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# Nonlinear Multipoint Boundary Value Problems for Second Order Differential Equations

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*Abstract.* In this paper we shall discuss nonlinear multipoint boundary value problems for second order differential equations when deviating arguments depend on the unknown solution. Sufficient conditions under which such problems have extremal and quasi-solutions are given. The problem of when a unique solution exists is also investigated. To obtain existence results, a monotone iterative technique is used. Two examples are added to verify theoretical results.

## 1 Introduction

Let points $t_i$ for $i = 0, 1, \ldots, r$ be given and $0 < t_1 < t_2 < \cdots < t_r \le T$. Let $y_0, z_0 \in C^2(J, \mathbb{R})$ and $z_0(t) \le y_0(t), t \in J$. Put

$$\Omega = \{(t, w) \in J \times \mathbb{R} : \ z_0(t) \le w \le y_0(t), t \in J\}.$$

Consider the multipoint boundary value problem

$$\text{(1.1)} \qquad \begin{cases} x''(t) = f\big(t, x\big(\beta(t, x(t))\big)\big) \equiv F(x, x)(t), \quad t \in J = [0, T], \\ x'(0) = k, \\ \quad 0 = g(x(0), x(t_1), \ldots, x(t_r)), \end{cases}$$

where

$$\text{(1.2)} \qquad F(x, y)(t) = f\big(t, x\big(\beta(t, y(t))\big)\big).$$

We assume that

(H$_1$) $f \in C(J \times \mathbb{R}, \mathbb{R}), g \in C(\mathbb{R}^{r+1}, \mathbb{R}), \beta \in C(\Omega, J)$.

In order to obtain existence results for differential equations, one may apply the monotone iterative method (see [19] for details). This technique can be applied successfully to boundary value problems for both first and second order differential equations with deviating arguments (see, for example [2–4, 11–15, 17, 18, 20, 22]). Note that in all the above mentioned papers a deviating argument $\beta$ depends only

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on $t$, so $\beta(t,x) = \bar{\beta}(t)$ on $J$ and a one-sided Lipschitz condition is assumed on $f$ with respect to the last argument with constant (see [2–4, 10–20, 22]) or functional coefficients (see [11–15]). See also [1, 5–9, 21, 23–25]. In this paper we are interested in finding sufficient conditions which guarantee the existence of a solution $x$ of problem (1.1) in a more general case than in the above mentioned papers, namely when the deviating argument $\beta$ depends also on the unknown solution $x$. Let us recall also [16], where the first order problem was investigated for $g(u, v_1, \dots, v_r) = -u+\lambda v_r+k$ with $t_r = T$. We extend the application of the monotone iterative technique to such general cases for the second order differential equations with nonlinear multipoint boundary conditions. In this paper, we also discuss problems of type (1.1) when we have more arguments $\beta$. Two examples are added to verify theoretical results.

## 2 Extremal Solutions of Problem (1.1)

We say that $y_0 \in C^2(J, \mathbb{R})$ is a *lower solution* of (1.1) if

$$\begin{cases} y_0''(t) \geq F(y_0, y_0)(t), \; t \in J, \\ y_0'(0) \geq k, \\ \qquad 0 \geq g(y_0(0), y_0(t_1), \dots, y_0(t_r)). \end{cases}$$

We say that $y_0 \in C^2(J, \mathbb{R})$ is an *upper solution* of (1.1) if the above inequalities are reversed. Indeed, $F$ is defined by (1.2).

A solution $y \in C^2(J, \mathbb{R})$ of problem (1.1) is called *maximal* if $x(t) \leq y(t)$, $t \in J$ for each solution $x$ of (1.1), and *minimal* if the reverse inequality holds. If both minimal and maximal solutions exist, we call them *extremal* solutions of (1.1).

If we know the existence of lower and upper solutions $y_0, z_0$ of problem (1.1) such that $z_0(t) \leq y_0(t)$, $t \in J$, then under corresponding conditions we can prove the existence of the extremal solutions of (1.1) in the sector

$$[z_0, y_0]_* = \{w \in C^2(J, \mathbb{R}) : z_0(t) \leq w(t) \leq y_0(t), \; t \in J\}.$$

It is the content of the following.

***Theorem 2.1*** *Suppose that assumption* $(\mathrm{H}_1)$ *holds and in addition assume that*

($\mathrm{H}_2$) *$f$ is nondecreasing with respect to the last argument and $k \geq 0$;*
($\mathrm{H}_3$) *$y_0$ and $z_0$ are lower and upper solutions of problem* (1.1), *respectively and $z_0(t) \leq y_0(t)$, $t \in J$;*
($\mathrm{H}_4$) *$\beta(t,u)$ is nondecreasing with respect to $u$;*
($\mathrm{H}_5$) *$y_0, z_0$ are nondecreasing and $f(t,u) \geq 0$ for $t \in J$ and $z_0(\beta(t, z_0(t))) \leq u \leq y_0(\beta(t, y_0(t)))$;*
($\mathrm{H}_6$) *$g$ is nondecreasing with respect to the last $r$ variables and*

$$g(u, v_1, \dots, v_r) - g(\bar{u}, v_1, \dots, v_r) \leq \bar{u} - u \tag{2.1}$$

*for $z_0(0) \leq u \leq \bar{u} \leq y_0(0)$, $z_0(t_i) \leq v_i \leq y_0(t_i)$, $i = 1, 2, \dots, r$.*

*Then problem* (1.1) *has minimal and maximal solutions in the sector* $[z_0, y_0]_*$.

**Proof** The method of proof is based on the construction of sequences $\{y_n, z_n\}$ of approximate solutions defined by

$$\begin{cases} y''_{n+1}(t) = F(y_n, y_n)(t), \ t \in J, \\ y'_{n+1}(0) = k, \\ y_{n+1}(0) = y_n(0) + g(y_n(0), y_n(t_1), \dots, y_n(t_r)), \end{cases}$$

$$\begin{cases} z''_{n+1}(t) = F(z_n, z_n)(t), \ t \in J, \\ z'_{n+1}(0) = k, \\ z_{n+1}(0) = z_n(0) + g(z_n(0), z_n(t_1), \dots, z_n(t_r)). \end{cases}$$

We first observe that elements $y_1, z_1$ are well defined as the unique solution of the corresponding problems. Note that if we put $p = z_0 - z_1$, then we see that

$$\begin{aligned} &p''(t) \le F(z_0, z_0)(t) - F(z_0, z_0)(t) = 0, \ t \in J, \\ &p'(0) \le 0, \\ &p(0) = z_0(0) - z_0(0) - g(z_0(0), \ z_0(t_1), \dots, z_0(t_r)) \le 0. \end{aligned}$$

This shows that $z_0(t) \le z_1(t), \ t \in J$. In the same way we can show that $y_1(0) \le y_0(t), \ t \in J$. Now we put $p = z_1 - y_1$. Then $p'(0) = 0$, and

$$p''(t) = F(z_0, z_0)(t) - F(y_0, y_0)(t) \le 0,$$

because $z_0(\beta(t, z_0(t))) \le z_0(\beta(t, y_0(t))) \le y_0(\beta(t, y_0(t)))$ (see assumption ($H_2$)). Moreover, it can be easily seen that assumption ($H_6$) guarantees that

$$\begin{aligned} p(0) &= z_0(0) - y_0(0) + g(z_0(0), z_0(t_1), \dots, z_0(t_r)) - g(y_0(0), y_0(t_1), \dots, y_0(t_r)) \\ &\le z_0(0) - y_0(0) + g(z_0(0), y_0(t_1), \dots, y_0(t_r)) - g(y_0(0), y_0(t_1), \dots, y_0(t_r)) \\ &\le z_0(0) - y_0(0) + y_0(0) - z_0(0) = 0. \end{aligned}$$

It proves that $y_1(t) \le z_1(t), \ t \in J$. Consequently,

$$z_0(t) \le z_1(t) \le y_1(t) \le y_0(t), \quad t \in J. \tag{2.2}$$

Note that $y_1$ is nondecreasing because $y_1''(t) \ge 0$, $y_1'(t) \ge y_1'(0) \ge 0$.

Now we show that $y_1$ is a lower solution of problem (1.1). Indeed,

$$y_1''(t) = F(y_0, y_0)(t) \ge F(y_1, y_1)(t),$$

because $y_0(\beta(t, y_0(t))) \ge y_0(\beta(t, y_1(t))) \ge y_1(\beta(t, y_1(t)))$. Moreover, in view of assumption $H_6$, we have

$$\begin{aligned} y_1(0) &= y_0(0) + g(y_0(0), y_0(t_1), \dots, y_0(t_r)) \\ &\ge y_0(0) + g(y_0(0), y_1(t_1), \dots, y_1(t_r)) - g(y_1(0), y_1(t_1), \dots, y_1(t_r)) \\ &\quad + g(y_1(0), y_1(t_1), \dots, y_1(t_r)) \ge y_1(0) + g(y_1(0), y_1(t_1), \dots, y_1(t_r)), \end{aligned}$$

so $g(y_1(0), y_1(t_1), \dots, y_1(t_r)) \le 0$. It shows that $y_1$ is a lower solution of (1.1). Similarly, we can show that $z_1$ is nondecreasing and it is an upper solution of (1.1).

By induction in $n$, we can show that

$$z_0(t) \le z_1(t) \le \cdots \le z_n(t) \le y_n(t) \le \cdots \le y_1(t) \le y_0(t), \quad t \in J$$

for $n = 1, 2, \dots$.

From the above, $y_n, z_n$ are uniformly bounded. Indeed, $y_n$ satisfies the integral equation

$$y_{n+1}(t) = y_n(0) + g(y_n(0), y_n(t_1), \dots, y_n(t_r)) + kt + \int_0^t \int_0^s F(y_n, y_n)(\tau) d\tau ds.$$

We see that $\{y_n, z_n\}$ are equicontinuous. The Arzeli–Ascoli theorem guarantees the existence of subsequences $\{y_{n_k}, z_{n_k}\}$ and functions $y, z \in C^2(J, \mathbb{R})$ with $y_{n_k}, z_{n_k}$ converging uniformly on $J$ to $y$ and $z$, respectively. Because $f$ and $g$ are continuous, the functions $y$ and $z$ are solutions of problem (1.1) and $z_0(t) \le z(t) \le y(t) \le y_0(t)$, $t \in J$.

Finally, it is easy to show that $z, y$ are minimal and maximal solutions of problem (1.1) in the sector $[z_0, y_0]_*$. It completes the proof. ■

## 3 Quasi-Solutions of Problem (1.1)

Put $S = \{t_1, t_2, \dots, t_r\}$, $Q = \{t_{m_1}, t_{m_2}, \dots, t_{m_d}\}$. The set $Q$ may be empty or $Q \subset S$. For example, if

$$g(a_0, a_1, \dots, a_r) = -a_0 + \sum_{i=1}^{r} k_i a_i + l$$

and if $k_s < 0$ for some fixed $s$, $1 \le s \le r$, then $t_s \in Q$. If $k_i \ge 0$ for all $i = 1, 2, \dots, r$, then $Q$ is empty. Let $G(a, b, c, w) = g(a(0), w(b, c; t_1), \dots, w(b, c; t_r))$ and

$$w(u, v; t_i) = \begin{cases} v(t_i) & \text{if } t_i \in Q, \\ u(t_i) & \text{if } t_i \notin Q, \end{cases} \qquad \bar{w}(u, v; t_i) = \begin{cases} u(t_i) & \text{if } t_i \in Q, \\ v(t_i) & \text{if } t_i \notin Q, \end{cases}$$

for $i = 1, 2, \dots, r$, see [10]. Note that if $Q = S$, then $w(u, v; t_i) = v(t_i)$, $\bar{w}(u, v; t_i) = u(t_i)$ for $i = 1, 2, \dots, r$; while if $Q$ is empty, then $w(u, v; t_i) = u(t_i)$, $\bar{w}(u, v; t_i) = v(t_i)$, $i = 1, 2, \dots, r$.

Now we are in the position to prove the following theorem.

***Theorem 3.1*** *Suppose that assumptions* $(\mathrm{H}_1)$, $(\mathrm{H}_2)$ *hold. In addition assume that*

$(\mathrm{H}_{31})$ $y_0, z_0 \in C^2(J, \mathbb{R})$ *satisfy the system*

$$\begin{cases} y_0''(t) \ge F(y_0, z_0)(t), \ t \in J, \\ y_0'(0) \ge k, \\ \quad 0 \ge G(y_0, y_0, z_0, w), \end{cases} \qquad \begin{cases} z_0''(t) \le F(z_0, y_0)(t), \ t \in J, \\ z_0'(0) \le k, \\ \quad 0 \le G(z_0, y_0, z_0, \bar{w}), \end{cases}$$

*and* $z_0(t) \le y_0(t)$, $t \in J$;

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($H_{41}$) *$\beta(t,u)$ is nonincreasing with respect to u;*

($H_{51}$) *$y_0, z_0$ are nondecreasing and $f(t,u) \geq 0$ for $t \in J$ and $z_0(\beta(t, y_0(t))) \leq u \leq y_0(\beta(t, z_0(t)))$;*

($H_{61}$) *g satisfies condition* (2.1) *and*

(i) *if $t_i \notin Q$ and $\bar{v}(t_i) \geq v(t_i)$ for fixed i, then*

$$g(u, v(t_1), \dots, v(t_i), \dots, v(t_r)) \leq g(u, v(t_1), \dots, \bar{v}(t_i), \dots, v(t_r)),$$

(ii) *if $t_i \in Q$ and $\bar{v}(t_i) \geq v(t_i)$ for fixed i, then*

$$g(u, v(t_1), \dots, v(t_i), \dots, v(t_r)) \geq g(u, v(t_1), \dots, \bar{v}(t_i), \dots, v(t_r)).$$

*Then problem* (1.1) *has a quasi-solution in the sector $[z_0, y_0]_*$* i.e., *there exist functions $y, z \in C^2(J, \mathbb{R})$ such that*

$$\text{(3.1)} \qquad \begin{cases} y''(t) = F(y,z)(t), \quad t \in J, \\ y'(0) = k, \\ \quad 0 = G(y,y,z,w), \end{cases} \qquad \begin{cases} z''(t) = F(z,y)(t), \quad t \in J, \\ z'(0) = k, \\ \quad 0 = G(z,y,z,\bar{w}) \end{cases}$$

*and $z_0(t) \leq z(t) \leq y(t) \leq y_0(t)$, $t \in J$.*

*Moreover, if problem* (1.1) *has a solution $q \in C^2(J, \mathbb{R})$ such that $z_0(t) \leq q(t) \leq y_0(t)$, $t \in J$, then $z(t) \leq q(t) \leq y(t)$, $t \in J$.*

**Proof** Define sequences $\{y_n, z_n\}$ by

$$\begin{cases} y''_{n+1}(t) = F(y_n, z_n)(t), \ t \in J, \\ y'_{n+1}(0) = k, \\ y_{n+1}(0) = y_n(0) + G(y_n, y_n, z_n, w), \end{cases} \qquad \begin{cases} z''_{n+1}(t) = F(z_n, y_n)(t), \ t \in J, \\ z'_{n+1}(0) = k, \\ z_{n+1}(0) = z_n(0) + G(z_n, y_n, z_n, \bar{w}). \end{cases}$$

In view of conditions (i) and (ii) of assumption ($H_{61}$), it is easy to show that

$$G(z_0, y_0, z_0, \bar{w}) \leq G(z_0, y_0, z_0, w).$$

Similarly as in the proof of Theorem 2.1, we can show relation (2.2). Now there is no problem verifying that assumption ($H_{31}$) holds with $(y_1, z_1)$ instead of $(y_0, z_0)$.

Consequently, we can show that

$$z_0(t) \leq z_1(t) \leq \cdots \leq z_n(t) \leq y_n(t) \leq \cdots \leq y_1(t) \leq y_0(t)$$

for $t \in J$ and $n = 1, 2, \dots$. Indeed, the limits

$$\lim_{n\to\infty} y_n(t) = y(t), \quad \lim_{n\to\infty} z_n(t) = z(t)$$

exist, where $y, z$ satisfy system (3.1).

To finish the proof it remains to prove that if problem (1.1) has a solution $q \in [z_0, y_0]_*$, then $z(t) \leq q(t) \leq y(t)$, $t \in J$. Notice that using the method of mathematical induction, we can show that $z_n(t) \leq q(t) \leq y_n(t)$, $t \in J$. Taking $n \to \infty$ in the last relation, we have the assertion. ∎

***Theorem 3.2*** *Suppose that assumptions* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_{61})$ *are satisfied. In addition, we assume the following:*

$(\mathrm{H}_{21})$ *$f$ is nonincreasing with respect to the last argument and $k \geq 0$.*
$(\mathrm{H}_{32})$ *$y_0, z_0 \in C^2(J, \mathbb{R})$ satisfy the system*

$$\begin{cases} y_0''(t) \geq F(z_0, z_0)(t), \quad t \in J, \\ y_0'(0) \geq k, \\ \qquad 0 \geq G(y_0, y_0, z_0, w), \end{cases} \qquad \begin{cases} z_0''(t) \leq F(y_0, y_0)(t), \quad t \in J. \\ z_0'(0) \leq k, \\ \qquad 0 \leq G(z_0, y_0, z_0, \bar{w}), \end{cases}$$

*and $z_0(t) \leq y_0(t)$, $t \in J$;*
$(\mathrm{H}_{42})$ *$\beta(t, u)$ is nondecreasing with respect to $u$.*
$(\mathrm{H}_{52})$ *$y_0, z_0$ are nondecreasing and $f(t, u) \geq 0$ for $t \in J$, $z_0(\beta(t, z_0(t))) \leq u \leq y_0(\beta(t, y_0(t)))$.*

*Then the assertion of Theorem* 3.1 *holds with*

$$\begin{cases} y''(t) = F(z, z)(t), \quad t \in J, \\ y'(0) = k, \\ \qquad 0 = G(y, y, z, w), \end{cases} \qquad \begin{cases} z''(t) = F(y, y)(t), \quad t \in J, \\ z'(0) = k, \\ \qquad 0 = G(z, y, z, \bar{w}) \end{cases}$$

*instead of system* (3.1).

**Proof** In this case, we define sequences $\{z_n, z_n\}$ by the following relations

$$\begin{cases} y_{n+1}''(t) = F(z_n, z_n)(t), \quad t \in J, \\ y_{n+1}'(0) = k, \\ y_{n+1}(0) = y_n(0) + G(y_n, y_n, z_n, w), \end{cases} \qquad \begin{cases} z_{n+1}''(t) = F(y_n, y_n)(t), \quad t \in J, \\ z_{n+1}'(0) = k, \\ z_{n+1}(0) = z_n(0) + G(z_n, y_n, z_n, \bar{w}). \end{cases}$$

Since the rest of the proof is similar to the proof of Theorem 3.1, it is omitted. ■

***Theorem 3.3*** *Suppose that assumptions* $(\mathrm{H}_1)$, $(\mathrm{H}_{21})$, $(\mathrm{H}_{41})$, $(\mathrm{H}_{61})$ *hold and in addition assume the following:*

$(\mathrm{H}_{33})$ *$y_0, z_0 \in C^2(J, \mathbb{R})$ satisfy the system*

$$\begin{cases} y_0''(t) \geq F(z_0, y_0)(t), \quad t \in J, \\ y_0'(0) \geq k, \\ \qquad 0 \geq G(y_0, y_0, z_0, w), \end{cases} \qquad \begin{cases} z_0''(t) \leq F(y_0, z_0)(t), \quad t \in J, \\ z_0'(0) \leq k, \\ \qquad 0 \leq G(z_0, y_0, z_0, \bar{w}), \end{cases}$$

*and $z_0(t) \leq y_0(t)$, $t \in J$;*
$(\mathrm{H}_{53})$ *$y_0, z_0$ are nondecreasing and $f(t, u) \geq 0$ for $t \in J, z_0(\beta(t, y_0(t))) \leq u \leq y_0(\beta(t, z_0(t)))$.*

*Then the assertion of Theorem* 3.1 *holds with*

$$\begin{cases} y''(t) = F(z, y)(t), & t \in J, \\ y'(0) = k, \\ \quad 0 = G(y, y, z, w), \end{cases} \qquad \begin{cases} z''(t) = F(y, z)(t), & t \in J, \\ z'(0) = k, \\ \quad 0 = G(z, y, z, \bar{w}) \end{cases}$$

*instead of system* (3.1).

Omitting the proof we define only sequences $\{y_n, z_n\}$ by

$$\begin{cases} y''_{n+1}(t) = F(z_n, y_n)(t), & t \in J, \\ y'_{n+1}(0) = k, \\ y_{n+1}(0) = y_n(0) + G(y_n, y_n, z_n, w), \end{cases} \begin{cases} z''_{n+1}(t) = F(y_n, z_n)(t), & t \in J, \\ z'_{n+1}(0) = k, \\ z_{n+1}(0) = z_n(0) + G(z_n, y_n, z_n, \bar{w}). \end{cases}$$

## 4 A Unique Solution of Problem (1.1)

In Sections 2 and 3, we formulated conditions which guarantee that problem (1.1) has the extremal or quasi-solutions. In this section, we shall discuss the existence of a unique solution of problem (1.1). We start from the following lemma.

***Lemma 4.1*** *Assume that* $K, L \in C(J, \mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty)$. *Let* $\beta \in C(\Omega, J)$. *In addition we assume that*

$(\mathrm{H}_0)$ *there exist constants* $a_i \in [0, 1)$, $i = 1, 2, \dots, r$, $\sum_{i=1}^{r} a_i < 1$ *and such that*

$$\sum_{i=1}^{r} a_i + \sum_{i=1}^{r} a_i \int_0^{t_i} \int_0^{s} L^*(\tau) d\tau ds + \Big(1 - \sum_{i=1}^{r} a_i\Big) \int_0^{T} \int_0^{s} L^*(\tau) d\tau\, ds < 1, \tag{4.1}$$

*where* $L^*(t) = K(t) + L(t)$.

*Let* $p \in C^2(J, \mathbb{R})$ *and*

$$\begin{cases} p''(t) \le K(t)p(t) + L(t)p(\beta(t, w(t))), & t \in J, \\ p'(0) \le 0, \\ p(0) \le \displaystyle\sum_{i=1}^{r} a_i p(t_i), \end{cases}$$

*where* $w \in [z_0, y_0]_*$. *Then* $p(t) \le 0$, $t \in J$.

**Proof** Assume that the assertion is not true. It means that there exists a point $t_0^* \in J$ such that $p(t_0^*) > 0$. Put $p(t_1^*) = \max_{t \in J} p(t)$. Then $K(t)p(t) + L(t)p(\beta(t, w(t))) \le p(t_1^*)L^*(t)$. Now, integrating two times the differential inequality for $p$, we have

$$p(t) \le p(0) + p(t_1^*) \int_0^{t} \int_0^{s} L^*(\tau)\, d\tau ds$$

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because $p'(0) \le 0$. Adding to this the boundary condition for $p(0)$, we obtain

$$p(t) \le p(t_1^*)\Big[\Big(1-\sum_{i=1}^{r} a_i\Big)^{-1}\sum_{i=1}^{r} a_i\int_0^{t_i}\int_0^{s} L^*(\tau)d\tau ds+\int_0^{t}\int_0^{s} L^*(\tau)d\tau ds\Big] \tag{4.2}$$

for $t \in J$. Put $t = t_1^*$ in formula (4.2) to obtain

$$\begin{aligned} p(t_1^*) &\le p(t_1^*)\Big[\Big(1-\sum_{i=1}^{r} a_i\Big)^{-1}\sum_{i=1}^{r} a_i\int_0^{t_i}\int_0^{s} L^*(\tau)d\tau ds+\int_0^{T}\int_0^{s} L^*(\tau)d\tau ds\Big] \\ &< p(t_1^*). \end{aligned}$$

This is a contradiction. ■

***Remark 1*** It is easy to see that condition (4.1) holds if we assume that

$$\sum_{i=1}^{r} a_i + \int_0^{T}\int_0^{s} L^*(\tau)d\tau ds < 1.$$

Moreover, if we assume that $L^*(t) = L^*$, then the last condition takes the place

$$2\sum_{i=1}^{r} a_i + L^*T^2 < 2.$$

***Theorem 4.2*** *Let all assumptions of Theorem* 2.1 *hold. In addition, we assume that*

($H_7$) *there exist functions* $L, M \in C(J, \mathbb{R}_+)$ *such that*

$$f(t,\bar u) - f(t,u) \le L(t)(\bar u - u) \qquad \beta(t,\bar v) - \beta(t,v) \le M(t)(\bar v - v)$$

*for* $z_0(t) \le v \le \bar v \le y_0(t)$, $\min_{t\in J} z_0(t) \le u \le \bar u \le \max_{t\in J} y_0(t)$;

($H_8$) $g(u, v_1,\dots,v_r) = -u + g_1(v_1,\dots,v_r)$, *where* $g_1$ *is nondecreasing with respect to all variables and there exist constants* $a_i \in [0,1)$, $i = 1,2,\dots,r$ *and*

$$g_1(\bar v_1,\dots,\bar v_r) - g_1(v_1,\dots,v_r) \le \sum_{i=1}^{r} a_i[\bar v_i - v_i]$$

*for* $z_0(t_i) \le v_i \le \bar v_i \le y_0(t_i)$, $i = 1,2,\dots,r$;

($H_9$) *condition* (4.1) *holds with* $L^*(t) = L(t)[1 + N_1M(t)]$, *where*

$$N_1 = k + \int_0^T F(y_0,y_0)(s)ds.$$

*Then problem* (1.1) *has a unique solution in the sector* $[z_0, y_0]_*$.

**Proof** From Theorem 2.1, we know that $z, y$ are the minimal and maximal solutions of (1.1) and moreover $z_0(t) \le z(t) \le y(t) \le y_0(t)$, $t \in J$. We need to show that $z = y$. Put $p = y - z$. Then in view of assumptions ($H_7$), ($H_9$), we obtain

$$\begin{aligned} p''(t) &= F(y,y)(t) - F(z,z)(t) \le L(t)\big[\, y\big(\beta(t,y(t))\big) - z\big(\beta(t,z(t))\big)\big] \\ &= L(t)\big[\, p\big(\beta(t,y(t))\big) + z\big(\beta(t,y(t))\big) - z\big(\beta(t,z(t))\big)\big] \\ &\le K(t)p(t) + L(t)p\big(\beta(t,y(t))\big) \end{aligned}$$

with $K(t) = L(t)N_1M(t)$.

Moreover, $p'(0) = 0$ and

$$p(0) = g_1(y(t_1),\dots,y(t_r)) - g_1(z(t_1),\dots,z(t_r)) \le \sum_{i=1}^{r} a_i p(t_i)$$

by assumption ($H_8$). As a consequence of Lemma 4.1, we obtain $p(t) \le 0$, $t \in J$, so $y(t) \le z(t)$, $t \in J$. It shows that $y = z$. ∎

***Remark 2*** Note that $N_1 \le y_0'(T)$. Indeed, knowing that $y_0$ is a lower solution of problem (1.1), we have

$$N_1 = k + \int_0^T F(y_0,y_0)(s)ds \le k + \int_0^T y_0''(s)ds \le y_0'(T).$$

***Example 1*** We consider the following problem

$$\text{(4.3)} \quad \begin{cases} x''(t) = \gamma_2(t)\exp[\gamma_1 x(\delta t x(t))] \equiv f\big(t, x(\beta(t,x(t)))\big), & t \in J = [0,1], \\ x'(0) = 0, \\ x(0) = \lambda x(1), \quad 0 < \lambda \le 2/5, \end{cases}$$

where $\gamma_2 \in C(J,(0,\frac{1}{2}])$, $0 < \gamma_1 \le \frac{1}{2}$, $0 < \delta \le \frac{1}{2}$. Here $\beta(t,x) = \delta t x$, $g(u,v) = -u + \lambda v$. Note that $f$ and $\beta$ are nondecreasing with respect to the last variable.

Let $y_0(t) = t^2 + 1$, $z_0(t) = 0$, $t \in J$. Then

$$\begin{aligned} &F(y_0,y_0)(t) = \gamma_2(t)\exp[\gamma_1\delta^2 t^2(t^2+1)^2 + \gamma_1] \le \tfrac{1}{2}e < 2 = y_0''(t), \\ &F(z_0,z_0)(t) = \gamma_2(t) > 0 = z_0''(t), \end{aligned}$$

and

$$g(y_0(0),y_0(1)) = g(1,2) = -1 + 2\lambda < 0, \qquad g(z_0(0),z_0(1)) = g(0,0) = 0$$

with $y_0'(0) = z_0'(0) = 0$. This proves that $y_0, z_0$ are lower and upper solutions of problem (4.3), respectively. Note that all assumptions of Theorem 2.1 hold, so problem (4.3) has extremal solutions in the sector $[z_0, y_0]_*$.

Moreover,

$$L(t) = \tfrac{1}{2}e\gamma_2(t), \quad M(t) = \tfrac{1}{2}t, \quad L^*(t) = \tfrac{1}{2}e\gamma_2(t)\Big[1 + \tfrac{1}{2}te\int_0^1 \gamma_2(s)ds\Big]$$

from assumptions ($H_7$) and ($H_9$). Note that

$$\lambda + \int_0^1 \int_0^s L^*(\tau)d\tau ds = \lambda + \tfrac{1}{2}e\int_0^1 \int_0^s \gamma_2(t)\Big[1 + \tfrac{1}{2}et\int_0^1 \gamma_2(\tau)d\tau\Big]dtds$$
$$\leq \tfrac{2}{5} + \tfrac{1}{8}e\big[1 + \tfrac{1}{4}e\big] \approx 0.97 < 1.$$

It proves that all assumptions of Theorems 2.1 and 4.2 are satisfied. Hence, problem (4.3) has a unique solution in the sector $[z_0, y_0]_*$.

Now, based on Theorems 3.1–3.3 and Lemma 4.1, we formulate corresponding conditions under which problem (1.1) has a unique solution. We omit the proofs because they are similar to the proof of Theorem 4.2.

***Theorem 4.3*** *Let all assumptions of Theorem* 3.1 *hold. In addition, we assume the following:*

($H_{71}$) *There exist functions $L, M \in C(J, \mathbb{R}_+)$ such that*

$$f(t,\bar{u}) - f(t,u) \leq L(t)(\bar{u} - u) \quad \beta(t,v) - \beta(t,\bar{v}) \leq M(t)(\bar{v} - v)$$

*for $z_0(t) \leq v \leq \bar{v} \leq y_0(t)$, $\min_{t\in J} z_0(t) \leq u \leq \bar{u} \leq \max_{t\in J} y_0(t)$;*

($H_{81}$) *$g(u, v_1, \dots, v_r) = -u + g_1(v_1, \dots, v_r)$, where $g_1 \in C(\mathbb{R}^r, \mathbb{R})$ and there exist constants $a_i \in [0,1)$, $i = 1,2,\dots,r$ such that*

$$g_1(w(y,z;t_1),\dots,w(y,z;t_r)) - g_1(\bar{w}(y,z;t_1),\dots,\bar{w}(y,z;t_r)) \leq \sum_{i=1}^r a_i[y(t_i) - z(t_i)]$$

*for $z_0(t_i) \leq z(t_i) \leq y(t_i) \leq y_0(t_i)$, $i = 1,2,\dots,r$;*

($H_{91}$) *assumption* ($H_9$) *holds with $N_1 = k + \int_0^T F(y_0, z_0)(s)ds$.*

*Then problem* (1.1) *has a unique solution in the sector $[z_0, y_0]_*$.*

***Theorem 4.4*** *Let all assumptions of Theorem* 3.2 *hold. Let assumption* ($H_{81}$) *hold. In addition, we assume the following:*

($H_{72}$) *There exist functions $L, M \in C(J, \mathbb{R}_+)$ such that*

$$f(t,u) - f(t,\bar{u}) \leq L(t)(\bar{u} - u) \quad \beta(t,\bar{v}) - \beta(t,v) \leq M(t)(\bar{v} - v)$$

*for $z_0(t) \leq v \leq \bar{v} \leq y_0(t)$, $\min_{t\in J} z_0(t) \leq u \leq \bar{u} \leq \max_{t\in J} y_0(t)$.*

($H_{92}$) *Assumption* ($H_9$) *holds with $N_1 = k + \int_0^T F(z_0, z_0)(s)ds$.*

*Then problem* (1.1) *has a unique solution in the sector $[z_0, y_0]_*$.*

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***Theorem 4.5*** *Let all assumptions of Theorem* 3.3 *hold. Let assumption* $(\mathrm{H}_{81})$ *hold. In addition, assume the following:*

$(\mathrm{H}_{73})$ *There exist functions* $L, M \in C(J, \mathbb{R}_+)$ *such that*

$$f(t,u) - f(t,\bar{u}) \leq L(t)(\bar{u} - u) \quad \beta(t,v) - \beta(t,\bar{v}) \leq M(t)(\bar{v} - v)$$

*for* $z_0(t) \leq v \leq \bar{v} \leq y_0(t)$, $\min_{t\in J} z_0(t) \leq u \leq \bar{u} \leq \max_{t\in J} y_0(t)$.

$(\mathrm{H}_{93})$ *Assumption* $(\mathrm{H}_9)$ *holds with* $N_1 = k + \int_0^T F(z_0, y_0)(s)ds$.

*Then problem* (1.1) *has a unique solution in the sector* $[z_0, y_0]_*$.

## 5 The General Case

The next lemma extends Lemma 4.1 to differential inequalities for $p$ having more arguments of type $\beta$. The proof is similar to the proof of Lemma 4.1 and therefore it is omitted.

***Lemma 5.1*** *Assume that* $K, L_j \in C(J, \mathbb{R}_+), \beta_j \in C(\Omega, J)$, $j = 1, 2, \ldots, q$. *In addition, we assume that there exist* $a_i \in [0, 1)$, $i = 1, 2, \ldots, r$, $\sum_{i=1}^r a_i < 1$ *and such that condition* (4.1) *holds with* $L^*(t) = K(t) + \sum_{j=1}^q L_j(t)$. *Let* $p \in C^2(J, \mathbb{R})$ *and*

$$\begin{cases} p''(t) \leq K(t)p(t) + \displaystyle\sum_{j=1}^{q} L_j(t)p(\beta_j(t, w_j(t))), \quad t \in J, \\ p'(0) \leq 0, \\ p(0) \leq \displaystyle\sum_{i=1}^{r} a_i p(t_i) \end{cases}$$

*for* $w_j \in [z_0, y_0]_*$, $j = 1, 2, \ldots, q$. *Then* $p(t) \leq 0$, $t \in J$.

In this section, we consider the problem of the form

(5.1)
$$\begin{cases} x''(t) = f\big(t, x\big(\beta(t, x(t))\big), x\big(\gamma(t, x(t))\big)\big) \equiv \mathcal{F}(x, x, x, x)(t), \quad t \in J = [0, T], \\ x'(0) = k, \\ 0 = g\big(x(0), x(t_1), \ldots, x(t_r)\big), \end{cases}$$

where $\mathcal{F}(x, y, u, w)(t) = f\big(t, x\big(\beta(t, y(t))\big), u\big(\gamma(t, w(t))\big)\big)$ and

$$f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \quad \beta, \gamma \in C(J \times \mathbb{R}, \mathbb{R}), \quad g \in C(\mathbb{R}^{r+1}, \mathbb{R}).$$

***Theorem 5.2*** *Assume the following hold:*

$(\mathrm{A}_1)$ $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\beta, \gamma \in C(J \times \mathbb{R}, \mathbb{R})$, $g_1 \in C(\mathbb{R}^r, \mathbb{R})$, *and* $g(u, v_1, \ldots, v_r) = -u + g_1(v_1, \ldots, v_r)$.

$(\mathrm{A}_2)$ $f$ *is nondecreasing with respect to the last two variables,* $k \geq 0$.

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(A$_3$) $y_0, z_0 \in C^2(J, \mathbb{R})$ *satisfy the system*

$$\begin{cases} y_0''(t) \geq \mathcal{F}(y_0, y_0, y_0, z_0)(t), \quad t \in J, \qquad y_0'(0) \geq k, \quad 0 \geq G(y_0, y_0, z_0, w), \\ z_0''(t) \leq \mathcal{F}(z_0, z_0, z_0, y_0)(t), \quad t \in J, \qquad z_0'(0) \leq k, \quad 0 \leq G(z_0, y_0, z_0, \bar{w}), \end{cases}$$

*and* $z_0(t) \leq y_0(t)$, $t \in J$.

(A$_4$) $\beta, \gamma \colon \Omega \to J$, $\beta(t,u)$ *is nondecreasing while* $\gamma(t,u)$ *is nonincreasing with respect to* $u$.

(A$_5$) $y_0, z_0$ *are nondecreasing,* $f(t,u,v) \geq 0$ *for* $t \in J$, $z_0\big(\beta(t, z_0(t))\big) \leq u \leq y_0\big(\beta(t, y_0(t))\big)$, $z_0\big(\gamma(t, y_0(t))\big) \leq v \leq y_0\big(\gamma(t, z_0(t))\big)$, $t \in J$.

(A$_6$) *Conditions* (i) *and* (ii) *of assumption* (H$_{61}$) *hold.*

(A$_7$) *Assumption* (H$_{81}$) *holds.*

(A$_8$) *There exist functions* $L_1, L_2, M_1, M_2 \in C(J, R_+)$ *such that*

$$\begin{gathered} f(t, \bar{u}_1, \bar{v}_1) - f(t, u_1, v_1) \leq L_1(t)(\bar{u}_1 - u_1) + L_2(t)(\bar{v}_1 - v_1), \\ \beta(t, \bar{v}) - \beta(t, v) \leq M_1(t)(\bar{v} - v) \\ \gamma(t, w) - \gamma(t, \bar{w}) \leq M_2(t)(\bar{w} - w). \end{gathered}$$

*if* $\min_{t\in J} z_0(t) \leq u_1 \leq \bar{u}_1 \leq \max_{t\in J} y_0(t)$, $\min_{t\in J} z_0(t) \leq v_1 \leq \bar{v}_1 \leq \max_{t\in J} y_0(t)$, $z_0(t) \leq v \leq \bar{v} \leq y_0(t)$, $z_0(t) \leq w \leq \bar{w} \leq y_0(t)$, $t \in J$.

(A$_9$) *Condition* (4.1) *holds for* $L^*(t) = L_1(t)[1 + N_1 M_1(t)] + L_2(t)[1 + N_1 M_2(t)]$ *with* $N_1 = k + \int_0^T \mathcal{F}(y_0, y_0, y_0, z_0)(s)ds$.

*Then problem* (5.1) *has a unique solution in the sector* $[z_0, y_0]_*$.

**Proof** To show this theorem we use the ideas from the proofs of Theorems 2.1 and 4.2. First of all, let the sequences $\{y_n, z_n\}$ be defined by

$$\begin{cases} y_{n+1}''(t) = \mathcal{F}(y_n, y_n, y_n, z_n)(t), \ t \in J, \\ y_{n+1}'(0) = k, \\ y_{n+1}(0) = y_n(0) + G(y_n, y_n, z_n, w), \end{cases} \quad \begin{cases} z_{n+1}''(t) = \mathcal{F}(z_n, z_n, z_n, y_n)(t), \ t \in J, \\ z_{n+1}'(0) = k, \\ z_{n+1}(0) = z_n(0) + G(z_n, y_n, z_n, \bar{w}) \end{cases}$$

for $n = 0, 1, \dots$.

Using assumption (A$_3$) and definition for $y_1, z_1$, we can show that $z_0(t) \leq z_1(t)$ and $y_1(t) \leq y_0(t)$ on $J$. To show that $z_1(t) \leq y_1(t)$ we put $p = z_1 - y_1$. Then in view of assumptions (A$_2$),(A$_4$), we see that

$$p''(t) = \mathcal{F}(z_0, z_0, z_0, y_0)(t) - \mathcal{F}(y_0, y_0, y_0, z_0)(t) \leq 0,$$

because $z_0(\beta(t, z_0(t))) \leq y_0(\beta(t, y_0(t)))$, $z_0(\gamma(t, y_0(t))) \leq y_0(\gamma(t, z_0(t)))$. Indeed, $p'(0) = 0$. Moreover, in view of assumption (A$_6$), we have $p(0) \leq 0$. This shows that $z_1(t) \leq y_1(t)$ on $J$, so $z_0(t) \leq z_1(t) \leq y_1(t) \leq y_0(t)$, $t \in J$. Using the definition for $y_1, z_1$ and assumption $f(t,u,v) \geq 0$, we see that $y_1, z_1$ are nondecreasing.

In the next step we must show that assumption ($A_3$) holds with $(y_1, z_1)$ instead of $(y_0, z_0)$. Note that in view of assumptions ($A_2$), ($A_4$), we have

$$y_1''(t) = \mathcal{F}(y_0, y_0, y_0, z_0)(t) \geq \mathcal{F}(y_1, y_1, y_1, z_1)(t),$$
$$z_1''(t) = \mathcal{F}(z_0, z_0, z_0, y_0)(t) \leq \mathcal{F}(z_1, z_1, z_1, y_1)(t),$$

because

$$y_0\big(\beta(t, y_0(t))\big) \geq y_1\big(\beta(t, y_1(t))\big), \qquad y_0\big(\gamma(t, z_0(t))\big) \geq y_1\big(\gamma(t, z_1(t))\big),$$
$$z_0\big(\beta(t, z_0(t))\big) \leq z_1\big(\beta(t, z_1(t))\big), \qquad z_0\big(\gamma(t, y_0(t))\big) \leq z_1\big(\gamma(t, y_1(t))\big).$$

Moreover,

$$y_1(0) = y_0(0) + G(y_0, y_0, z_0, w) \geq y_1(0) + G(y_1, y_1, z_1, w),$$
$$z_1(0) = z_0(0) + G(z_0, y_0, z_0, \bar{w}) \leq z_1(0) + G(z_1, y_1, z_1, \bar{w}),$$

in view of assumption ($A_6$). This means that assumption ($A_3$) holds with $(y_1, z_1)$ instead of $(y_0, z_0)$.

Sequences $\{y_n, z_n\}$ converge to limit functions $y, z$ (see the proof of Theorem 2.1). Indeed, $z_0(t) \leq z(t) \leq y(t) \leq y_0(t)$, $t \in J$ and $y, z$ are solutions of the system

$$\begin{cases} y''(t) = \mathcal{F}(y, y, y, z)(t), & t \in J, \\ y'(0) = k, \\ \quad 0 = G(y, y, z, w), \end{cases} \qquad \begin{cases} z''(t) = \mathcal{F}(z, z, z, y)(t), & t \in J, \\ z'(0) = k, \\ \quad 0 = G(z, y, z, \bar{w}). \end{cases}$$

To show that $y = z$ we put $p = y - z$. Then using assumptions ($A_8$), ($A_9$), we obtain

$$\begin{aligned} p''(t) &= \mathcal{F}(y, y, y, z)(t) - \mathcal{F}(z, z, z, y)(t) \\ &\leq L_1(t)\big[p\big(\beta(t, y(t))\big) + z\big(\beta(t, y(t))\big) - z\big(\beta(t, z(t))\big)\big] \\ &\quad + L_2(t)\big[p\big(\gamma(t, z(t))\big) + z\big(\gamma(t, z(t))\big) - z\big(\beta(t, y(t))\big)\big] \\ &\leq L_1(t)p(\beta(t, y(t))) + L_2(t)p(\gamma(t, z(t))) + K(t)p(t) \end{aligned}$$

with $K(t) = L_1(t)N_1M_1(t) + L_2(t)N_1M_2(t)$. Moreover, $p'(0) = 0$ and

$$0 = G(y, y, y, z, w) - G(z, z, z, y, \bar{w}) \leq -p(0) + \sum_{i=1}^{r} a_i p(t_i)$$

in view of assumption ($A_7$). Then this, assumption ($A_9$), and Lemma 5.1 prove that $y(t) \leq z(t)$ on $J$. It means that $y = z$, so $y$ is the unique solution of problem (5.1). ∎

***Example 2*** Consider the following problem:

$$(5.2)\qquad \begin{cases} x''(t) = A(t)x(\beta(t,x(t))) + B(t)x\big(\gamma(t,x(t))\big) + C(t), & t \in J = [0,1], \\ x'(0) = 0, \\ x(0) = \lambda_1 x\big(\frac{1}{2}\big) + \lambda_2 x(1) + \lambda_3, \end{cases}$$

where $A, B, C \in C(J, [0,\infty))$, $\lambda_1, \lambda_2 > 0$, $\lambda_3 \geq 0$, and $\lambda_1 + \lambda_2 < 1$.

Here $f(t,u,v) = A(t)u + B(t)v + C(t)$, $\beta(t,x) = x$, $\gamma(t,x) = \frac{1}{1+x}$, $g(u,v_1,v_2) = -u + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3$. Note that $f$ is nondecreasing with respect to the last two variables.

Put $y_0(t) = \frac{1}{2}(t^2+1)$, $z_0(t) = 0$, $t \in J$. Then $\Omega = \{(t,w) : 0 \leq w \leq \frac{1}{2}(t^2+1), t \in J\}$. We see that $\beta, \gamma \in C(\Omega, J)$.

We assume that

$$(5.3)\qquad \begin{cases} 1 \geq \frac{1}{2}A(t)\big[\frac{1}{4}(t^2+1)^2 + 1\big] + B(t) + C(t), & t \in J, \\ \frac{1}{2} \geq \frac{5}{8}\lambda_1 + \lambda_2 + \lambda_3. \end{cases}$$

In view of (5.3), functions $y_0, z_0$ satisfy assumption $(A_3)$. It is easy to check that assumptions $(A_4)$, $(A_5)$, $(A_6)$ are satisfied. Assumptions $(A_7)$, $(A_8)$ hold with

$$a_1 = \lambda_1,\ a_2 = \lambda_2,\ L_1(t) = A(t),\ L_2(t) = B(t),\ M_1(t) = M_2(t) = 1, \quad t \in J.$$

If we also assume that

$$(5.4)\qquad \lambda_1 + \lambda_2 + \lambda_1 \int_0^{1/2}\int_0^s L^*(\tau)d\tau ds + (1-\lambda_1)\int_0^1\int_0^s L^*(\tau)d\tau ds < 1$$

with

$$L^*(t) = [A(t) + B(t)]\Big\{1 + \int_0^1 \Big[\frac{1}{8}A(s)((s^2+1)^2 + 4) + B(s) + C(s)\Big]ds\Big\},$$

then problem (5.2) has a unique solution in the sector $[z_0, y_0]_*$ by Theorem 5.2.

Let $\lambda_1 = \lambda_2 = 1/4$, $A(t) = B(t) = C(t) = \alpha$, $t \in J$. Then $L^*(t) = \frac{\alpha(30+7\alpha)}{15}$. If we take $0 \leq \lambda_3 \leq \frac{3}{32}$ and $0 < \alpha \leq 1/3$, then conditions (5.3) and (5.4) are satisfied.

***Theorem 5.3*** *Let assumptions* $(A_1)$, $(A_4)$ *hold. Assume the following:*

$(A'_2)$ $f$ *is nonincreasing with respect to the last two variables,* $k \geq 0$.

$(A'_3)$ $y_0, z_0 \in C^2(J, \mathbb{R})$ *satisfy the system*

$$\begin{cases} y_0''(t) \geq \mathcal{F}(z_0,z_0,z_0,y_0)(t),\ t \in J, & y_0'(0) \geq k,\ 0 \geq G(y_0,y_0,z_0,w), \\ z_0''(t) \leq \mathcal{F}(y_0,y_0,y_0,z_0)(t),\ t \in J, & z_0'(0) \leq k,\ 0 \leq G(z_0,y_0,z_0,\bar{w}), \end{cases}$$

*and* $z_0(t) \leq y_0(t)$, $t \in J$.

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($A'_4$) *$y_0, z_0$ are nondecreasing, $f(t,u,v) \geq 0$ for $t \in J$,*

$$z_0(\beta(t, z_0(t)) \leq u \leq y_0(\beta(t, y_0(t)), \ z_0(\gamma(t, y_0(t)) \leq v \leq y_0(\gamma(t, z_0(t)), t \in J.$$

($A'_5$) *Assumptions* ($A_6$) *and* ($A_7$) *hold,*

($A'_6$) *There exist functions $L_1, L_2, M_1, M_2 \in C(J, R_+)$, such that*

$$f(t, u_1, v_1) - f(t, \bar{u}_1, \bar{v}_1) \leq L_1(t)(\bar{u}_1 - u_1) + L_2(t)(\bar{v}_1 - v_1),$$
$$\beta(t, \bar{v}) - \beta(t, v) \leq M_1(t)(\bar{v} - v)$$
$$\gamma(t, w) - \gamma(t, \bar{w}) \leq M_2(t)(\bar{w} - w)$$

*if* $\min_{t \in J} z_0(t) \leq u_1 \leq \bar{u}_1 \leq \max_{t \in J} y_0(t)$, $\min_{t \in J} z_0(t) \leq v_1 \leq \bar{v}_1 \leq \max_{t \in J} y_0(t)$, $z_0(t) \leq v \leq \bar{v} \leq y_0(t)$, $z_0(t) \leq w \leq \bar{w} \leq y_0(t)$, $t \in J$.

($A'_7$) *Assumption* ($A_9$) *holds with $N_1 = k + \int_0^T \mathcal{F}(z_0, y_0, z_0, y_0)(s)ds$.*

*Then problem* (5.1) *has a unique solution in the sector* $[z_0, y_0]_*$.

The sequences $\{y_n, z_n\}$ are now defined by

$$\begin{cases} y''_{n+1}(t) = \mathcal{F}(z_n, z_n, z_n, y_n)(t), \\ y_{n+1}(0) = k, \\ y_{n+1}(0) = y_n(0) + G(y_n, y_n, z_n, w), \end{cases} \qquad \begin{cases} z''_{n+1}(t) = \mathcal{F}(y_n, y_n, y_n, z_n)(t), \\ z_{n+1}(0) = k, \\ z_{n+1}(0) = z_n(0) + G(z_n, y_n, z_n, \bar{w}) \end{cases}$$

for $t \in J$, $n = 0, 1, \ldots$. The proof is similar to the proof of Theorem 5.2 and therefore it is omitted.

***Remark 3*** We can also discuss problem (5.1) assuming that, for example, $f(t,u,v)$ is nondecreasing with respect to $u$ and nonincreasing with respect to $v$. Note that in problem (1.1) there can be more arguments of type $\beta$ and $\gamma$.

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