

ON THE EMBEDDING INTO A RING OF AN ARCHIMEDEAN l -GROUP

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We shall prove the following about the “ringification” ρA of [2] and [5] of an archimedean l -group A : (a) Any “minimal ring” containing A is ρA ; (b) $A \mapsto \rho A$ is a reflector; (c) ρA need not be laterally complete when A is. These constitute the solutions to the problems posed in [2] by Paul Conrad.

1. The embedding into a ring. Let \mathcal{L} be the category which has objects archimedean l -groups A with distinguished positive weak unit e_A , and morphisms l -group homomorphisms $h: A \rightarrow B$ with $h(e_A) = e_B$. Let \mathcal{R} be the category with objects archimedean f -rings R with identity 1_R which is a weak unit, and morphisms l -ring homomorphisms $h: R \rightarrow S$ with $h(1_R) = 1_S$.

1.1 THEOREM. *If $A \in \mathcal{L}$, then there is $\rho A \in \mathcal{R}$ and an \mathcal{L} -embedding $\rho_A: A \rightarrow \rho A$ with the universal mapping property: If $h: A \rightarrow R$ is an \mathcal{L} -morphism, with $R \in \mathcal{R}$, then there is a unique \mathcal{R} -morphism $\rho h: \rho A \rightarrow R$ with $(\rho h) \circ \rho_A = h$.*

It seems appropriate to credit this theorem to Conrad and the present authors: In [2], Conrad creates a “ringification” $c_A: A \rightarrow cA$ by embedding A into its essential closure $D(Q)$ (Q being the Stone space of the polar algebra of A) with $e_A \mapsto 1$; then cA is the subring generated by A . He then shows that any f -ring B (archimedean or not) in which A is large (see § 4) with e_A an identity for B , is essentially cA ; this uniqueness is a weak version of the mapping property of 1.1. (Actually, Conrad proceeded more generally, with a fixed order basis rather than simply a weak unit; this will not concern us at present). In ignorance of [2], we proved Theorem 1.1 in [5], but assuming scalar multiplication on the groups and rings. The appendix to [5] discusses these things, and points out that the scalar multiplication is not needed.

We shall see in 3.2 below, that for each A , $(\rho_A, \rho A) \cong (c_A, cA)$. The consequences are discussed there.

Remark on reflections. There is the obvious forgetful functor $F: \mathcal{R} \rightarrow \mathcal{L}$, and 1.1 says exactly that F has a left adjoint ρ for which the adjunctions ρ_A are embeddings. In fact, F is one-to-one. (I.e., each \mathcal{L} -object admits at most one compatible multiplication making the weak unit the ring identity; this is proved in 2.2 of [2], and again, differently, in §4 of [5].) This permits inter-

preting \mathcal{R} as a subcategory of \mathcal{L} , and with this interpretation, 1.1 says that \mathcal{R} is reflective in \mathcal{L} . (Terminology from, e.g., [6].)

While we shall use this terminology below— ρA will be called the \mathcal{R} -reflection of A , and ρ_A the *reflection embedding*—we would like to point out that 1.1 and (a) of the introduction (= 3.1 below) permit another proof of the uniqueness of multiplication: Given $R \in \mathcal{R}$, the identity $1_R: R \rightarrow R$ is a “minimal \mathcal{L} -embedding” of $R \in \mathcal{L}$ into $R \in \mathcal{R}$, hence by 3.1 has the universal mapping property of 1.1. Thus, if R' denotes the \mathcal{L} -object R with another multiplication, we have

$$\begin{array}{ccc} R & \xrightarrow{1_R} & R \\ \downarrow 1_R & & \swarrow f \\ R' & & \end{array}$$

for a unique \mathcal{R} -morphism f . Clearly $f = 1_R$, which says the multiplications of R' and R are the same.

2. Point-separating representation. Wide use has been made of various representations of archimedean l -groups and rings as structures of continuous extended-real-valued functions defined over spaces of ideals or something similar. In our proof [5] of 1.1 (sketched in § 3) we employed a version of the Yosida representation [9] (which, it seems to us, has been under-exploited), and we shall need it further here.

Given $A \in \mathcal{L}$, let X_A denote the space of ideals of A (also called *solid subgroups*) which are maximal with respect to the property of not containing the weak unit e_A , given the hull-kernel topology.

If X is any topological space, let

$$D(X) = \{f: X \rightarrow [-\infty, +\infty] \mid f \text{ continuous, } f^{-1}(-\infty, +\infty) \text{ dense}\}.$$

2.1 THEOREM. *If $A \in \mathcal{L}$, then X_A is non-void compact Hausdorff, and there is an \mathcal{L} -isomorphism $A \mapsto \hat{A}$ onto an “ l -subgroup of $D(X_A)$ ” such that \hat{e}_A is the function constantly 1 and \hat{A} separates the points of X_A .*

If $A \mapsto \bar{A}$ is an \mathcal{L} -isomorphism onto an l -subgroup of $D(X)$, with X compact Hausdorff, such that $\bar{e}_A \equiv 1$ and \bar{A} separates points, then there is a homeomorphism $\tau: X \rightarrow X_A$ such that $\hat{a} \circ \tau = \bar{a}$ for each $a \in A$.

2.2 THEOREM. *The representation $A \mapsto \hat{A} \subset D(X_A)$ of 2.1 has the property: If $h: A \rightarrow B$ is an \mathcal{L} -morphism, then there is a unique continuous map $\tau: X_B \rightarrow X_A$ such that $h(a)^\wedge = \hat{a} \circ \tau$ for each $a \in A$.*

The proofs of 2.1 and 2.2 are in [5] for vector lattices; the necessary modifications (described in [5]) are easy.

2.3 PROPOSITION. *With h and τ as in 2.2, h is one-to-one if and only if τ is onto.*

Proof. We need to know that (for any A), \hat{A} separates points from closed sets in X_A (equivalently, the sets $\text{coz } \hat{a} = \{p | \hat{a}(p) \neq 0\}$ form a base). This follows by a compactness argument from the point-separating. Then, $\tau(X_B) \neq X_A$ is equivalent to the existence of $a \in A$, $a \neq 0$, with $(\text{coz } \hat{a}) \cap \tau(X_B) = \emptyset$. For such a , we have $\hat{a}(\tau(x)) = 0$ for each x , hence $h(a) = 0$ and h is not one-to-one. The converse is now clear.

3. Strong uniqueness of ρA . The principal result of this section (and the paper) is

3.1 THEOREM. *Let $A \in \mathcal{L}$, $R \in \mathcal{R}$, let $h : A \rightarrow R$ be an \mathcal{L} -embedding, and suppose there is no \mathcal{R} -object properly between $h(A)$ and R . Then there is an \mathcal{R} -isomorphism $i : \rho A \rightarrow R$ with $i \circ \rho_A = h$.*

This is the exact statement of (a) of the introduction. Before proceeding to the proof, we give some corollaries.

3.2 COROLLARY. *For each $A \in \mathcal{L}$, Conrad's ringification (c_A, cA) and $(\rho_A, \rho A)$ (of 1.1) are related by a commuting \mathcal{R} -isomorphism (per 3.1).*

Proof. Conrad's construct satisfies the condition in 3.1, by 1.1 of [2].

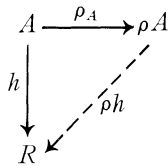
3.3 COROLLARY. *For each $A \in \mathcal{L}$, Conrad's (c_A, cA) has the universal mapping property of 1.1.*

Proof. By 3.2 and 1.1.

Thus referring to the questions at the end of [2]: 3.1 and 3.2 yield an affirmative answer to (1); 3.1 and 3.3 yield an affirmative answer to (2).

We now present the proof to 3.1, in several steps.

3.4 Given $A \in \mathcal{L}$, consider a diagram



with $A \xrightarrow{h} R$ as in the hypotheses of 3.1; the unique \mathcal{R} -morphism ρh with $(\rho h) \circ \rho_A = h$ exists by 1.1. Thus, the image $(\rho h)(\rho A) \in \mathcal{R}$, and by minimality, ρh is onto. We seek to prove that ρh is 1 - 1. Then ρh will be the \mathcal{R} -isomorphism of 3.1.

Applying 2.2 to the above diagram yields the “dual” commuting diagram

$$\begin{array}{ccc} X_A & \xleftarrow{\alpha} & X_{\rho A} \\ \beta \uparrow & & \nearrow \gamma \\ X_R & & \end{array}$$

in the category of compact Hausdorff spaces. (Here, $\hat{a} \circ \alpha = \rho_A(a)^\wedge$ for each $a \in A$; $\hat{a} \circ \beta = h(a)^\wedge$ for each $a \in A$; $\hat{b} \circ \gamma = (\rho h)(b)$ for each $b \in \rho A$.) According to 2.3, α and β are onto because ρ_A and h are one-to-one; and, ρh will be one-to-one if γ is onto. We prove this in two steps:

3.5 LEMMA. *Consider a commuting diagram*

$$\begin{array}{ccc} X & \xleftarrow{\alpha} & Y \\ \beta \uparrow & & \nearrow \gamma \\ Z & & \end{array}$$

in compact Hausdorff spaces. If α is irreducible and β is onto, then γ is onto.

(A continuous function is called *irreducible* if it is onto, and maps *no* proper closed subspace of the domain onto the range.)

Proof. $\gamma(Z)$ is compact, hence closed. If γ were not onto, then $\gamma(Z)$ would be a proper closed subspace; hence $\alpha(\gamma(Z)) \neq X$, because α is irreducible. But $\alpha(\gamma(Z)) = \beta(Z) = X$, because β is onto.

3.6 PROPOSITION. *For each $A \in \mathcal{L}$, the continuous map $X_A \xleftarrow{\alpha} X_{\rho A}$ which “induces” the reflection embedding $A \xrightarrow{\rho A} \rho A$ per 2.2 is irreducible.*

Proof. We need to recall the construction of ρA from § 6 of [5]:

Let L be the set of principal ideals (= solid subgroups) which contain the weak unit e_A , directed by set inclusion. For each $I \in L$, ρI is constructed (below) with the universal mapping property of 1.1. Then, if $J \supset I$, the inclusion “lifts” to an \mathcal{R} -morphism $\varphi_I^J : \rho I \rightarrow \rho J$, which is shown to be one-to-one. This process provides a direct limit system in \mathcal{R} . ρA is defined as

$$\varinjlim \{\rho I \mid I \in L\} \quad (\text{the direct limit in } \mathcal{R}),$$

and the reflection embedding $\rho_A : A \rightarrow \rho A$ results from the fact that $A = \bigcup \{I \mid I \in L\}$.

For $I \in L$, the construction of ρI is this: For some $a \geq e_A$, we have $I = \{b \in A \mid |b| \leq na \text{ for some integer } n\}$. Letting $R_a = \hat{a}^{-1}(-\infty, +\infty)$, each

$b \in I$ is real-valued on R_a , and we may view \hat{I} as a subset of $C(R_a)$ (the continuous real-valued functions on R_a). Since $C(R_a) \in \mathcal{R}$, there is a least " \mathcal{R} -subobject" of $C(R_a)$ which contains I ; this is taken as ρI , with $I \rightarrow \hat{I} \subset \rho I$ the reflection embedding.

Now I has weak unit e_A and \hat{I} separates the points of X_A . By 2.1, $X_I = X_A$. Since $\rho I \subset C(R_a)$, there is a compactification of R_a over which all functions in ρI extend (with values in $[-\infty, +\infty]$) and with the extensions separating points (take a quotient of the Stone-Ćech compactification βR_a). By 2.1, this is $X_{\rho I}$. Evidently, there is a map $\alpha_I : X_{\rho I} \rightarrow X_I = X_A$ extending the inclusion $R_a \subset X_A$.

For $J \supset I$, $\varphi_I^J : \rho I \rightarrow \rho J$ is, by 2.2, "dually induced" by a map $\alpha_I^J : X_{\rho J} \rightarrow X_{\rho I}$. We now have an inverse limit system of compact Hausdorff spaces.

Let

$$Z = \lim_{\leftarrow} \{X_{\rho I} | I \in L\}.$$

For $I \in L$, let $\Pi_I : Z \rightarrow X_{\rho I}$ be the projection and let $\alpha_I \circ \Pi_I \equiv \alpha : Z \rightarrow X_I = X_A$. This is easily seen to not depend on the choice of I . We now shall show that α is irreducible, and then that $Z = X_{\rho A}$.

Let F be a proper closed set in Z . We may as well suppose that $F = Z - U$, where U is nonvoid and basic, of the form

$$Z \cap \bigcap_{i=1}^n \Pi_{I_i}^{-1}(U_i),$$

U_i open in $X_{\rho I_i}$. For each i , let a_i be a generator for I_i with $a_i \geq e_A$, and let I be generated by $a_1 \vee \cdots \vee a_n$. Since $I \supset I_i$ for each i , U collapses to $Z \cap \Pi_I^{-1}(G)$, where

$$G = \bigcap_{i=1}^n (\alpha_{I_i})^{-1}(U_i).$$

Then

$$\alpha(F) = \alpha_I \left(\prod_I (F) \right) = \alpha_I \left(\prod_I (Z - U) \right) \cong \alpha_I (X_{\rho I} - G).$$

Now $\alpha_I (X_{\rho I} - R_a) = X_A - R_a$ ($a = a_1 \vee \cdots \vee a_n$) by 6.11 of [3]. Since $G \cap R_a \neq \emptyset$, $\alpha_I (X_{\rho I} - G)$ misses some points of R_a , hence is a proper subset of X_A . Thus, α is irreducible.

To see that $Z = X_{\rho A}$, we create a point-separating representation of ρA on Z and use 2.2. If $f \in \rho A$, then $f \in \rho I$ for some I and is viewed as a $[-\infty, +\infty]$ -valued function on $X_{\rho I}$. Let $\hat{f} = f \circ \Pi_I$; this is independent of I . We want to show that

$$\hat{f}^{-1}(-\infty, +\infty) = \prod_I^{-1}(f^{-1}(-\infty, +\infty))$$

is dense, and it is easy to check that irreducible maps inversely preserve dense sets. So, it suffices that Π_I be irreducible. But $\alpha_I \circ \Pi_I = \alpha$ is irreducible,

and so is α_I (from the end of the last paragraph); thus \prod_I must be. To see that the \hat{f} 's separate points, suppose $p \neq q$ in Z . For some coordinate I , $p_I \neq q_I$, and there is $f \in \rho I$ with $f(p_I) \neq f(q_I)$; then $\hat{f}(p) \neq \hat{f}(q)$.

The proof of 3.6 is complete.

4. On irreducibility and strong uniqueness. We make a few remarks on the material of the preceding section. The first gives an algebraic version of the irreducibility condition, and the second points out a generalization of 3.1.

4.1 PROPOSITION. *Let $h : A \rightarrow B$ be an \mathcal{L} -embedding. Then, $h(A)$ is large in B if and only if the map $\alpha : X_B \rightarrow X_A$ “dually inducing” h (per 2.2) is irreducible.*

(“Large” means that each nonzero ideal of B meets $h(A)$ non-trivially, or that whenever $0 < b \in B$, then there are $a \in A$ and an integer n , with $0 < h(a) < nb$.)

Proof. First, irreducibility of α is easily seen to be equivalent to: For O open and nonvoid in X_B , there is an open nonvoid V in X_A with $\alpha^{-1}(V) \subset O$. The sets O and V may be taken to be basic; a convenient basis for an X_A is $\{\text{coz } \hat{a} | a \in A^+\}$ (as in the proof of 2.2).

So, let $h(A)$ be large in B , let $0 = \text{coz } \hat{b}$ be given and choose a and n with $0 < h(a) < nb$. Obviously, $\text{coz } h(a)^\wedge \subset \text{coz } (nb)^\wedge = \text{coz } \hat{b}$. But $h(a)^\wedge = \hat{a} \circ \alpha$, and $\text{coz } h(a)^\wedge = \alpha^{-1}(\text{coz } \hat{a})$. Thus $V = \text{coz } \hat{a}$ has $\alpha^{-1}(V) \subset O$.

Conversely, let α be irreducible and let $0 < b \in B$. Choose n and open O with $\hat{b}|O \geq 1/n$. Now take $V = \text{coz } \hat{a}_1$ ($a_1 \in A^+$) with $\alpha^{-1}(V) \subset O$, and let $a = a_1 \wedge e_A$ (so $\hat{a} = \hat{a}_1 \wedge 1$). Then $\text{coz } h(a)^\wedge = \alpha^{-1}(\text{coz } \hat{a}) = \alpha^{-1}(\text{coz } \hat{a}_1) \subset O \subset \text{coz } \hat{b} = \text{coz } (nb)^\wedge$. Clearly, $0 < h(a) < nb$.

Combining 4.1 with the obvious generalizations of 3.4 and 3.5 (whose proofs work here), we have

4.2 PROPOSITION. *Let \mathcal{S} be a subcategory of \mathcal{L} , and let $A \in \mathcal{L}$ have an \mathcal{S} -reflection $S_A : A \rightarrow sA$ with S_A an \mathcal{L} -embedding. If $S_A(A)$ is large in sA , then any \mathcal{L} -embedding $h : A \rightarrow S$ for which there is no \mathcal{S} -object properly between $h(A)$ and S is, up to \mathcal{S} -isomorphism (as in 3.1), the \mathcal{S} -reflection.*

Remarks. (a) We do not suppose \mathcal{S} is reflective, only that there be $S_A : A \rightarrow sA$ ($sA \in \mathcal{S}$) with the required universal mapping property (as in 1.1).

(b) Conrad showed that for his (c_A, cA) , $c_A(A)$ is large in cA . But without knowing reflectivity (one of Conrad's questions), we don't see how to get the conclusion of 4.2 (another of his questions).

The conclusion of 4.2 (or 3.1) seems a striking property. It is well known to fail for “almost reflections” like Dedekind or lateral completion, and is very uncommon for *topological* embedding — reflections. (The only such examples we know are in Hausdorff uniform spaces; those *epi*-reflective subcategories

which contain all complete spaces.) On the other hand, we know of no embedding—reflection in \mathcal{L} for which it fails.

4.3 *Question.* It it true that whenever \mathcal{S} is an embedding-reflective subcategory of \mathcal{L} , then for each reflection $S_A : A \rightarrow sA$, $S_A(A)$ is large in sA ?

5. Lateral completeness. We present two examples of laterally complete l -groups A for which ρA fails to be laterally complete; thus Conrad's question (3) is answered negatively. The first example below was contributed by the referee, who called it “the discrete version of the (second) example . . .”; this is quite simpler then the second example. The second example (our original one) is, however, a vector lattice, and since we don't see how to get a simpler vector lattice example, we include it here. (These examples are also orthocomplete with ρA failing to be (the first directly, the second by [8]); this was part of Conrad's question).

5.1. *Example.* Let T be the subgroup of the additive reals R generated by 1 and π . For each i , let T_i be a copy of T and let

$$A = \prod_{i=1}^{\infty} T_i.$$

Then A is laterally complete, but the subring A' of $\prod_{i=1}^{\infty} R_i$ generated by A is not laterally complete. And $A' = \rho A$ either by 3.1 or by 1.1 of [2] (since A is large in $\prod_{i=1}^{\infty} R_i$).

5.2. *Example.* Let A be the lateral completion of the vector lattice P of continuous piecewise linear functions on the unit interval. We shall see that A is not laterally complete. We require a description of A .

Let X be the projective cover of the unit interval [4]: X is compact, extremally disconnected, and there is an irreducible map $\alpha : X \rightarrow [0, 1]$. (Or, X is the Stone space of the Boolean algebra of polars of P .) A familiar [7; 1] representation of P is created, with

$$\bar{P} = \{p \circ \alpha \mid p \in P\}.$$

(As in § 3, the sets $(P \circ \alpha)^{-1}(-\infty, +\infty)$ are dense, by irreducibility of α .) Then the lateral completion A emerges as the set of all $f \in D(X)$ such that for some disjoint family \mathcal{U} of clopen sets with $\cup \mathcal{U}$ dense, $f|U \in \bar{P}$ for each $U \in \mathcal{U}$ (Theorem 11 of [8]).

The description of these functions can be simplified a bit: First, such \mathcal{U} 's are at most countable because $[0, 1]$ has the countable chain condition, and irreducible maps inversely preserve this property. Second, then, if $f \in A$ there is $\mathcal{U} = \{U_n\}$ countable, and for each n , $P_n \in P$ such that $f|U_n = \bar{P}_n|U_n$. Associated with P_n is a “partition” of $[0, 1]$, say $0 = x_0 < x_1 < \cdots < x_{k+1} = 1$, such that on each $[x_i, x_{i+1}]$, p_n is linear. Then $C_i^n = \text{cl}_X \alpha^{-1}(x_i, x_{i+1})$ is clopen in X (by external disconnectivity). Thus, for each i , there is *linear* q_i , such

that $\overline{P_n}|C_i^n = q_i|C_i^n$. So, the system $\mathcal{V} = \{U_n \cap C_i^n\}_{n,i}$ consists of disjoint clopen sets, the union is still dense, and for each $V \in \mathcal{V}$ there is a linear q_V such that $f|V = \overline{q_V}|V$.

Changing the notation, we have the following characterizing description of those $f \in A$: There is a countable dense disjoint family $\{U_n\}$ of clopen sets, and for each n , linear $q_n \in P$, such that $f|U_n = \overline{q_n}|U_n$.

By extremal disconnectivity, $D(X) \in \mathcal{R}$, so by 3.1 or by § 1 of [2] ρA is the smallest \mathcal{R} -subobject (i.e., l -subring) of $D(X)$ which contains A . This, it is easy to see, consists of those $f \in D(X)$ for which there is a positive integer d and a countable, dense, disjoint family $\{U_n\}$ of clopen sets, and for each n a polynomial p_n of degree $\leq d$, with $f|U_n = \overline{p_n}|U_n$.

Thus we see that ρA is not laterally complete: Take any such infinite system $\{U_n\}$, let \mathcal{X}_n be the characteristic function of U_n , and for each n choose a polynomial p_n of degree $\geq n$. Then each $\mathcal{X}_n \overline{p_n} \in \rho A$ but $\bigvee_n \mathcal{X}_n \overline{p_n} \notin \rho A$.

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