Bull. Aust. Math. Soc. **87** (2013), 406–424
doi:10.1017/S0004972712000536

NEW REDUCTIONS AND LOGARITHMIC LOWER BOUNDS FOR THE NUMBER OF CONJUGACY CLASSES IN FINITE GROUPS

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(Received 27 April 2012; accepted 27 May 2012; first published online 7 September 2012)

Abstract

The unsolved problem of whether there exists a positive constant c such that the number $k(G)$ of conjugacy classes in any finite group G satisfies $k(G) \geq c \log_2 |G|$ has attracted attention for many years. Deriving bounds on $k(G)$ from (that is, reducing the problem to) lower bounds on $k(N)$ and $k(G/N)$, $N \trianglelefteq G$, plays a critical role. Recently Keller proved the best lower bound known for solvable groups:

$$k(G) > c_0 \frac{\log_2 |G|}{\log_2 \log_2 |G|} \quad (|G| \geq 4)$$

using such a reduction. We show that there are many reductions using $k(G/N) \geq \beta[G:N]^\alpha$ or $k(G/N) \geq \beta(\log[G:N])^t$ which, together with other information about G and N or $k(N)$, yield a *logarithmic* lower bound on $k(G)$.

2010 *Mathematics subject classification*: primary 20D60.
Keywords and phrases: number of conjugacy classes, solvable groups.

1. Introduction

Let $k(G)$ denote the number of conjugacy classes of the finite group G. Answering a question of Frobenius, E. Landau observed in 1903 that for a fixed k_0 only a finite number of finite groups G satisfy $k(G) = k_0$. In 1968 Erdős and Turán [ET] (and independently Newman [Ne]) made this explicit by proving that $k(G) > \log_2 \log_2 |G|$ always holds. Ongoing since around 1910, the classification of finite groups according to their number of conjugacy classes is now complete for $k \leq 14$ [VS, VV1, VV2]. Of the more than 350 nonisomorphic groups with $k(G) \leq 14$, 25 satisfy $k(G) < \log_2 |G|$. Exactly five of the latter are solvable, and these satisfy $k(G) > \frac{4}{5} \log_2 |G|$. In fact *all* groups with $k(G) \leq 14$ satisfy $k(G) > \log_3 |G|$, and thus $k(G) > \log_3 |G|$ whenever $|G| \leq 3^{15}$. Perhaps $k(G) > \log_3 |G|$ for all finite groups G.

A simple induction beginning with $k(G/Z(G))$ shows that $k(G) > \log_2 |G|$ whenever G is nilpotent. In 1985 Cartwright [Ca] proved that $k(G) \geq \frac{3}{5} \log_2 |G|$ when G is supersolvable, but there are important families of groups, for example Frobenius groups as well as G with $|G| = p^\alpha q^\beta$ or G' nilpotent, for which the best known bound so far is $k(G) > c \log_2 |G|/\log_2 \log_2 |G|$. On the other hand, for each prime p there is a p-group P of order p^p with $k(P) < (\log_2 |P|)^3$. But no collection $\{G\}$ is known with $|G| \to \infty$ and $k(G) < (\log_2 |G|)^2$. See [Be3] for a more complete history and bibliography.

Keller [Ke] proved in 2011 the best general lower bounds to date, improving on those of Pyber [Py] 20 years ago. Keller proved that:

(i) there exists an (explicitly computable) constant $\epsilon_1 > 0$ such that for *every* finite group G with $|G| \geq 4$,

$$k(G) > \frac{\epsilon_1 \log_2 |G|}{(\log_2 \log_2 |G|)^7}.$$

Moreover:

(ii) if G is *solvable* ($|G| \geq 4$), then

$$k(G) > \frac{\epsilon_1 \log_2 |G|}{\log_2 \log_2 |G|}.$$

Pyber had obtained like bounds with exponent 8 instead of 7 in (i), and a denominator of $(\log_2 \log_2 |G|)^3$ in (ii) (see [Be3]). One of the main results underlying these improvements is when G is solvable and the Frattini subgroup $\Phi(G) = 1$. Here Pyber proved that (when $|G| \geq 4$)

$$k(G) \geq |G|^{\gamma/(\log_2 \log_2 |G|)^2},$$

where γ is a positive constant ($\gamma < 2^{-12}$). Keller's improvement finds a polynomial lower bound when $\Phi(G) = 1$: $k(G) \geq |G|^\beta$, where β is a positive constant.

2. Background and preliminaries

Perhaps the simplest reduction arises when G itself is nilpotent. Here $Z(G) \neq 1$ and G/Z is nilpotent. For any group G and $N \trianglelefteq G$, $k(G) = k_G(G - N) + k_G(N) \geq k(G/N) + k_G(N) - 1$, where $k_G(S)$ is the number of G-conjugacy classes that partition the normal subset S, so $k(G) \geq k(G/Z) + |Z| - 1$. Thus if $k(G/Z) \geq \log_2[G : Z]$, then $k(G) \geq \log_2 |G|$. When G is supersolvable, Cartwright [Ca] began his proof that $k(G) \geq \frac{3}{5} \log_2 |G|$ with a reduction lemma having the hypothesis that $k(G/M) \geq \frac{3}{5} \log_2[G : M]$ and $k(G/N) \geq \frac{3}{5} \log_2[G : N]$, assuming the existence of certain normal subgroups M, N of G. Next is Pyber's reduction lemma, which also plays a key role in Keller's recent results mentioned above (here $\log(\cdot) = \log_2(\cdot)$):

Lemma 2.1 [Py, Lemma 2.2]. *Let G be any group ($|G| \geq 4$) and $N \trianglelefteq G$ with N nilpotent. If $k(G/N) \geq 2^{x(\log[G:N])^{1/t}}$ for constants $0 < x \leq 1$, $t \geq 1$, then*

$$k(G) \geq \left(\frac{x^t}{2}\right) \frac{\log |G|}{(\log \log |G|)^t}.$$

From Pyber's lemma with $t = 1$ we conclude that when N is a nilpotent normal subgroup of G and $k(G/N) \geq [G:N]^\alpha$ $(0 < \alpha \leq 1)$, then $k(G) \geq \alpha/2 \log_2 |G|/\log_2 \log_2 |G|$. As we will see, there are general situations where $N \trianglelefteq G$ and $k(G/N) \geq \beta[G:N]^\alpha$ (or even $k(G/N) \geq \beta(\log[G:N])^t$) which, together with other information about N or $k(N)$, yield *logarithmic* lower bounds for $k(G)$.

In 2004 (see [Be3]) the author presented several general 'logarithmic reductions' including [Be3, Lemma 4.5]. Suppose that $N \trianglelefteq G$, $\alpha, \beta > 0$ and $(\beta/(1+\alpha))^{\beta/(1+\alpha)} \leq b$ (the base of the logarithm). If:

(i) $k(N) \geq |N|^\alpha$;
(ii) $k(G/N) \geq \beta \log[G:N]$; and
(iii) $|G|^{\alpha-((1+\alpha)/\beta)} \geq \log |G|$,

then $k(G) \geq \log |G|$. But (iii) implies that $\beta/(1+\alpha) > 1/\alpha$, so the smaller α is the larger the base b. Here we remove any relation between b and the other parameters (except in one useful situation). We know that when $|G| \leq 3^{15}$ then $k(G) > \log_3 |G|$, so assuming that $|G|$ is large is natural. But the requirement of (iii) that $\beta > (1+\alpha)/\alpha$ and that $|G|$ be 'large enough' depending on α, β will be avoided in many important situations.

We often use the following lemma.

Lemma 2.2.

(a) *When G is solvable, $F'(G) \leq \Phi(G) < F(G)$* [Hu, III, 3.11, 4.2].
(b) *If $|G| = \prod p_i^{\alpha_i}$ and $s = \max\{\alpha_i\} \geq 3$, then the nilpotence class $c(\Phi) \leq (s-1)/2$* [HP].
(c) *If G is nilpotent with nilpotence class c, then $k(G) \geq c|G|^{1/c} - c + 1$* [Sh].

We will also use results from [Be2, Be3]. The first part of Lemma 2.3(a) also appears in [Ca].

Lemma 2.3. *Suppose that $N \trianglelefteq G$. Then:*

(a) $k(G) \geq k(G/N) + (k(N)-1)/[G:N]$; *(Note that equality occurs if and only if G is a Frobenius group with kernel N.)*
(b) *if $k(N) \geq |N|^\alpha$ and $k(G/N) \geq [G:N]^\beta$ $(\alpha, \beta > 0)$, then $k(G) \geq |G|^{\alpha\beta/(\alpha+\beta+1)}$;*
(c) $k(G/N \cap G') = [N : N \cap G']k(G/N)$.

As mentioned, it follows from the classification of finite groups according to their number k of conjugacy classes (now complete for $k \leq 14$) that $k(G) > \log_3 |G|$ when $|G| \leq 3^{15}$. Using this and Lemma 2.3(a), we have the following corollary.

Corollary 2.4. *Suppose that $N \trianglelefteq G$, together with* (i) $k(N) \geq |N|^\alpha$ $(0 < \alpha \leq 1)$ *and* (ii) $k(G/N) \geq (1+\alpha)\log[G:N]$. *Then* (a) $k(G) \geq (\alpha - \log\log|G|/\log|G|)\log|G|$; *and* (b) *when $\log(\cdot) = \log_3(\cdot)$,* (i), (ii) *and* (iii) $|N|^{\alpha-c} \geq \log|N|$ $(0 < c < \alpha)$ *imply that* $k(G) \geq \min\{c, 0.39\}\log|G|$.

PROOF. (a) If $|N|^{1+\alpha} \geq |G| \log |G|$, then assumption (i) and Lemma 2.3(a) imply that $k(G) > k(N)/[G:N] \geq |N|^{1+\alpha}/|G| \geq \log |G|$. Otherwise $|N|^{1+\alpha} < |G| \log |G|$ and thus $(1+\alpha)\log |G| - (1+\alpha)\log |N| > \alpha \log |G| - \log\log |G|$. Also, (ii) and Lemma 2.3(a) yield $k(G) > k(G/N) \geq (1+\alpha)\log[G:N] \geq \alpha \log |G| - \log\log |G|$, so

$$k(G) > \left(\alpha - \frac{\log\log |G|}{\log |G|}\right)\log |G|.$$

(b) Since $\log_3 \log_3 n/\log_3 n$ decreases for $n \geq 20$, when $|N| \geq 20$ then (iii) yields

$$\alpha - c \geq \frac{\log_3 \log_3 |N|}{\log_3 |N|} > \frac{\log_3 \log_3 |G|}{\log_3 |G|}$$

that is,

$$\alpha - \frac{\log_3 \log_3 |G|}{\log_3 |G|} > c,$$

so from (a) $k(G) > c \log_3 |G|$. For $|N| \leq 19$, N is abelian ($\alpha = 1$) except possibly when $|N| = 2n$ ($3 \leq n \leq 9$). We check (for example, [VV1]) that for such N, $k(N) > |N|^{3/5}$ except when $N = \mathrm{Alt}(4)$, where $k(N) = 4 > |N|^{0.557}$. Since $|G| > 3^{15}$, $\log_3 \log_3 |G|/\log_3 |G| < 0.165$. Thus for $|N| \leq 19$, $k(N) \geq |N|^{\alpha}$, where

$$\alpha - \frac{\log_3 \log_3 |G|}{\log_3 |G|} > 0.557 - 0.165 > 0.39.$$

The conclusion follows. □

If G is nilpotent-by-nilpotent (both N and G/N are nilpotent), Keller's general result gives $k(G) \geq \epsilon_1 \log_2 |G|/\log_2 \log_2 |G|$, where ϵ_1 is a small constant. Given more information about $k(N)$, we improve this in many cases. Recall, for example, that when N is nilpotent of nilpotence class c, then $k(N) \geq |N|^{1/c}$ [Sh].

COROLLARY 2.5. *If $N \trianglelefteq G$, G/N is nilpotent and $k(N) \geq |N|^{\alpha}$ $(0 < \alpha \leq 1)$ then $k(G) \geq \alpha \log_b |G| - \log_b \log_b |G|$ $(b = 2^{4/3})$.*

PROOF. Since G/N is nilpotent, $k(G/N) \geq \frac{3}{2}\log_2[G:N]$ [Ca]. When $b = 2^{4/3}$, $\frac{3}{2}\log_2 n = 2\log_b n \geq (1+\alpha)\log_b n$, so assumptions (i) and (ii) of Corollary 2.4 are met. Thus $k(G) \geq \alpha \log_b |G| - \log_b \log_b |G|$. □

We return to G/N nilpotent and $k(N) \geq |N|^{\alpha}$ in Corollary 3.11(c), finding a logarithmic lower bound with coefficient depending on α, but *without* having to assume that $|G|$ is large enough, depending on α.

As mentioned earlier, the best bound known when G' is nilpotent is [Ca] $k(G) \geq \log_2 |G|/\log_2 \log_2 |G|$, but the author knows of no such example with $k(G) < \log_2 |G|$. Taking into account the prime factorisations of $|G|$ and $|G'|$, we note the following improvements. Corollary 2.6(a) generalises [Be3, Proposition 4.10(b)], removing any restriction on how large $|G|$ is.

Corollary 2.6. *Suppose that G' is nilpotent.*

(a) *If $|G| = \prod p_i^{\alpha_i}$ and $s := \max\{\alpha_i\}$, then $k(G) \geq |G|^{1/2s+1}$.*
(b) *If $|G'| = \prod p_i^{\beta_i}$ and $r := \max\{\beta_i\} \geq 2$, then $k(G) \geq |G|^{1/2r-1}$.*

Proof. (a) Since G' is nilpotent, $G'' \leq F' \leq \Phi(G)$ (Lemma 2.2(a)), so G/Φ is metabelian and, by [Be2], $k(G/\Phi) \geq [G : \Phi]^{1/3}$. By Lemma 2.2(c), $k(\Phi) \geq |\Phi|^{1/c}$ where c is the nilpotence class of Φ. If $s \leq 2$, then all Sylow subgroups of G' are abelian, G' is abelian and $k(G) \geq |G|^{1/3}$. If $s \geq 3$, then by Lemma 2.2(b), $k(\Phi) \geq |\Phi|^{2/s-1}$. Using Lemma 2.3(b) with $\alpha = 2/(s-1)$ and $\beta = 1/3$ gives the result.

(b) The nilpotence class $c(G')$ is the maximum of the classes of its Sylow p-subgroups. The class of a group of order p^n, $n \geq 2$, is at most $n-1$, so $c(G') \leq r-1$. Again by Lemma 2.2(c), $k(G') \geq |G'|^{1/c} \geq |G'|^{1/r-1}$. From Lemma 2.3(b) with $N = G'$, $\alpha = 1/(r-1)$ and $\beta = 1$, the result follows. □

Remark 2.7. When G is solvable, $|G| = \prod p_i^{\alpha_i}$ and $\alpha_i \leq 2$, each Sylow subgroup of G is abelian and the derived length $d(G) \leq 3$ [Ta]. Since $k(G) \geq |G|^{1/2^d-1}$ [Be2], here $k(G) > |G|^{1/7}$. Furthermore, when $n = \prod p_i^{\alpha_i}$ and $s(n) := \max\{\alpha_i\}$, Niven [Ni] proved that the *average order* of $s(n)$ lies between 1 and 2, that is, $\lim_{n\to\infty}(1/n)\sum_{j=1}^{n} s(j)$ is approximately 1.7.

3. New reductions

We begin with a general reduction related to Pyber's lemma above, when $t = 1$. Instead of assuming that N is nilpotent, we make an assumption on $k(N)$ which leads to the conclusion that $k(G) \geq \log|G|$ when $|G|$ is large enough. Unless otherwise noted, $\log(\cdot) = \log_b(\cdot)$, where $b \geq 2$. Note that Lemma 3.1 may be used when $N \geq G'$, putting $k(G/N) \geq \beta[G : N]^{\alpha}$ where for example $\alpha = 1/2, \beta = \sqrt{2}$ when $[G : N] = 2$ and $\alpha = 1 - 1/n, \beta = 1 + 1/n$ $(n \geq 2)$ when $[G : N] \geq 3$.

Lemma 3.1. *Suppose that $N \trianglelefteq G$, with*

(i) $k(G/N) \geq \beta[G : N]^{\alpha}$ *$(0 < \alpha < 1 < \beta)$ and*
(ii) $k(N) \geq (\log|N|)^{1+1/\alpha}$.

Then $k(G) \geq \log|G|$ for all $|G|$ large enough (depending only on α, β). In particular, when N is solvable and $\Phi(N)$ is abelian, together with (i), the conclusion follows for all $|G|$ large enough, depending only on α, β.

Proof. Since $k(G) > \max\{k(G/N), k(N)/[G : N]\}$, the conclusion follows from hypothesis (i) when $\beta[G : N]^{\alpha} \geq \log|G|$, that is, when $|N| \leq \beta^{1/\alpha}|G|/(\log|G|)^{1/\alpha}$. So we may assume that $|N| > \beta^{1/\alpha}|G|/(\log|G|)^{1/\alpha}$. From $k(G) > k(N)/[G : N]$ it follows from hypothesis (ii) that

$$k(G) > \frac{\beta^{1/\alpha}(\log(\beta^{1/\alpha}) + \log|G| - \frac{1}{\alpha}\log\log|G|)^{1+1/\alpha}}{(\log|G|)^{1/\alpha}}.$$

But $\beta^{1/\alpha} > \beta > 1$, so $k(G) > \log|G|$ as long as

$$\beta\Big(\log|G| - \frac{1}{\alpha}\log\log|G|\Big)^{1+\alpha} \geq (\log|G|)^{1+\alpha},$$

that is, when

$$\frac{\log|G|}{\log\log|G|} \geq \frac{1 + (\beta^{1/(\alpha+1)} - 1)^{-1}}{\alpha},$$

which is true for all sufficiently large $|G|$, depending only on α, β.

Now suppose that N is solvable and $\Phi(N)$ is abelian. From (i) we have $k(G) > k(G/N) > [G:N]^{\alpha} \geq \log|G|$, if $|N| \leq |G|/(\log|G|)^{1/\alpha}$. So assume that $|N| > |G|/(\log|G|)^{1/\alpha}$. Now $\Phi(N)$ is abelian, so [Be3, Proposition 2.3] $k(N) \geq (\log|N|)^{1+1/\alpha}$ (that is (ii) holds) when $|N|$ is large enough, and hence when $|G|$ is large enough, depending only on α. (This also follows from Keller's Theorem 3.1, applied to $N/\Phi(N)$, and Lemma 2.3(b) above with N replaced by $\Phi(N)$ and G replaced by N.) □

Corollary 3.2.

(a) *Suppose that $N \trianglelefteq G$ with G/N nilpotent of nilpotence class $c(G/N) \geq 2$ and $k(N) \geq (\log|N|)^{c+1}$. Then $k(G) \geq \log|G|$ as long as $|G|$ is large enough, depending only on c.*

(b) *Suppose that G' is nilpotent and $|G| \geq 2^{56}$.*

   (i) *If $k(\Phi(G)) \geq (\log_2|\Phi|)^4$, then $k(G) \geq \log_2|G|$.*

   (ii) *If $\Phi(G)$ has nilpotence class $c(\Phi) \leq \frac{1}{4}\log_2|\Phi|/\log_2\log_2|\Phi|$, then $k(G) \geq \log_2|G|$.*

Proof. (a) Since G/N is nilpotent of class c, $k(G/N) \geq c[G:N]^{1/c} - (c-1)$ [Sh], and the latter is greater than or equal to $(1+1/c)[G:N]^{1/c}$ for $c \geq 2$. In Lemma 3.1 set $\beta = 1 + 1/c$ and $\alpha = 1/c$. From the proof we see that $k(G) > \log|G|$ as long as $\log|G|/\log\log|G| \geq (1 + (\beta^{1/(1+\alpha)} - 1)^{-1})/\alpha$, that is,

$$\frac{\log|G|}{\log\log|G|} \geq \bigg(1 + \bigg(\Big(1 + \frac{1}{c}\Big)^{1/(1+1/c)} - 1\bigg)^{-1}\bigg)c.$$

(b) (i) When G' is nilpotent $G'' \leq F'(G) \leq \Phi(G)$ (Lemma 2.2(a)), so $(G/\Phi)'' = \{1\}$. Thus $k(G/\Phi) \geq (\frac{9}{2}[G:\Phi])^{1/3}$ [Be1], and the conclusion follows from Lemma 3.1 with $N = \Phi$, $\alpha = 1/3$ and $\beta = (9/2)^{1/3}$, after checking that $|G|$ is large enough. (ii) Since $k(\Phi) \geq |\Phi|^{1/c}$ (Lemma 2.2(c)) the conclusion follows from the assumed upper bound on $c(\Phi)$, and (i). □

When G' is nilpotent, so far we only know that $k(G) \geq \log_2|G|/\log_2\log_2|G|$ [Ca]. It is thus worthwhile to record further reduction theorems in this area which conclude that $k(G) \geq \log|G|$. We may assume that G' is not abelian, since then $k(G) \geq (\frac{9}{2}|G|)^{1/3}$ [Be1], and we will often need the following lemma.

Lemma 3.3 [Be3, Corollary 3.2(a)–(c)].

(a) *If* $N \trianglelefteq G$ *and* N *is nonabelian, then* $k_G(N) - 1 \geq 2|C_G(N)|/[G:N]$. *Thus for any* $N \trianglelefteq G$, $k_G(N) - 1 \geq (|N| - 1)/[G : C_G(N)]$.

(b) *If* G *is solvable and* N *is a minimal normal subgroup of* G *such that* (i) $k(G/N) \geq \log[G : N]$ *and* (ii) $[G : F] \leq (|N| - 1)/\log|N|$, *then* $k(G) \geq \log|G|$.

(c) *If* G' *is nilpotent and* N *is a minimal normal subgroup of* G *such that* $k(G/N) \geq \log[G : N]$ *and* $(|N| - 1)/\log|N| \geq \log|G|$, *then* $k(G) \geq \log|G|$.

Lemma 3.4. *Suppose that* $N \trianglelefteq G$, *with* $k(G/N) \geq (1 + \epsilon)\log[G : N]$ *and* $|N| \leq (\log|G|)^t$ $(\epsilon, t > 0)$. *Then* $k(G) \geq \log|G|$ *for* $|G|$ *large enough, depending only on* ϵ, t. *In fact, if also* $k(G) \geq t(1 + 1/\epsilon)\log\log|G|$ *then* $k(G) \geq \log|G|$, *without restriction on* $|G|$.

Proof. Since $k(G) > k(G/N) \geq (1 + \epsilon)\log[G : N]$, and $|N| \leq (\log|G|)^t$, we have $k(G) > (1 + \epsilon)(\log|G| - t\log\log|G|)$. The conclusion follows when the latter is greater than or equal to $\log|G|$, which is equivalent to $\log|G|/\log\log|G| \geq t(1 + 1/\epsilon)$. This is true for all large enough $|G|$, depending only on ϵ, t. And when it is false, assuming $k(G) \geq t(1 + 1/\epsilon)\log\log|G|$ yields $k(G) > \log|G|$. □

Theorem 3.5.

(a) *Suppose that* $C_G(G') \ntrianglelefteq G'$ *and* $k(G/C_G(G')) \geq \log[G : C_G(G')]$. *Then* $k(G) \geq \log|G|$ *when* $|G| \geq 2^{13}$ *(in base* 3 *we may assume that* $|G| > 3^{15}$*).*

(b) *Given* $\epsilon > 0$, *for all large enough solvable groups* G *(depending only on* ϵ*), if* $k(G/C_G(G')) \geq (1 + \epsilon)\log[G : C_G(G')]$ *then* $k(G) \geq \log|G|$.

Proof. (a) Since $C_G(G') \trianglelefteq G$, when $|C_G(G')| \leq |G|^{1/2}$ we are done using our assumptions on $C_G(G')$ and [Be3, Corollary 3.9(a)]. On the other hand, when $|C_G(G')| \geq |G|^{1/2}$, by Lemma 3.3(a),

$$k(G) > k_G(G') - 1 \geq \frac{2|C_G(G')|}{[G : G']} \geq \frac{2|G|^{1/2}}{[G : G']}.$$

But $k(G) \geq [G : G'] + 1$, so we may assume that $[G : G'] < \log|G| - 1$. Since $2|G|^{1/2} \geq \log_2^2|G| - 1$ for $|G| \geq 2^{13}$, we conclude that $k(G) > \log|G|$.

(b) Using Lemma 3.4, if $|C_G(G')| \leq (\log|G|)^2$ then our hypothesis yields $k(G) \geq \log|G|$. Next assume that $|C_G(G')| \geq (\log|G|)^2$. From Lemma 3.3(a),

$$k_G(C_G(G')) - 1 \geq \frac{|C_G(C_G(G'))|(|C_G(G')| - 1)}{|G|} \geq \frac{|C_G(G')| - 1}{[G : G']} \geq \frac{(\log|G|)^2 - 1}{[G : G']}.$$

Again, we may assume that $[G : G'] < \log|G| - 1$, so $k(G) > \log|G| + 1$. □

We have remarked that no collection of groups is known for which $|G| \to \infty$ and $k(G) < (\log|G|)^2$. If $0 < \delta < 1$, then for all $|G|$ large enough (depending only on δ) $k(G') > (\log|G'|)^2$ implies that $k(G) > \delta\log|G|$ [Be3, Lemma 3.5]. In Corollary 3.7 (another application of Lemma 3.1) we prove that, for each $n \geq 2$ and all $|G|$ large enough depending only on n, if $k(G^{(n)}) \geq (\log|G^{(n)}|)^{2^n}$ then $k(G) \geq \log|G|$. We must first extend a result of [Be2] that if G has derived length d, then $k(G) \geq |G|^{1/2^d - 1}$. Lemma 3.6 is a slight improvement over a result of M. Herzog communicated to the author.

LEMMA 3.6. *If G is a finite solvable group of derived length d, then*

$$k(G) \geq \left(\frac{3}{2} - \frac{1}{2^d}\right)|G|^{1/(2^d-1)}. \tag{3.1}$$

PROOF. If $d = 1$, then (3.1) holds with equality. In [Be1] we proved that $k(G) \geq (\frac{9}{2}|G|)^{\frac{1}{3}}$ when G is metabelian, and since $(\frac{9}{2})^{\frac{1}{3}} > 5/4$, (3.1) is true when $d = 2$. Thus we may suppose that $d \geq 3$ and (3.1) holds with $d - 1$ replacing d. Using our inductive assumption,

$$k(G') \geq \left(\frac{3}{2} - \frac{1}{2^{d-1}}\right)|G'|^{1/(2^{d-1}-1)}.$$

Lemma 2.3(a) with $N = G'$ yields

$$k(G) \geq [G : G'] + \frac{k(G')|G'|}{|G|} - \frac{1}{2}.$$

Setting $|G'| = x$, $|G| = g$ and

$$a = 1 + \frac{1}{2^{d-1} - 1}, \quad b = \frac{3}{2} - \frac{1}{2^{d-1}},$$

we arrive at

$$k(G) \geq \frac{g}{x} + \frac{b}{g}x^a - \frac{1}{2}. \tag{3.2}$$

Let $f(x) = (g/x) + (b/g)x^a$. Then $f'(x) = -(g/x^2) + (ab/g)x^{a-1}$, and since $f''(x) > 0$ for $x > 0$ the solution x_0 to $f'(x) = 0$ corresponds to a minimum for $f(x)$. From $f'(x_0) = 0$ we obtain $(g/x_0)^2 = abx_0^{a-1}$, that is, $x_0 = (g^2/ab)^{1/(a+1)}$. Thus $g/x_0 = (ab)^{1/(a+1)}g^{1-2/(a+1)} = (ab)^{1/(a+1)}g^{1/(2^d-1)}$. Furthermore,

$$\frac{b}{g}x_0^a = \frac{b}{g}\left(\frac{g^2}{ab}\right)^{a/(a+1)} = \frac{(ab)^{1/(a+1)}}{a}g^{1-2/(a+1)},$$

so, from (3.2), $k(G) \geq (ab)^{1/(a+1)}(1 + (1/a) - (1/2))g^{1/(2^d-1)}$.

It remains only to show that when $d \geq 3$, $(ab)^{1/(a+1)}(\frac{1}{2} + (1/a)) \geq \frac{3}{2} - 1/2^d$. First check that

$$ab = \left(1 + \frac{1}{2^{d-1} - 1}\right)\left(\frac{3}{2} - \frac{1}{2^{d-1}}\right) = \frac{(\frac{3}{2})2^{d-1} - 1}{2^{d-1} - 1},$$

which decreases to $\frac{3}{2}$ as $d \to \infty$. Also $1/(a + 1)$ increases as $d \to \infty$. Thus $(ab)^{1/(a+1)} > (\frac{3}{2})^{\frac{3}{7}} > \frac{9}{8}$, and finally

$$(ab)^{1/(a+1)}\left(\frac{1}{2} + \frac{1}{a}\right) > \frac{9}{8}\left(\frac{1}{2} + \frac{1}{a}\right) = \frac{9}{8}\left(\frac{3}{2} - \frac{1}{2^{d-1}}\right) > \frac{3}{2} - \frac{1}{2^d},$$

since $d \geq 3$. □

COROLLARY 3.7. *For each $n \geq 2$, let $\{G\}_n$ denote the class of solvable groups G for which $k(G^{(n)}) \geq (\log |G^{(n)}|)^{2^n}$. If $G \in \{G\}_n$ and $|G|$ is large enough (depending only on n), then $k(G) \geq \log |G|$.*

PROOF. Since $G/G^{(n)}$ has derived length n, by Lemma 3.6 we have $k(G/G^{(n)}) \geq (3/2 - 1/2^n)[G : G^{(n)}]^{1/2^n-1}$. In Lemma 3.1 set $N = G^{(n)}$, $\alpha = 1/2^n - 1$, and $\beta = 3/2 - 1/2^n$ (which is greater than 1 since $n \geq 2$). Also $1 + 1/\alpha = 2^n$, and hypotheses (i) and (ii) are satisfied. Thus $\log |G|/\log \log |G| \geq (2^n - 1)(1 + ((3/2 - 1/2^n)^{1-1/2^n} - 1)^{-1})$ yields $k(G) \geq \log |G|$. □

The logarithmic reductions in [Be3, Lemma 4.5 and Theorem 4.8], while assuming that $k(N) \geq |N|^\alpha$ and $k(G/N) \geq \beta \log[G : N]$, also require that $|G|$ be 'large enough', depending on the parameters involved. Theorem 3.9 below shows that by relating α, β and $[N : N \cap G']$ in a single inequality, the requirement that $|G|$ is large can be avoided. This has important consequences. First we need the following lemma.

LEMMA 3.8. *If $k(G/N) \geq \beta \log[G : N]$ and $\beta \geq \log |G|/\log \log |G|$, then $k(G) \geq \log |G|$.*

PROOF. We always have $k(G) \geq k(G/N)$, so we may assume (using our hypothesis) that $\beta \leq k(G/N)/\log[G : N] < \log |G|/\log [G : N]$. If $[G : N] \geq \log |G|$, it follows that $\beta < \log |G|/\log \log |G|$, contradicting our assumption. If $[G : N] < \log |G|$ then $\beta \leq [G : N]/\log [G : N] < \log |G|/\log \log |G|$, since $x/\log x$ increases for $x \geq 3$ and we may assume that $\log |G| > k(G) \geq 4$. Again $\beta < \log |G|/\log \log |G|$, contradicting our assumption. □

THEOREM 3.9. *Suppose that $N \trianglelefteq G$, with*

(i) $k(N) \geq |N|^\alpha$ *($0 < \alpha \leq 1$) and*
(ii) $k(G/N) \geq \beta \log[G : N]$ ($\beta > 0$).

If also either

(iii) $(\beta\alpha - 1)[N : N \cap G'] \geq 1 + \alpha$ *or*
(iv) $|G|^{\alpha-(1+\alpha)/\beta[N:N\cap G']} \geq \log |G|$,

then $k(G) \geq \log |G|$.

PROOF. From Lemma 2.3(a),

$$k(G) \geq k(G/N) + \frac{k(N) - 1}{[G : N]} > \frac{k(N)}{[G : N]},$$

so (i) yields $k(G) > |N|^{1+\alpha}/|G|$. By Lemma 3.8 and (ii) we may assume that $|G|^{1/\beta} \geq \log |G|$, so if $|N|^{1+\alpha}/|G| \geq |G|^{1/\beta}$ we are done. If $|N|^{1+\alpha} < |G|^{1+1/\beta}$

then $[G : N] > |G|^{(\alpha-1/\beta)/(\alpha+1)}$, so by Lemma 2.3(c) and (iii),

$$\begin{aligned} k(G) &\geq k(G/N \cap G') = [N : N \cap G']k(G/N) \\ &\geq \beta[N : N \cap G'] \log[G : N] \\ &> \beta[N : N \cap G']\Big(1 - \frac{1 + 1/\beta}{1 + \alpha}\Big) \log |G| \\ &= \frac{(\beta\alpha - 1)[N : N \cap G']}{1 + \alpha} \log |G| \geq \log |G|. \end{aligned}$$

Concerning (iv), note that as before (i) yields $k(G) > |N|^{1+\alpha}/|G|$, and we may assume that $|N|^{1+\alpha}/|G| < \log |G|$, that is, $[G : N] > (|G|^{\alpha}/ \log |G|)^{1/(1+\alpha)}$. From (ii) we obtain

$$k(G/N) \geq \frac{\beta}{1 + \alpha}(\alpha \log |G| - \log \log |G|).$$

By Lemma 2.3(c),

$$\begin{aligned} k(G) \geq k(G/N \cap G') &= [N : N \cap G']k(G/N) \\ &\geq \Big(\frac{\beta}{1 + \alpha}\Big)[N : N \cap G'](\alpha \log |G| - \log \log |G|). \end{aligned}$$

The latter is greater than or equal to $\log |G|$ when

$$\Big(\frac{\alpha\beta[N : N \cap G']}{1 + \alpha} - 1\Big) \log |G| \geq \frac{\beta[N : N \cap G']}{1 + \alpha} \log \log |G|,$$

which is (iv). □

Note that G' is nilpotent in Corollary 2.6(a), where we used $k(G/\Phi) \geq [G : \Phi]^{1/3}$ along with $s = \max\{\alpha_i\}$ in $|G| = \prod p_i^{\alpha_i}$ to conclude that $k(G) \geq |G|^{1/2s+1}$. In particular, $k(G) \geq \log |G|$ when $(2s + 1) \log \log |G| \leq \log |G|$. We always have $k(G) \geq \log \log |G|$ [ET], but here if we *also* know that $k(G) \geq (2s + 1) \log \log |G|$, then again $k(G) \geq \log |G|$.

Assuming only that G is solvable, Keller [Ke, Theorem 3.1] has proved that when $\Phi(G) = 1$ then $k(G) \geq |G|^{\beta}$ for some universal constant $\beta > 0$ (a specific value for β is not provided). In Corollary 3.10 we use $k(G/\Phi)$ and $s = \max\{\alpha_i\}$ to conclude that $k(G)$ has a logarithmic lower bound, in three different ways. As discussed in Remark 2.7, if $s \leq 2$ then $k(G) > |G|^{1/7}$ when G is solvable.

Corollary 3.10. *Suppose that G is solvable, $|G| = \prod p_i^{\alpha_i}$ (p_i distinct primes, $\alpha_i \geq 1$) and $s = \max\{\alpha_i\} \geq 3$.*

(a) *If $k(G/\Phi) \geq [G : \Phi]^{\alpha}$, $\alpha > 0$ and $k(G) \geq (s + 1) \log \log |G|$, then $k(G) \geq \alpha \log |G|$.*
(b) *If $k(G/\Phi) \geq [G : \Phi]^{\alpha}$ and $k(G) \geq s^{1+\epsilon}$ ($\epsilon > 0$), then $k(G) \geq \log |G|$ when G is sufficiently large (depending only on α, ϵ).*
(c) *If $k(G/\Phi) \geq s \log[G : \Phi]$, then $k(G) \geq \log |G|$.*

Proof. Since $s \geq 3$, we use Lemma 2.2(b) and (c) as in the proof of Corollary 2.6(a) to conclude that $k(\Phi) \geq |\Phi|^{2/(s-1)}$.

(a) From Lemma 2.3(a),

$$k(G) \geq k(G/\Phi) + \frac{k(\Phi) - 1}{[G : \Phi]} > \max\left\{k(G/\Phi), \frac{k(\Phi)}{[G : \Phi]}\right\}.$$

If $|\Phi|$ is 'large', that is, $|\Phi| \geq |G|^{1-1/(s+1)}$, then

$$k(G) > \frac{k(\Phi)}{[G : \Phi]} \geq \frac{|\Phi|^{2/(s-1)}}{[G : \Phi]} = \frac{|\Phi|^{(s+1)/(s-1)}}{|G|} \geq |G|^{1/(s-1)}.$$

When $s - 1 \leq \log|G|/\log\log|G|$ we conclude that $k(G) > \log|G|$. If $s - 1 \geq \log|G|/\log\log|G|$ then by assumption $k(G) \geq (s + 1)\log\log|G| > \log|G|$. Finally, if $|\Phi|$ is 'small', that is, $[G : \Phi] \geq |G|^{1/s+1}$, then $k(G) > k(G/\Phi) \geq [G : \Phi]^{\alpha} \geq |G|^{\alpha/s+1}$, and the latter is greater than or equal to $\log|G|$ as long as $\alpha/s + 1 \geq \log\log|G|/\log|G|$. Otherwise, $\alpha/(s + 1) \leq \log\log|G|/\log|G|$ and it follows from our assumption that $k(G) \geq (s + 1)\log\log|G| \geq \alpha\log|G|$.

(b) If $|\Phi| \geq |G|^{1-1/s}$, then $k(G) > |\Phi|^{(s+1)/(s-1)}/|G| \geq |G|^{1/s}$. If $s \leq \log|G|/\log\log|G|$ then $k(G) > \log|G|$. Otherwise

$$s > \frac{\log|G|}{\log\log|G|} \quad \text{and} \quad k(G) \geq s^{1+\epsilon} > \left(\frac{\log|G|}{\log\log|G|}\right)^{1+\epsilon} \geq \log|G|,$$

when $\log|G| \geq (\log\log|G|)^{1+1/\epsilon}$. On the other hand, if $|\Phi| < |G|^{1-1/s}$, then $k(G) > k(G/\Phi) \geq [G : \Phi]^{\alpha} > |G|^{\alpha/s}$. Here if $s \leq \alpha(\log|G|/\log\log|G|)$ then $k(G) > \log|G|$. Otherwise

$$s > \alpha\left(\frac{\log|G|}{\log\log|G|}\right) \quad \text{and} \quad k(G) \geq s^{1+\epsilon} > \alpha^{1+\epsilon}\left(\frac{\log|G|}{\log\log|G|}\right)^{1+\epsilon},$$

so $k(G) > \log|G|$ when $\log|G| \geq (\log\log|G|/\alpha)^{1+1/\epsilon}$.

(c) Since $k(\Phi) \geq |\Phi|^{2/(s-1)}$ and $k(G/\Phi) \geq s\log[G : \Phi]$, set $N = \Phi$, $\alpha = 2/(s - 1)$ and $\beta = s$ in Theorem 3.9. Then $\alpha(\beta - 1) = 2$, that is, $\beta\alpha - 1 = 1 + \alpha$ so (i)–(iii) are satisfied and $k(G) \geq \log|G|$. □

Comment. Theorem 3.9 implies that if (i) $k(N) \geq |N|^{\alpha}$, (ii) $k(G/N) \geq \beta\log[G : N]$ ($\beta > 0$) *and* $N \nleq G'$, then *either* (iii) *or* (iv) yield $k(G) \geq \log|G|$: (iii) $\beta \geq (1 + 3/\alpha)/2$ (≥ 2), (iv) N is abelian, $\beta > 1$ and $|G|^{1-1/\beta} \geq \log|G|$ (or $k(G) \geq \beta/(\beta - 1)\log\log|G|$). But whether or not $N \leq G'$, $|G|$ need not be large, as we see next.

Corollary 3.11. *Suppose that $N \trianglelefteq G$ and $k(N) \geq |N|^{\alpha}$ ($0 < \alpha \leq 1$).*

(a) *If $k(G/N) \geq \beta\log[G : N]$, $\beta \geq 1 + 2/\alpha$ (≥ 3), then $k(G) \geq (\beta/(1 + 2/\alpha))\log|G|$.*

(b) *Suppose that $k(G/N) \geq (1 + \alpha)\log_a[G : N]$, ($a := 1/\alpha > 1$). Then*

$$k(G) \geq \left(\frac{a + 1}{2a^2 + a}\right)\log_a|G| > \frac{\alpha}{2}\log_a|G|.$$

(Note that $(a + 1)/(2a^2 + a) < 2\alpha/3$.)

(c) *Suppose also that G/N is nilpotent. If $\alpha = 1$ then $k(G) \geq \frac{3}{4}\log_2|G|$* [Ca]. *If $\alpha = \frac{1}{2}$ then $k(G) \geq \frac{3}{10}\log_2|G|$. In general, let $n \geq 1$ be the smallest integer such that $k(N) \geq |N|^{1/2^n}$. Then $k(G) \geq (1/n2^{n+1})\log_2|G|$.*

Proof. (a) Suppose $k(G/N) \geq \beta \log_b[G:N]$. Choose c such that $\beta\log_b[G:N] = (1+2/\alpha)\log_c[G:N]$, that is, $\beta/(1+2/\alpha) = \log_c[G:N]/\log_b[G:N] = \log_c b$. Then hypotheses (i)–(iii) of Theorem 3.9 are satisfied (whether $N \leq G'$ or not) where 'β' in the Theorem equals $1+2/\alpha$, and the base of the logarithm is c. Thus $k(G) \geq \log_c|G| = (\beta/(1+2/\alpha))\log_b|G|$.

(b) Here we set $b := (a^a)^{(2a+1)/(a+1)}$. Since $a := 1/\alpha$,

$$\frac{1+\alpha}{1+2/\alpha} = \frac{a+1}{2a^2+a} = \log_b a.$$

Thus

$$\begin{aligned} k(G/N) &\geq (1+\alpha)\log_a[G:N] \\ &= \left(\left(1+\frac{2}{\alpha}\right)(\log_b a)\right)\log_a[G:N] \\ &= \left(1+\frac{2}{\alpha}\right)\log_b[G:N]. \end{aligned}$$

With $\beta := 1+2/\alpha$, hypotheses (i)–(iii) of Theorem 3.9 are satisfied, so

$$k(G) \geq \log_b|G| = \frac{\log_a|G|}{\log_a b} = \left(\frac{a+1}{2a^2+a}\right)\log_a|G|.$$

(c) Since G/N is nilpotent, $k(G/N) \geq \frac{3}{2}\log_2[G:N]$ [Ca], so we set $a = 2$ and $\alpha = \frac{1}{2}$ in (b), obtaining $k(G) \geq \frac{3}{10}\log_2|G|$ when $k(N) \geq |N|^{1/2}$. If $k(N) \geq |N|^{1/2^n}$ we use $a = 2^n$ and $\alpha = 1/2^n$ in (b). Thus

$$k(G/N) \geq \tfrac{3}{2}\log_2[G:N] \geq (1+\alpha)\log_a[G:N]$$

so, again using (b), we conclude that $k(G) > (\alpha/2)\log_a|G|$. Finally, $\log_{2^n}|G| = (1/n)\log_2|G|$ and $\alpha/2 = 1/2^{n+1}$. □

It follows from Theorem 3.9 that when $N \trianglelefteq G$, $N \not\leq G'$ and N is abelian, with $k(G/N) \geq (1+\epsilon)\log[G:N]$ $(\epsilon > 0)$, then $k(G) \geq \log|G|$ for $|G|$ large enough (depending only on ϵ). But what if $N \leq G'$? Generally, when G is solvable and N is a minimal normal subgroup of G, Theorem 3.12 gives the same conclusion.

Theorem 3.12. *For each $\epsilon > 0$ and all solvable groups G with $|G|$ large enough (depending only on ϵ), if N is a minimal normal subgroup of G and $k(G/N) \geq (1+\epsilon)\log[G:N]$, then $k(G) \geq \log|G|$.*

Proof. For ease of presentation we give a proof when $\log(\cdot) = \log_2(\cdot)$, but a careful examination of the proof shows that Theorem 3.12 holds in any base at least 2.

Among solvable groups we first consider G for which $[G:\Phi] \geq |G|^{1/\sqrt{\log_2|G|}}$.[1] It is always true that $F'(G) \leq \Phi(G)$, so $[G:F'] \geq [G:\Phi] \geq |G|^{1/\sqrt{\log_2|G|}}$. Among such G, and with γ the constant from Pyber's theorem, suppose G large enough so that $(\log_2\log_2|G|)^3 < \gamma(\log_2|G|)^{1/2}$, and thus $[G:\Phi] > |G|^{(\log_2\log_2|G|)^3/\gamma\log_2|G|}$. Since $\Phi(G/\Phi) = \{1\}$, by Pyber's theorem,

$$\begin{aligned} k(G) > k(G/\Phi) &\geq [G:\Phi]^{\gamma/(\log_2\log_2[G:\Phi])^2} \\ &> |G|^{(\log_2\log_2|G|/\log_2\log_2[G:\Phi])^2(\log_2\log_2|G|/\log_2|G|)} > \log_2|G|, \end{aligned}$$

the desired result.

Next we consider those solvable groups G satisfying $[G:\Phi] < |G|^{1/\sqrt{\log_2|G|}}$, and hence $[G:F] < |G|^{1/\sqrt{\log_2|G|}}$. By assumption, N is a minimal normal subgroup of G and $k(G/N) \geq (1+\epsilon)\log_2[G:N]$. If $(|N|-1)/\log_2|N| \geq |G|^{1/\sqrt{\log_2|G|}}$, then

$$\frac{|N|-1}{\log_2|N|} > [G:F],$$

and from Lemma 3.3(b) we conclude that $k(G) \geq \log_2|G|$. So finally we assume that $(|N|-1)/\log_2|N| < |G|^{1/\sqrt{\log_2|G|}}$. If $|N| \leq 25$, then

$$k(G) > k(G/N) \geq (1+\epsilon)\log_2[G:N] \geq (1+\epsilon)(\log_2|G| - \log_2 25),$$

and the latter is greater than or equal to $\log_2|G|$ if $|G| \geq 5^{2(1+1/\epsilon)}$. If $|N| \geq 25$, then

$$|N|^{1/2} \leq \frac{|N|-1}{\log_2|N|} < |G|^{1/\sqrt{\log_2|G|}},$$

which implies that $[G:N] > |G|^{1-2/\sqrt{\log_2|G|}}$, and

$$k(G) > k(G/N) \geq (1+\epsilon)\log_2[G:N] > (1+\epsilon)(\log_2|G| - 2\sqrt{\log_2|G|}).$$

Here $k(G) > \log_2|G|$ when $|G| \geq 2^{4(1+1/\epsilon)^2}$. □

As mentioned earlier, Theorem 3.12 holds when the base of the logarithm is 2 or greater. For example, we have the following corollary.

COROLLARY 3.13. *For all solvable groups G with $|G|$ large enough, if N is a minimal normal subgroup of G and $k(G/N) \geq \frac{3}{4}\log_2[G:N]$, then $k(G) \geq \log_3|G|$.*

PROOF. As in the theorem, first consider solvable G for which $[G:\Phi] \geq |G|^{1/\sqrt{\log_3|G|}}$. Since G may be assumed nonnilpotent, $[G:\Phi] \geq 6$ so

$$\left(\frac{\log_2\log_2[G:\Phi]}{\log_3\log_3[G:\Phi]}\right)^2 < 10.$$

[1] With considerably more effort (see Proposition 2.3 and its corollary in [Be3]), we have shown that under the latter condition on $|\Phi|$, for all large enough $|G|$ (depending only on $t > 0$) $k(G) > (\log_2|G|)^t$. Even more follows from Theorem 3.1 of Keller [Ke], but the proof here is much simpler and this is all we need.

(Note that $\log_2 \log_2 n = \log_2 \log_2 3 + (\log_2 3) \log_3 \log_3 n$ always holds.) Set $\beta_0 := \gamma/10$ (γ being Pyber's constant), so by Pyber's theorem,

$$\begin{aligned} k(G) > k(G/\Phi) &\geq [G:\Phi]^{\gamma/(\log_2 \log_2 [G:\Phi])^2} \\ &> [G:\Phi]^{\beta_0/(\log_3 \log_3 [G:\Phi])^2} \\ &> [G:\Phi]^{\beta_0/(\log_3 \log_3 |G|)^2} \\ &\geq |G|^{\beta_0/(\log_3 \log_3 |G|)^2 \sqrt{\log_3 |G|}}. \end{aligned}$$

If $|G|$ is so large that $(\log_3 \log_3 |G|)^3 \leq \beta_0 \sqrt{\log_3 |G|}$, then

$$k(G) > |G|^{\beta_0/\sqrt{\log_3 |G|}(\log_3 \log_3 |G|)^2} \geq |G|^{\log_3 \log_3 |G|/\log_3 |G|} = \log_3 |G|.$$

Working in base 3, when $[G:\Phi] < |G|^{1/\sqrt{\log_3 |G|}}$ the remainder of the proof goes through, since Lemma 3.3(b) makes no reference to the base. □

Example 3.14.

(a) Let G be solvable, with $N \leq M \trianglelefteq G$, N minimal normal in G, M/N abelian and G/M nilpotent. Then G/N is abelian-by-nilpotent so $k(G/N) \geq \frac{3}{4} \log_2 [G:N]$ [Ca]. By Corollary 3.13, $k(G) \geq \log_3 |G|$ for $|G|$ large enough.

(b) Suppose that G is solvable, N is a minimal normal subgroup of G and G/N is supersolvable. Then $k(G/N) \geq \frac{3}{5} \log_2 [G:N] = (1 + \frac{1}{5}) \log_4 [G:N]$ [Ca]. By Theorem 3.9, for $|G|$ large enough, $k(G) \geq \log_4 |G| = \frac{1}{2} \log_2 |G|$.

In [Be3, Proposition 2.3] we proved that for all solvable groups G with *abelian* Frattini subgroup $\Phi(G)$, if $|G|$ is large enough (depending only on $t > 0$) then $k(G) > (\log_2 |G|)^t$, and Keller [Ke, Theorem 4.1] proved that $k(G) > |G|^{\beta/2+\beta}$. Here we obtain a $(\log |G|)^t$ lower bound for $k(G)$ assuming only that the nilpotence class of $\Phi(G)$ is 'small enough' with respect to $\log |G|$, and $|G|$ is large enough, depending only on t.

Theorem 3.15. *For all solvable groups G with $|G|$ large enough (depending only on $t \geq 1$), if the class $c(\Phi)$ satisfies $c \leq \sqrt{\log |G|}(1 - 1/\log\ \log |G|)$ then $k(G) > (\log |G|)^t$.*

Proof. As in the proofs of Proposition 2.3 and its corollary in [Be3], when $[G:\Phi] \geq |G|^{1/\sqrt{\log |G|}}$ we use Pyber's theorem to prove that when $|G|$ is large enough, depending only on t, $k(G) > (\log |G|)^t$.

Suppose on the other hand that $[G:\Phi] < |G|^{1/\sqrt{\log |G|}}$. By assumption, $\Phi(G)$ has nilpotence class c, so

$$k(\Phi) \geq |\Phi|^{1/c} > |G|^{(1/c)(1-1/\sqrt{\log |G|})}.$$

Now

$$k(G) \geq k(G/\Phi) + \frac{k(\Phi) - 1}{[G:\Phi]} > \frac{k(\Phi)}{[G:\Phi]} > \frac{|G|^{(1/c)(1-1/\sqrt{\log |G|})}}{[G:\Phi]}.$$

Again using our assumption that $[G : \Phi] < |G|^{1/\sqrt{\log |G|}}$,

$$k(G) > |G|^{(1/c)(1-1/\sqrt{\log |G|})-1/\sqrt{\log |G|}}.$$

Thus $k(G) > (\log |G|)^t$ as long as

$$\frac{1}{c}\Big(1 - \frac{1}{\sqrt{\log |G|}}\Big) > \frac{t \log\log |G|}{\log |G|} + \frac{1}{\sqrt{\log |G|}},$$

that is, as long as

$$c(\sqrt{\log |G|} + t \log\log |G|) < \log |G| - \sqrt{\log |G|}. \tag{3.3}$$

Finally, for all large enough $|G|$ (depending only on t), $(t \log\log |G|)^2 < \sqrt{\log |G|}$, that is,

$$(t \log\log |G| - 1)(t \log\log |G| + \sqrt{\log |G|}) < (\sqrt{\log |G|} - 1)(t \log\log |G|),$$

which is equivalent to

$$1 - \frac{1}{t \log\log |G|} < \frac{\sqrt{\log |G|} - 1}{t \log\log |G| + \sqrt{\log |G|}}.$$

By hypothesis,

$$c \le \sqrt{\log |G|}\Big(1 - \frac{1}{\log\log |G|}\Big) \le \sqrt{\log |G|}\Big(1 - \frac{1}{t \log\log |G|}\Big),$$

since $t \ge 1$. But the latter is less than $(\log |G| - \sqrt{\log |G|})/(\sqrt{\log |G|} + t \log\log |G|)$ so (3.3) is indeed satisfied, and $k(G) > (\log |G|)^t$ in each case. □

Remark 3.16. Keller [Ke, Theorem 3.1] proved that when $\Phi(G) = 1$, $k(G) \ge |G|^\beta$, where $\beta < 1$ is a positive constant. Thus $k(G) > k(G/\Phi) \ge [G : \Phi]^\beta$, and if $|\Phi| \le |G|^{1-1/\sqrt{\log |G|}}$ we have $k(G) > |G|^{\beta/\sqrt{\log |G|}} > (\log |G|)^t$ for all sufficiently large $|G|$ (depending only on t). On the other hand, if $|\Phi| > |G|^{1-1/\sqrt{\log |G|}}$, then (as shown in the proof above) $k(G) > |G|^{(1/c)(1-1/\sqrt{\log |G|})-1/\sqrt{\log |G|}}$. For $c \le \frac{2}{3}\sqrt{\log |G|}$ it is straightforward to show that this lower bound for $k(G)$ is (for all large enough $|G|$) greater than $((c/2)|G|^{1/c})^{\beta/3}$, the lower bound given in [Ke, Theorem 4.1].

We are now able to generalise the last statement of Lemma 3.1, no longer assuming that $\Phi(N)$ is abelian.

Corollary 3.17. *Suppose N is solvable and $N \trianglelefteq G$, with*

(i) $k(G/N) \ge \beta[G : N]^\alpha$ *($0 < \alpha < 1 < \beta$) and*
(ii) *the nilpotence class* $c(\Phi(N)) \le \sqrt{\log |N|}(1 - 1/\log\ \log |N|)$.

Then $k(G) \ge \log |G|$ when $|G|$ is large enough (depending only on α, β).

PROOF. We will show that hypothesis (ii) of Lemma 3.1 is also satisfied for the pair (G, N), and thus the conclusion follows. By hypothesis (i), $k(G) > k(G/N) > [G : N]^{\alpha}$, and the latter is greater than or equal to $\log |G|$ when $|N| \le |G|/(\log |G|)^{1/\alpha}$. So suppose that $|N| \ge |G|/(\log |G|)^{1/\alpha}$. According to the proof of Theorem 3.15 (with N replacing G and $t = 1 + 1/\alpha$), we only need $\sqrt{\log |N|}/(\log\log |N|)^2 > (1 + 1/\alpha)^2$ to ensure that $k(N) \ge (\log |N|)^{1+1/\alpha}$ and hence that hypothesis (ii) of Lemma 3.1 is also satisfied. Since $|N| \ge |G|/(\log |G|)^{1/\alpha}$ and $\sqrt{\log x}/(\log\log x)^2$ is an increasing function for $\log\log x > 4$, we conclude that when $|G|$ is large enough (depending on α), hypothesis (ii) of Lemma 3.1 is satisfied along with hypothesis (i), and the desired conclusion follows. □

4. $k(G/N) \ge (\log[G : N])^t$

Up to this point we have assumed that either $k(G/N) \ge \beta[G : N]^{\alpha}$ or $k(G/N) \ge \beta \log[G : N]$, β a positive constant. But sometimes (the best) we may assume is that $k(G/N) \ge (\log[G : N])^t$, $t \ge 2$. (Again, we note that no collection $\{G\}$ is known with $|G| \to \infty$ and $k(G) < (\log |G|)^2$.)

LEMMA 4.1. *Let $N \trianglelefteq G$, N nilpotent and $k(G/N) \ge (\log[G : N])^t$, $t \ge 2$. If N has nilpotence class $c \ge 1$, then $k(G) > (\log |G|)^{t-1}$ for all such G with $|G|$ large enough, depending only on c, t.*

PROOF. We prove that with these hypotheses $k(G) > (\log |G|)^{t-1}$ as long as $\{|G|, c, t\}$ satisfy

$$(\log |G|)^{1-1/t}((\log |G|)^{1/t} - (c + 1)) \ge c(t - 1) \log\log |G|. \tag{4.1}$$

With $\log(\cdot) = \log_b(\cdot)$, we first note that $(\log[G : N])^t > (\log |G|)^{t-1}$ if and only if $[G : N] > b^{(\log |G|)^{1-1/t}}$. So we assume that $|N| \ge |G|/b^{(\log |G|)^{1-1/t}}$, which is equivalent to

$$\frac{|N|^{1+1/c}}{|G|} \ge \frac{|G|^{1/c}}{b^{(1+1/c)(\log |G|)^{1-1/t}}}.$$

But $k(N) \ge |N|^{1/c}$, and hence, by Lemma 2.3(a), $k(G) > |G|^{1/c}/b^{(1+1/c)(\log |G|)^{1-1/t}}$. Thus $k(G) > (\log |G|)^{t-1}$ as long as $|G| \ge (\log |G|)^{c(t-1)} b^{(c+1)(\log |G|)^{1-1/t}}$, which is equivalent to (4.1). □

Note. Suppose that N is abelian ($c = 1$). It is easy to check that (4.1) follows from $|G| \ge b^{3^t}$ and

$$1 - \frac{1}{t} \ge \frac{\log\log\log |G| + \log(t - 1)}{\log\log |G|}.$$

If $b = 3$, the latter follows from $|G| \ge 3^{3^t}$ and $t \ge 2$. If $b = 2$, (4.1) follows from $|G| \ge 2^{2^{2t}}$ and $t \ge 2$.

COROLLARY 4.2. *Suppose that N is a nilpotent normal subgroup of G and the nilpotence class c of N satisfies $2c + 1 \le (\log |G|)^{1/2}$. If also $k(G/N) \ge (\log[G : N])^2$, then $k(G) > \log |G|$ for all such G with $|G|$ large enough.*

Proof. Our assumption on c yields (4.1) of Lemma 4.1, with $t = 2$. Hence $k(G) > \log |G|$. □

Question 4.3. When $\Phi(G) = 1$ (or more generally when $F(G)$ is abelian) does $k(G/F) \geq (\log[G:F])^2$ hold? If so, then $k(G/F) \geq (\log[G:F])^2$ *always*, since $\Phi(G/\Phi) = 1$ and $F(G/\Phi) = F(G)/\Phi(G)$ (is abelian) so $G/F(G) \cong G/\Phi/F(G/\Phi)$. In general, Corollary 4.2 implies that when $|G|$ is large enough, $k(G/F) \geq (\log[G:F])^2$ and the nilpotence class $c(F)$ satisfies $c \leq ((\log |G|)^{1/2} - 1)/2$, then $k(G) \geq \log |G|$.

Corollary 4.4. *If* $k(G/N) \geq (\log[G:N])^t$ $(t \geq 2)$, *then* $k(G/N') \geq \log[G:N']^{t-1}$, *whenever* $[G:N']$ *is large enough, depending only on* t.

Proof. In Lemma 4.1, replace G by G/N' and N by N/N'. The conclusion follows as long as $[G:N']$ satisfies (4.1) with respect to t, when $c = 1$. □

Lemma 4.5. *Let* $y > x \geq b^e$, $t \geq 2$, *and*

(i) $(\log x)^{1-1/t}((\log x)^{1/t} - 2) \geq (t-1) \log\log x$, *where* $\log(\cdot) = \log_b(\cdot)$.

Then

(ii) $(\log y)^{1-1/(t-1)}((\log y)^{1/(t-1)} - 2) \geq (t-2) \log\log y$.

Proof. Note that (ii) is automatically satisfied when $t = 2$, since $y > b^2$. So assume that $t \geq 3$, and we first check that (i) $\Longrightarrow$ (ii) follows from

$$\frac{(\log y)(1 - 2(\log y)^{-1/t-1})}{(t-2)\log\log y} > \frac{(\log x)(1 - 2(\log x)^{-1/t})}{(t-1)\log\log x} \geq 1. \tag{4.2}$$

Since $\log x/\log\log x$ is an increasing function for $x \geq b^e$,

$$\frac{\log y}{\log\log y} > \frac{\log x}{\log\log x}.$$

Also, $\log y > (\log x)^{(t-1)/t}$ implies that $1 - 2(\log y)^{-1/t-1} > 1 - 2(\log x)^{-1/t}$, and (4.2) follows. □

Theorem 4.6. *Suppose that* G *is solvable,* $N \trianglelefteq G$ *and* $k(G/N) \geq (\log[G:N])^{d(N)+1}$, $d(N)$ *the derived length of* N. *Then* $k(G) \geq \log |G|$, *as long as* $[G:N']$ *is large enough, depending only upon* $d(N)$.

Proof. We will prove that $k(G) \geq \log |G|$ as long as (4.1) of Lemma 4.1 is satisfied, with $[G:N']$ replacing $|G|$ and $d(N) + 1$ replacing t, always with $c = 1$.

When N is abelian, the conclusion follows from Lemma 4.1, with $c = 1$ and $t = 2$. When $d(N) = 2$ the assumption is that $k(G/N) \geq (\log[G:N])^3$. If $[G:N']$ satisfies (4.1) with $t = 3$, then from Corollary 4.4 $k(G/N') \geq (\log[G:N'])^2$. Here N' is abelian so we may again apply Lemma 4.1 with $c = 1$, $t = 2$ and conclude that $k(G) \geq \log |G|$ as long as $|G|$ satisfies (4.1) with $t = 2$. From Lemma 4.5 with $t = 3$, $[G:N']$ replacing x and $[G:N''] = |G|$ replacing y, we see that $|G|$ indeed satisfies (4.1) with $t = 2$.

Assume for an inductive proof that the theorem is true whenever $d(N) = n$. Now let $d(N) = n + 1$ and $k(G/N) \geq (\log[G : N])^{d(N)+1} = (\log[G : N])^{n+2}$. Suppose also that $[G : N']$ satisfies (4.1) with $t = n + 2$. From Corollary 4.4,

$$k(G/N') \geq (\log[G : N'])^{n+1} = (\log[G : N'])^{d(N')+1}.$$

From our inductive hypothesis ($d(N') = n$), $k(G) \geq \log |G|$ as long as $[G : N'']$ satisfies (4.1) with $t = n + 1$. But Lemma 4.5, with $t = n + 2$, $[G : N']$ replacing x, and $[G : N'']$ replacing y, guarantees that $[G : N'']$ indeed satisfies (4.1) with $t = n + 1$. Thus the theorem is also true when $d(N) = n + 1$. □

As mentioned, Keller [Ke, Theorem 3.1] proved that when G is solvable and $\Phi(G) = 1$, $k(G) \geq |G|^{\beta}$, where $\beta < 1$ is a positive constant. We now use this to significantly improve the result of [Be2, Theorem 1] that if G has derived length $d(G)$, then $k(G) \geq |G|^{1/2^{d}-1}$, shifting attention to $d(F(G))$.

Theorem 4.7. *Suppose that G is a solvable group with Fitting subgroup $F(G)$. Then for each $n \geq 1$,*

$$k(G/F^{(n)}(G)) \geq [G : F^{(n)}(G)]^{1/(1+1/\beta)2^{n}-1} \tag{4.3}$$

where β is the constant from Keller's theorem. In particular,

$$k(G) \geq |G|^{1/(1+1/\beta)2^{d}-1}$$

where $d = d(F)$ is the derived length of $F(G)$.

Proof. From Keller's theorem, $k(G/\Phi) \geq [G : \Phi]^{\beta}$. If $F(G)$ is abelian, so is $\Phi(G)$, and using Lemma 2.3(b) with $N = \Phi$ and $\alpha = 1$ we obtain the inequality for $k(G)$ when $d = 1$. If $N \trianglelefteq G$ and $N \leq \Phi(G)$, then $\Phi(G/N) = \Phi(G)/N$ and $F(G/N) = F(G)/N$ [Hu, III. 3.4, 4.2]. Thus

$$k((G/F')/\Phi(G/F')) = k(G/\Phi) \geq [G : \Phi]^{\beta} = [G/F' : \Phi(G/F')]^{\beta},$$

and $\Phi(G/F')$ is abelian (Lemma 2.3(a)). As before, now with $N = \Phi(G/F')$, we conclude that $k(G/F') \geq [G : F']^{1/(1+2/\beta)}$, and thus inequality (4.3) with $n = 1$. If, in addition, $F'(G)$ is abelian, another use of Lemma 2.3(b) with $N = F'(G)$, $\alpha = 1$ and β replaced by $(1 + 2/\beta)^{-1}$ yields the desired inequality when $d = 2$.

To complete the proof of (4.3) by induction, we assume that $n \geq 2$, and for all solvable groups G,

$$k(G/F^{(n-1)}(G)) \geq [G : F^{(n-1)}(G)]^{1/(1+1/\beta)2^{n-1}-1}. \tag{4.4}$$

First note that $F'(G/F^{(n)}) = (F/F^{(n)})' = F'/F^{(n)}$ so $F''(G/F^{(n)}) = F''/F^{(n)} \ldots$ and finally $F^{(n-1)}(G/F^{(n)}) = F^{(n-1)}/F^{(n)}$ is abelian. Next substitute $G/F^{(n)}(G)$ for G in (4.4), and use $G/F^{(n-1)} \cong (G/F^{(n)})/F^{(n-1)}(G/F^{(n)})$ to obtain

$$k((G/F^{(n)})/F^{(n-1)}(G/F^{(n)})) \geq [(G/F^{(n)}) : F^{(n-1)}(G/F^{(n)})]^{1/(1+1/\beta)2^{n-1}-1}.$$

Since $F^{(n-1)}(G/F^{(n)})$ is abelian we use Lemma 2.3(b) with $N = F^{(n-1)}(G/F^{(n)})$, $\alpha = 1$ and β replaced by $1/(1 + 1/\beta)2^{n-1} - 1$ to obtain inequality (4.3). □

Setting $\beta_0 = \beta/(\beta+1)$ immediately leads to the following corollary.

COROLLARY 4.8. *If* $2^{d(F)} \le \beta_0(\log|G|/\log\log|G|+1)$*, then* $k(G) \ge \log|G|$.

REMARK 4.9. If G is a nilpotent group of nilpotence class c, then $d(G) \le \lfloor \log_2 c \rfloor + 1$ [Hu, III. 2.12], so $k(G) \ge \log_2|G|$ when

$$c(F(G)) \le \frac{\beta_0}{2}\left(\frac{\log|G|}{\log\log|G|}+1\right).$$

This may be compared to Corollary 4.2, and more importantly to Theorem 3.15, and Corollary 3.2(b)(ii).

References

[Be1] E. A. Bertram, 'Large centralizers in finite solvable groups', *Israel J. Math.* **47** (1984), 335–344.
[Be2] E. A. Bertram, 'Lower bounds for the number of conjugacy classes in finite solvable groups', *Israel J. Math.* **75** (1991), 243–255.
[Be3] E. A. Bertram, 'Lower bounds for the number of conjugacy classes in finite groups', in: *Ischia Group Theory 2004*, Contemporary Mathematics, 402 (American Mathematical Society, Providence, RI, 2006), pp. 95–117.
[Ca] M. Cartwright, 'The number of conjugacy classes of certain finite groups', *J. Lond. Math. Soc.* (2) **36** (1985), 393–404.
[ET] P. Erdős and P. Turán, 'On some problems of a statistical group theory IV', *Acta. Math. Acad. Sci. Hung.* **19** (1968), 413–435.
[HP] W. M. Hill and D. B. Parker, 'The nilpotence class of the Frattini subgroup', *Israel J. Math.* **15** (1973), 211–215.
[Hu] B. Huppert, *Endliche Gruppen I* (Springer, Berlin, 1967).
[Ke] T. M. Keller, 'Finite groups have even more conjugacy classes', *Israel J. Math.* **181** (2011), 433–444.
[Ne] M. Newman, 'A bound for the number of conjugacy classes in a group', *J. Lond. Math. Soc.* **43** (1968), 108–110.
[Ni] I. Niven, 'Averages of exponents in factoring integers', *Proc. Amer. Math. Soc.* **22** (1969), 356–360.
[Py] L. Pyber, 'Finite groups have many conjugacy classes', *J. Lond. Math. Soc.* (2) **46** (1992), 239–249.
[Sh] G. J. Sherman, 'A lower bound for the number of conjugacy classes in a finite nilpotent group', *Pacific J. Math.* **80** (1979), 253–254.
[Ta] D. R. Taunt, 'On *A*-groups', *Proc. Cambridge Philos. Soc.* **45** (1949), 24–42.
[VS] A. Vera-López and J. Sangroniz, 'The finite groups with thirteen and fourteen conjugacy classes', *Math. Nach.* **280**(5–6) (2007), 676–694.
[VV1] A. Vera-López and J. Vera-López, 'Classification of finite groups according to the number of conjugacy classes', *Israel J. Math.* **51** (1985), 305–338.
[VV2] A. Vera-López and J. Vera-López, 'Classification of finite groups according to the number of conjugacy classes II', *Israel J. Math.* **56** (1986), 188–221.

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