_2 \in R \ ; \ \ w \in F)$,

(2) $\qquad r \circ [w_1 w_2] = [r w_1] \circ w_2 - [r w_2] \circ w_1 + \xi_{w_1}(r \circ w_2) - \xi_{w_2}(r \circ w_1)$

$\qquad\qquad (r \in R \ ; \ \ w_1, w_2 \in F)$.

These formulae will now be used to show that

(3) $\qquad \phi(rw + \underline{\underline{r}}^2\underline{\underline{f}}) = r \circ w \qquad (r \in R \ , \ \ w \in F)$.

In order to do this, let $r \in R$ and suppose

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$$r = \sum \lambda_i \, y_i + \text{higher terms in the } y_j \quad (\lambda_i \in \Lambda \, ; \, y_i, y_j \in Y) \, .$$

Then, by (1), $r \circ w = \sum \lambda_i \, (y_i \circ w) \quad (w \in F)$. Also by expanding the higher terms of r as associative polynomials, one sees that $r \equiv \sum \lambda_i \, y_i \pmod{\underline{\underline{r}}^2}$. Hence

$$\phi(rx + \underline{\underline{r}}^2\underline{\underline{f}}) = r \circ x \qquad (x \in X) \, . \tag{4}$$

The equations (3) will then follow by linearity and induction on the degree of Lie monomials when it has been shown that

$$\phi(r[w_1 w_2] + \underline{\underline{r}}^2\underline{\underline{f}}) = r \circ [w_1 w_2] \qquad (w_1, w_2 \in F) \, . \tag{5}$$

For this purpose, suppose that $\phi(sw_i + \underline{\underline{r}}^2\underline{\underline{f}}) = s \circ w_i$ for all $s \in R$ and certain elements $w_1, w_2 \in F$. Then

$$r[w_1 w_2] = [rw_1]w_2 - [rw_2]w_1 + w_1 r w_2 - w_2 r w_1 \, ,$$

and so, by the assumption on w_1, w_2 , equation (5) follows from (2), since ϕ is $U(L)$-linear.

By equation (3), it follows that

$$\phi(rs + \underline{\underline{r}}^2\underline{\underline{f}}) = r \circ s = 0 \qquad (r, s \in R) \, . \tag{6}$$

Since the products $rs \quad (r, s \in R)$ span $\underline{\underline{r}}^2$ over $U(F)$, it follows that

$$\phi(\underline{\underline{r}}^2/\underline{\underline{r}}^2\underline{\underline{f}}) = \{0\} \, , \quad \text{i.e.} \quad \phi \in \operatorname{Ker} i_3^* \, .$$

In order to see that the cohomology class $\Phi = \Phi(\chi)$ of ϕ in $H^3(L; K_B)$ is independent of the original homomorphisms ξ , η , first consider another homomorphism $\eta' : R \to B$ such that $\mu\eta' = \xi | R = \mu\eta$. Then $\eta' - \eta$ maps into K_B , and one may define a $U(L)$-linear map $\psi : \underline{\underline{r}}/\underline{\underline{r}}^2 \to K_B$ by letting $\psi(y + \underline{\underline{r}}^2) = \eta'_y - \eta_y \quad (y \in Y)$. It will now be shown that

$$\psi(s + \underline{\underline{r}}^2) = \eta'_s - \eta_s \qquad (s \in R) \, . \tag{7}$$

This follows since, as with equation (4), if $s = \sum \lambda_i \, y_i$ + higher terms $(\lambda_i \in \Lambda \, , \, y_i \in Y)$ then

$$\psi(s + \underline{\underline{r}}^2) = \sum \lambda_i \left(\eta'_{y_i} - \eta_{y_i} \right) = \eta'_s - \eta_s , \tag{8}$$

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because (easily)

(9) $$\eta'_{[r_1r_2]} - \eta_{[r_1r_2]} = 0 \qquad (r_1, r_2 \in R) .$$

It follows that

(10) $$\psi(rw+\underline{\underline{r}}^2) = \eta'_{[rw]} - \eta_{[rw]} + \xi_w (\eta'_r - \eta_r) \qquad (r \in R , w \in F) ,$$

since ψ is $U(L)$-linear. If ϕ' is the obstruction determined by ξ and η' , one then gets

$$\phi'(rw+\underline{\underline{r}}^2\underline{\underline{f}}) = \eta'_{[rw]} + \xi_w (\eta'_r) = \phi(rw+\underline{\underline{r}}^2\underline{\underline{f}}) + \psi(rw+\underline{\underline{r}}^2) ,$$

i.e. $\phi' = \phi + i_2^*\psi$. (Given *any* $\psi' \in \mathrm{Hom}_{U(L)} (\underline{\underline{r}}/\underline{\underline{r}}^2, K_B)$, the map $\phi + i_2^*\psi'$ can be realized as the obstruction determined by ξ and η'' where $\eta''_y = \eta_y + \psi'(y+\underline{\underline{r}}^2)$ for $y \in Y$.)

Now let $\xi' : F \to \mathrm{Der}B$ be any other homomorphism lifting $\chi : L \to \mathrm{Der}\ B/\mu B$. Then $\xi' - \xi$ maps into μB . Choosing $\delta_x \in B$ such that $\xi'_x - \xi_x = \mu(\delta_x)$ $(x \in X)$, observe that the map $x \to \delta_x$ may be extended to an "extended derivation" $\delta : F \to B$, i.e. a Λ-linear map such that

(11) $$\delta_{[w_1w_2]} = \xi_{w_1} (\delta_{w_2}) - \xi_{w_2} (\delta_{w_1}) + \left|\delta_{w_1}, \delta_{w_2}\right| \qquad (w_1, w_2 \in F) .$$

(Maps of this type have been considered in [7] and [9], for example. In order to construct δ , one may, for example, form the split extension S of B by F corresponding to $\xi : F \to \mathrm{Der}\ B$. Then let $\theta : F \to S$ be the homomorphism such that $\theta_x = (\delta_x, x)$ for $x \in X$. By [9], Proposition 3.3, or directly, one obtains an extended derivation δ by letting $\delta_w = \theta_w - (0, w)$ $(w \in F)$.)

Letting $\eta^* = \eta + \delta|R$, one then obtains an algebra homomorphism $R \to B$ such that $\mu\eta^* = \xi'|R$. The theorem will follow when it has been shown that ξ' and η^* determine the same obstruction ϕ as ξ and η .

In order to do this, we first show that

(12) $$\xi'_w - \xi_w = \mu(\delta_w) \qquad (w \in F) .$$

For this purpose, suppose that (12) holds for certain elements $w_1, w_2 \in F$.

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Then

(13) $$\delta_{[w_1w_2]} = \xi_{w_1}(\delta_{w_2}) - \xi'_{w_2}(\delta_{w_1}) ,$$

and so, since $\mu(D_b) = [D,\mu(b)]$ for $D \in \mathrm{Der}\, B$, $b \in B$,

$$\mu\left(\delta_{[w_1w_2]}\right) = [\xi_{w_1},\mu(\delta_{w_2})] - [\xi'_{w_2},\mu(\delta_{w_1})]$$
$$= [\xi'_{w_1},\xi'_{w_2}] - [\xi_{w_1},\xi_{w_2}] .$$

Thus (12) holds for $[w_1w_2]$, and hence it follows for all $w \in F$, by linearity and induction on the degree of Lie monomials. Finally, with the aid of (11) one obtains

(14) $$\eta^*_{[rw]} + \xi'_w(\eta^*_r) = \eta_{[rw]} + \xi_w(\eta_r) = row \qquad (r \in R , w \in F) .$$

THEOREM 2.2. *There exists an extension*

$$0 \to B \to E \to L \to 0$$

inducing the homomorphism χ *if and only if the obstruction class*

$$\Phi(\chi) = 0 \text{ in } H^3(L;K_B) .$$

Proof. If such an extension exists, then there exists a commutative diagram of algebra homomorphisms:

$$\begin{array}{ccccccccc} 0 & \to & B & \to & E & \to & L & \to & 0 \\ & & \eta\uparrow & & \theta\uparrow & & \| & & \\ 0 & \to & R & \to & F & \xrightarrow{\pi} & L & \to & 0 . \end{array}$$

Define $\xi = \bar{\mu}\,\theta : F \to \mathrm{Der}\, B$ where $\bar{\mu} : E \to \mathrm{Der}\, B$ is the algebra homomorphism $e \to \mu_e|B$ $(e \in E)$. Then ξ lifts χ and $\mu\eta = \xi|R$, and

(15) $$\eta_{[rw]} + \xi_w(\eta_r) = 0 \qquad (r \in R , w \in F) .$$

Thus ξ and η determine the zero obstruction map.

Conversely, suppose that $\Phi(\chi) = 0$ in $H^3(L;K_B)$. Then, for any choice of homomorphisms ξ , η , the corresponding obstruction map $\phi \in \mathrm{Im}\, i_2^*$. By a remark following equation (10) above, there exists a homomorphism $\eta' : R \to B$ such that ξ and η' determine the zero obstruction map. This implies that

(16) $$\eta'_{[rw]} + \xi_w(\eta'_r) = 0 \qquad (r \in R , w \in F) .$$

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Now form the split extension S of B by F via ξ, and let M be the ideal of S consisting of all elements of the form $(\eta'_r, -r)$ $(r \in R)$. Then the monomorphism $B \to S$ and the epimorphism $\pi : F \to L$ induce an extension $0 \to B \to S/M \to L \to 0$. Further, this extension gives back the original homomorphism $\chi : L \to \text{Der } B/\mu B$.

3. Realization of cohomology classes

THEOREM 3.1. *Let* Λ *be a field, and* A *be any* L*-module. Then every element of* $H^3(L;A)$ *can be represented as the obstruction class of some homomorphism* $\chi : L \to \text{Der } B/\mu B$ *such that the centre* K_B *of* B *is* L*-isomorphic to* A.

Proof. If Λ is a field, it is always possible to choose a free presentation $0 \to R \to F \overset{\pi}{\to} L \to 0$ in which R is a free Lie algebra of rank > 1. Hence R has centre $\{0\}$, and so the direct sum $B = R \oplus A$ has centre A, if A is regarded as a zero algebra.

Now represent $H^3(L;A)$ as $\text{Ker } i_3^*/\text{Im } i_2^*$, as before, and consider an arbitrary map $\phi \in \text{Ker } i_3^*$. Define an algebra homomorphism $\xi : F \to \text{Der } B$ by letting

(17) $\xi_w(r,a) = ([wr], [\pi_w\, a] + \phi(rw + \underline{\underline{r}}^2\underline{\underline{f}}))$ $(w \in F\,,\ r \in R\,,\ a \in A)$,

where $[\pi_w a]$ denotes a operated on by π_w under the L-module structure for A. Then $\xi_w - \xi_{w+r} \in \mu B$ for $w \in F$, $r \in R$, and so ξ determines a homomorphism $\chi : L \to \text{Der} B/\mu B$. If $\eta : R \to B$ is the inclusion map, then $\mu\eta = \xi|R$ and

(18) $\eta_{[rw]} + \xi_w(\eta_r) = (0, \phi(rw + \underline{\underline{r}}^2\underline{\underline{f}}))$ $(r \in R\,,\ w \in F)$.

Therefore ϕ is the obstruction determined by ξ and η.

REMARKS. Theorem 3.1 completes possibly the most interesting part of the obstruction theory. The results can, of course, be refined so as to yield an isomorphism of $H^3(L;A)$ with a vector space of similarity classes of "kernels" $\chi : L \to \text{Der } B/\mu B$ such that $K_B \cong A$ (cf. [10], [6]). One can also examine the 'naturality' of the definition of obstruction, relative to any presentation morphism:

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$$\begin{array}{ccccccccc} 0 & \to & R & \to & F & \to & L & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & R' & \to & F' & \to & L' & \to & 0\ . \end{array}$$

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